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TIMELIKE HORIZONTAL BIHARMONIC $\mathcal{S}$-CURVES ACCORDING TO SABBAN FRAME IN $\mathbb{H}$

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ABSTRACT. In this paper, we study timelike horizontal biharmonic curves according to Sabban frame in the $\mathbb{H}$. We characterize the timelike horizontal biharmonic curves in terms of their geodesic curvature. Finally, we find out their explicit parametric equations according to Sabban Frame.

1. Introduction

The theory of biharmonic functions is an old and rich subject. Biharmonic functions have been studied since 1862 by Maxwell and Airy to describe a mathematical model of elasticity. The theory of polyharmonic functions was developed later on, for example, by E. Almansi, T. Levi-Civita and M. Nicolescu.

This study is organised as follows: Firstly, we study timelike horizontal biharmonic curves accordig to Sabban frame in the Heisenberg group Heis^3. Secondly, we characterize the timelike horizontal biharmonic curves in terms of their geodesic curvature. Finally, we find out their explicit parametric equations according to Sabban Frame.

2. The Lorentzian Heisenberg Group $\mathbb{H}$

Heisenberg group $\mathbb{H}$ can be seen as the space $\mathbb{R}^3$ endowed with the following multipilcation:

$$(\overline{x},\overline{y},\overline{z})(x,y,z)=(\overline{x}+x,\overline{y}+y,\overline{z}+z-\frac{1}{2}\overline{x}y+\frac{1}{2}x\overline{y})$$

Heis^3 is a three-dimensional, connected, simply connected and 2-step nilpotent Lie group.

The identity of the group is $(0,0,0)$ and the inverse of (x,y,z) is given by $(-x,-y,-z)$. The left-invariant Lorentz metric on $\mathbb{H}$ is

$$g=-dx^2+dy^2+(xdy+dz)^2.$$

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The following set of left-invariant vector fields forms an orthonormal basis for the corresponding Lie algebra:

$$\left\{\mathbf{e}_1 = \frac{\partial}{\partial z},\ \mathbf{e}_2 = \frac{\partial}{\partial y} - x\frac{\partial}{\partial z},\ \mathbf{e}_3 = \frac{\partial}{\partial x}\right\}. \tag{2.1}$$

The characterising properties of this algebra are the following commutation relations:

$$g(\mathbf{e}_1, \mathbf{e}_1) = g(\mathbf{e}_2, \mathbf{e}_2) = 1, \quad g(\mathbf{e}_3, \mathbf{e}_3) = -1.$$

Proposition 2.1. *For the covariant derivatives of the Levi-Civita connection of the left-invariant metric g, defined above the following is true:*

$$\nabla = \frac{1}{2}\begin{pmatrix} 0 & \mathbf{e}_3 & \mathbf{e}_2 \\ \mathbf{e}_3 & 0 & \mathbf{e}_1 \\ \mathbf{e}_2 & -\mathbf{e}_1 & 0 \end{pmatrix}, \tag{2.2}$$

where the (i, j)-element in the table above equals $\nabla_{\mathbf{e}_i}\mathbf{e}_j$ for our basis

$$\{\mathbf{e}_k, k = 1, 2, 3\}.$$

The unit pseudo-Heisenberg sphere (Lorentzian Heisenberg sphere) is defined by

$$\left(\mathbb{S}_1^2\right)_{\mathbb{H}} = \{\beta \in \mathbb{H} : g(\beta, \beta) = 1\}.$$

3. Timelike Horizontal Biharmonic $\mathcal{S}$-Curves According To Sabban Frame In The $\left(\mathbb{S}_1^2\right)_{\mathbb{H}}$

Let $\gamma : I \longrightarrow \mathbb{H}$ be a timelike curve in the Lorentzian Heisenberg group $\mathbb{H}$ parametrized by arc length. Let $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ be the Frenet frame fields tangent to the Lorentzian Heisenberg group $\mathbb{H}$ along γ defined as follows:

