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BIHARMONIC CURVES IN THE $\widetilde{\mathcal{SL}_2}(\mathbb{R})$

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Abstract. In this paper, we study biharmonic curves in the $\widetilde{\mathcal{SL}_2}(\mathbb{R})$. We characterize the biharmonic curves in terms of their curvature and torsion and we prove that all of biharmonic curves are helices in the $\widetilde{\mathcal{SL}_2}(\mathbb{R})$. Finally, we find out their explicit parametric equations.

1. Introduction

Harmonic maps $f:(M,g)\longrightarrow(N,h)$ between manifolds are the critical points of the energy

$$E(f)=\frac{1}{2}\int_M e(f)\,v_g, \tag{1.1}$$

where v_g is the volume form on (M,g) and

$$e(f)(x):=\frac{1}{2}\left\|df(x)\right\|^2_{T^*M\otimes f^{-1}TN}$$

is the energy density of f at the point $x\in M$.

Critical points of the energy functional are called harmonic maps.

The first variational formula of the energy gives the following characterization of harmonic maps: the map f is harmonic if and only if its tension field $\tau(f)$ vanishes identically, where the tension field is given by

$$\tau(f)=\operatorname{trace}\nabla df. \tag{1.2}$$

As suggested by Eells and Sampson in [3], we can define the bienergy of a map f by

$$E_2(f)=\frac{1}{2}\int_M\left\|\tau(f)\right\|^2 v_g, \tag{1.3}$$

and say that is biharmonic if it is a critical point of the bienergy.

Jiang derived the first and the second variation formula for the bienergy in [6,7], showing that the Euler–Lagrange equation associated to E_2 is

$$\begin{aligned}\tau_2(f)&=-\mathcal{J}^f(\tau(f))=-\Delta\tau(f)-\operatorname{trace}R^N(df,\tau(f))\,df\\&=0,\end{aligned} \tag{1.4}$$

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where $\mathcal{J}^f$ is the Jacobi operator of f . The equation $\tau_2(f) = 0$ is called the biharmonic equation. Since $\mathcal{J}^f$ is linear, any harmonic map is biharmonic. Therefore, we are interested in proper biharmonic maps, that is non-harmonic biharmonic maps.

In this paper, we study biharmonic helices in the $\widetilde{\mathcal{SL}_2(\mathbb{R})}$. We characterize the biharmonic curves in terms of their curvature and torsion and we prove that all of biharmonic curves are helices in the $\widetilde{\mathcal{SL}_2(\mathbb{R})}$. Finally, we find out their explicit parametric equations.

2. $\widetilde{\mathcal{SL}_2(\mathbb{R})}$

We identify $\widetilde{\mathcal{SL}_2(\mathbb{R})}$ with

$$\mathbb{R}^3_+ = \left\{(x,y,z) \in \mathbb{R}^3 : z > 0\right\}$$

endowed with the metric

$$g = ds^2 = (dx + \frac{dy}{z})^2 + \frac{dy^2 + dz^2}{z^2}.$$

The following set of left-invariant vector fields forms an orthonormal basis for $\widetilde{\mathcal{SL}_2(\mathbb{R})}$

$$\mathbf{e}_1 = \frac{\partial}{\partial x}, \mathbf{e}_2 = z\frac{\partial}{\partial y} - \frac{\partial}{\partial x}, \mathbf{e}_3 = z\frac{\partial}{\partial z}. \tag{2.1}$$

The characterising properties of g defined by

$$g(\mathbf{e}_1,\mathbf{e}_1) = g(\mathbf{e}_2,\mathbf{e}_2) = g(\mathbf{e}_3,\mathbf{e}_3) = 1,$$
$$g(\mathbf{e}_1,\mathbf{e}_2) = g(\mathbf{e}_2,\mathbf{e}_3) = g(\mathbf{e}_1,\mathbf{e}_3) = 0.$$

The Riemannian connection ∇ of the metric g is given by

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) \\ &\quad - g(X,[Y,Z]) - g(Y,[X,Z]) + g(Z,[X,Y]), \end{aligned}$$

which is known as Koszul's formula.

Using the Koszul's formula, we obtain

$$\begin{aligned} &\nabla_{\mathbf{e}_1}\mathbf{e}_1 = 0, && \nabla_{\mathbf{e}_1}\mathbf{e}_2 = \frac{1}{2}\mathbf{e}_3, && \nabla_{\mathbf{e}_1}\mathbf{e}_3 = -\frac{1}{2}\mathbf{e}_2, \\ &\nabla_{\mathbf{e}_2}\mathbf{e}_1 = \frac{1}{2}\mathbf{e}_3, && \nabla_{\mathbf{e}_2}\mathbf{e}_2 = \mathbf{e}_3, && \nabla_{\mathbf{e}_2}\mathbf{e}_3 = -\frac{1}{2}\mathbf{e}_1 - \mathbf{e}_2, \\ &\nabla_{\mathbf{e}_3}\mathbf{e}_1 = -\frac{1}{2}\mathbf{e}_2, && \nabla_{\mathbf{e}_3}\mathbf{e}_2 = \frac{1}{2}\mathbf{e}_1, && \nabla_{\mathbf{e}_3}\mathbf{e}_3 = 0. \end{aligned} \tag{2.2}$$

Moreover we put

$$R_{ijk} = R(\mathbf{e}_i,\mathbf{e}_j)\mathbf{e}_k, \quad R_{ijkl} = R(\mathbf{e}_i,\mathbf{e}_j,\mathbf{e}_k,\mathbf{e}_l),$$

where the indices i, j, k and l take the values $1, 2$ and 3

$$R_{1212} = R_{1313} = \frac{1}{4}, \quad R_{2323} = -\frac{7}{4}. \tag{2.3}$$

3. Biharmonic Curves in $\widetilde{\mathcal{SL}_2(\mathbb{R})}$

Biharmonic equation for the curve γ reduces to

$$\nabla^3_{\mathbf{T}}\mathbf{T} - R(\mathbf{T}, \nabla_{\mathbf{T}}\mathbf{T})\mathbf{T} = 0, \tag{3.1}$$

that is, γ is called a biharmonic curve if it is a solution of the equation (3.1).