$\mathbf{T}$ is the unit vector field γ' tangent to γ, $\mathbf{N}$ is the unit vector field in the direction of $\nabla_{\mathbf{T}}\mathbf{T}$ (normal to γ), and $\mathbf{B}$ is chosen so that $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ is a positively oriented orthonormal basis. Then, we have the following Frenet formulas:

$$\begin{aligned} \nabla_{\mathbf{T}}\mathbf{T} &= \kappa\mathbf{N}, \\ \nabla_{\mathbf{T}}\mathbf{N} &= \kappa\mathbf{T} + \tau\mathbf{B}, \\ \nabla_{\mathbf{T}}\mathbf{B} &= -\tau\mathbf{N}, \end{aligned} \tag{3.1}$$

where κ is the curvature of γ and τ is its torsion,

$$\begin{aligned} &g(\mathbf{T}, \mathbf{T}) = -1,\ g(\mathbf{N}, \mathbf{N}) = 1,\ g(\mathbf{B}, \mathbf{B}) = 1, \\ &g(\mathbf{T}, \mathbf{N}) = g(\mathbf{T}, \mathbf{B}) = g(\mathbf{N}, \mathbf{B}) = 0. \end{aligned}$$

Now we give a new frame different from Frenet frame. Let $\alpha : I \longrightarrow \left(\mathbb{S}_1^2\right)_{\mathbb{H}}$ be unit speed spherical timelike curve. We denote σ as the arc-length parameter of α . Let us denote $\mathbf{t}(\sigma) = \alpha'(\sigma)$, and we call $\mathbf{t}(\sigma)$ a unit tangent vector of α. We now set a vector $\mathbf{s}(\sigma) = \alpha(\sigma) \times \mathbf{t}(\sigma)$ along α. This frame is called the Sabban frame of α on $\left(\mathbb{S}_1^2\right)_{\mathbb{H}}$. Then we have the following spherical Frenet-Serret formulae of α :

$$\begin{aligned} \alpha' &= \mathbf{t}, \\ \mathbf{t}' &= \alpha + \kappa_g \mathbf{s}, \\ \mathbf{s}' &= \kappa_g \mathbf{t}, \end{aligned} \tag{3.2}$$

where κ_g is the geodesic curvature of the timelike curve α on the $\left(\mathbb{S}_1^2\right)_{\mathbb{H}}$ and

$$\begin{aligned} g(\mathbf{t},\mathbf{t}) = -1,\ g(\alpha,\alpha) = 1,\ g(\mathbf{s},\mathbf{s}) = 1, \\ g(\mathbf{t},\alpha) = g(\mathbf{t},\mathbf{s}) = g(\alpha,\mathbf{s}) = 0. \end{aligned}$$

With respect to the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, we can write

$$\begin{aligned} \alpha &= \alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2 + \alpha_3 \mathbf{e}_3, \\ \mathbf{t} &= t_1 \mathbf{e}_1 + t_2 \mathbf{e}_2 + t_3 \mathbf{e}_3, \\ \mathbf{s} &= s_1 \mathbf{e}_1 + s_2 \mathbf{e}_2 + s_3 \mathbf{e}_3. \end{aligned} \tag{3.3}$$

To separate a biharmonic curve according to Sabban frame from that of Frenet-Serret frame, in the rest of the paper, we shall use notation for the curve defined above as biharmonic $\mathcal{S}$-curve.

Theorem 3.1. *$\alpha : I \longrightarrow \left(\mathbb{S}_1^2\right)_{\mathbb{H}}$ is a timelike biharmonic $\mathcal{S}$-curve if and only if*

$$\begin{aligned} \kappa_g &= \text{constant} \neq 0, \\ 1 + \kappa_g^2 &= [-\frac{1}{4} + \frac{1}{2} s_1^2] + \kappa_g [\alpha_1 s_1], \\ \kappa_g^3 &= \alpha_3 s_3 - \kappa_g [\frac{1}{4} - \frac{1}{2} \alpha_1^2]. \end{aligned} \tag{3.4}$$

Proof. Using (2.1) and Sabban formulas (3.2), we have (3.4).

Corollary 3.2. *All of timelike biharmonic $\mathcal{S}$-curves in $\left(\mathbb{S}_1^2\right)_{\mathbb{H}}$ are helices.*

Consider a nonintegrable 2-dimensional distribution $(x, y) \longrightarrow \mathcal{H}_{(x,y)}$ in $\mathbb{H}$ defined as $\mathcal{H} = \ker \omega$, where $\omega = xdy + dz$ is a 1-form on $\mathbb{H}$. The distribution $\mathcal{H}$ is called the horizontal distribution.

A curve $\alpha : I \longrightarrow \mathbb{H}$ is called horizontal curve if $\gamma'(s) \in \mathcal{H}_{\gamma(s)}$, for every s.

Lemma 3.4. *Let α be a horizontal curve. Then,*

$$t_1(\sigma) = 0. \tag{3.5}$$

Proof. Using first equation of the system (3.3), we have

$$\omega(\alpha'(\sigma)) = \omega(t_1(\sigma)\mathbf{e}_1 + t_2(\sigma)\mathbf{e}_2 + t_3(\sigma)\mathbf{e}_3). \tag{3.6}$$

Thus, from (2.1) and (3.6), we obtain (3.5), which completes the proof.

Theorem 3.3. *Let α be a unit speed non-geodesic timelike biharmonic $\mathcal{S}$-curve. Then, the parametric equations of α are*

$$\begin{aligned} x^{\mathcal{S}}(\sigma) &= \frac{1}{\sqrt{1+\kappa_g^2}}\sinh[\sqrt{1+\kappa_g^2}\sigma+\mathcal{B}_1]+\mathcal{B}_2, \\ y^{\mathcal{S}}(\sigma) &= \frac{1}{\sqrt{1+\kappa_g^2}}\cosh[\sqrt{1+\kappa_g^2}\sigma+\mathcal{B}_1]+\mathcal{B}_3, \\ z^{\mathcal{S}}(\sigma) &= \frac{1}{2\left(1+\kappa_g^2\right)}[\sqrt{1+\kappa_g^2}\sigma+\mathcal{B}_1]-\frac{1}{4\left(1+\kappa_g^2\right)}\sinh 2[\sqrt{1+\kappa_g^2}\sigma+\mathcal{B}_1] \\ &\quad -\frac{\mathcal{B}_2}{\sqrt{1+\kappa_g^2}}\cosh[\sqrt{1+\kappa_g^2}\sigma+\mathcal{B}_1]+\mathcal{B}_4, \end{aligned} \tag{3.7}$$

where $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4$ are constants of integration.

Proof. From [9] and (3.5), we have

$$\mathbf{t} = \sinh[\sqrt{1+\kappa_g^2}\sigma+\mathcal{B}_1]\mathbf{e}_2+\cosh[\sqrt{1+\kappa_g^2}\sigma+\mathcal{B}_1]\mathbf{e}_3. \tag{3.8}$$

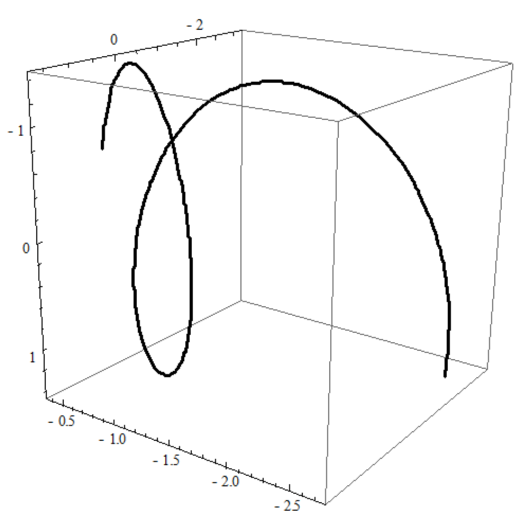
Using (2.1) in (3.8), we obtain

$$\begin{aligned} \mathbf{t} = (&\cosh[\sqrt{1+\kappa_g^2}\sigma+\mathcal{B}_1], \sinh[\sqrt{1+\kappa_g^2}\sigma+\mathcal{B}_1], \\ &(\frac{1}{\sqrt{1+\kappa_g^2}}\sinh[\sqrt{1+\kappa_g^2}\sigma+\mathcal{B}_1]+\mathcal{B}_2)\sinh[\sqrt{1+\kappa_g^2}\sigma+\mathcal{B}_1]), \end{aligned}$$

where $\mathcal{B}_1, \mathcal{B}_2$ are constants of integration.

Integrating both sides, we have (3.7). This proves our assertion. Thus, the proof of theorem is completed.

We can use Mathematica in above theorem, yields



References

[1] M. Babaarslan and Y. Yayli: *The characterizations of constant slope surfaces and Bertrand curves,* International Journal of the Physical Sciences 6(8) (2011), 1868-1875.

[2] R. Caddeo and S. Montaldo: *Biharmonic submanifolds of* $\mathbb{S}^3$, Internat. J. Math. 12(8) (2001), 867–876.

[3] J. Eells and J. H. Sampson: *Harmonic mappings of Riemannian manifolds,* Amer. J. Math. 86 (1964), 109–160.

[4] A. Gray: *Modern Differential Geometry of Curves and Surfaces with Mathematica,* CRC Press, 1998.

[5] S. Izumiya and N. Takeuchi: *Special Curves and Ruled Surfaces,* Contributions to Algebra and Geometry 44 (2003), 203-212.

[6] G. Y.Jiang: *2-harmonic isometric immersions between Riemannian manifolds,* Chinese Ann. Math. Ser. A 7(2) (1986), 130–144.

[7] G. Y.Jiang: *2-harmonic isometric immersions between Riemannian manifolds,* Chinese Ann. Math. Ser. A 7(2) (1986), 130–144.

[8] T. Körpınar and E. Turhan: *On characterization of B-canal surfaces in terms of biharmonic B-slant helices according to Bishop frame in Heisenberg group* $Heis^3$, J. Math. Anal. Appl. 382 (2011), 57–65.

[9] T. Körpınar and E. Turhan: *Timelike Biharmonic S-Curves According to Sabban Frame in Lorentzian Heisenberg Group* $Heis^3$, (in press)

[10] DY. Kwon , FC. Park, DP Chi: *Inextensible flows of curves and developable surfaces,* Appl. Math. Lett. 18 (2005), 1156-1162.

[11] Y. Ou and Z. Wang: *Linear Biharmonic Maps into Sol, Nil and Heisenberg Spaces,* Mediterr. j. math. 5 (2008), 379–394.

[12] S. Rahmani: *Metriqus de Lorentz sur les groupes de Lie unimodulaires, de dimension trois*, Journal of Geometry and Physics 9 (1992), 295-302.

[13] E. Turhan and T. Körpınar, *On spacelike biharmonic new type b-slant helices with timelike m2 according to Bishop frame in Lorentzian Heisenberg group* H^3, Advanced Modeling and Optimization, 14 (2) (2012), 297-302.

[14] E. Turhan and T. Körpınar: *On Characterization Of Timelike Horizontal Biharmonic Curves In The Lorentzian Heisenberg Group* $Heis^3$, Zeitschrift für Naturforschung A- A Journal of Physical Sciences 65a (2010), 641-648.

[15] E. Turhan and T. Körpınar: *Horizontal geodesics in Lorentzian Heisenberg group* $Heis^3$, Advanced Modeling and Optimization, 14 (2) (2012), 311-319.

[16] M. Turgut and S. Yilmaz: *Smarandache Curves in Minkowski Space-time,* International Journal of Mathematical Combinatorics 3 (2008), 51-55.

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