Let us consider biharmonicity of curves in $\widetilde{\mathcal{SL}_2(\mathbb{R})}$. Let $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ be the Frenet frame field along γ. Then, the Frenet frame satisfies the following Frenet–Serret equations:

$$\begin{aligned} \nabla_{\mathbf{T}}\mathbf{T} &= \kappa\mathbf{N}, \\ \nabla_{\mathbf{T}}\mathbf{N} &= -\kappa\mathbf{T} + \tau\mathbf{B}, \\ \nabla_{\mathbf{T}}\mathbf{B} &= -\tau\mathbf{N}, \end{aligned} \tag{3.2}$$

where $\kappa = |\mathcal{T}(\gamma)| = |\nabla_{\mathbf{T}}\mathbf{T}|$ is the curvature of γ and τ its torsion and

$$\begin{aligned} &g(\mathbf{T}, \mathbf{T}) = 1,\ g(\mathbf{N}, \mathbf{N}) = 1,\ g(\mathbf{B}, \mathbf{B}) = 1, \\ &g(\mathbf{T}, \mathbf{N}) = g(\mathbf{T}, \mathbf{B}) = g(\mathbf{N}, \mathbf{B}) = 0. \end{aligned}$$

With respect to the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, we can write

$$\begin{aligned} \mathbf{T} &= T_1\mathbf{e}_1 + T_2\mathbf{e}_2 + T_3\mathbf{e}_3, \\ \mathbf{N} &= N_1\mathbf{e}_1 + N_2\mathbf{e}_2 + N_3\mathbf{e}_3, \\ \mathbf{B} &= \mathbf{T} \times \mathbf{N} = B_1\mathbf{e}_1 + B_2\mathbf{e}_2 + B_3\mathbf{e}_3. \end{aligned} \tag{3.3}$$

Theorem 3.1. *$\gamma : I \longrightarrow \widetilde{\mathcal{SL}_2(\mathbb{R})}$ is a biharmonic curve if and only if*

$$\begin{aligned} \kappa &= \text{constant} \neq 0, \\ \kappa^2 + \tau^2 &= -\frac{1}{4} + \frac{15}{4}B_1^2, \\ \tau' &= 2N_1B_1. \end{aligned} \tag{3.4}$$

Proof. Using (3.1) and Frenet formulas (3.2), we have (3.4).

Theorem 3.2. *All of biharmonic curves in $\widetilde{\mathcal{SL}_2(\mathbb{R})}$ are helices.*

Theorem 3.3. *Let $\gamma : I \longrightarrow \widetilde{\mathcal{SL}_2(\mathbb{R})}$ be a unit speed non-geodesic curve. Then, the parametric equations of γ are*

$$x(s) = \frac{1}{\aleph}\sin\varphi\sin[\aleph s + C] + \frac{1}{\aleph}\sin\varphi\cos[\aleph s + C] + \wp_2, \tag{3.5}$$

$$y(s) = \frac{1}{\aleph^2 + \cos^2\varphi}\sin\varphi\wp_1 e^{\cos\varphi s}(-\aleph\cos[\aleph s + C] + \cos\varphi\sin[\aleph s + C]),$$

$$z(s) = \wp_1 e^{\cos\varphi s},$$

where $\wp_1$, $\wp_2$ are constants of integration.

Proof. Since γ is biharmonic, γ is a helix. So, without loss of generality, we take the axis of γ is parallel to the vector $\mathbf{e}_3$. Then,

$$g(\mathbf{T}, \mathbf{e}_3) = T_3 = \cos\varphi, \tag{3.6}$$

where φ is constant angle.

The tangent vector can be written in the following form

$$\mathbf{T} = T_1\mathbf{e}_1 + T_2\mathbf{e}_2 + T_3\mathbf{e}_3. \tag{3.7}$$

On the other hand the tangent vector $\mathbf{T}$ is a unit vector, so the following condition is satisfied

$$T_1^2 + T_2^2 = 1 - \cos^2\varphi. \tag{3.8}$$

Noting that $\cos^2\varphi + \sin^2\varphi = 1$, we have

$$T_1^2 + T_2^2 = \sin^2\varphi. \tag{3.9}$$

The general solution of (3.9) can be written in the following form

$$\begin{aligned} T_1 &= \sin\varphi\cos\mu, \\ T_2 &= \sin\varphi\sin\mu, \end{aligned} \tag{3.10}$$

where μ is an arbitrary function of s.

So, substituting the components T_1, T_2 and T_3 in the equation (3.7), we have the following equation

$$\mathbf{T} = \sin\varphi\cos\mu\mathbf{e}_1 + \sin\varphi\sin\mu\mathbf{e}_2 + \cos\varphi\mathbf{e}_3. \tag{3.11}$$

Also, without loss of generality, we take

$$\mu = \aleph s + C, \tag{3.12}$$

where $\aleph, C \in \mathbb{R}$.

Thus (3.11) and (3.12), imply

$$\mathbf{T} = \sin\varphi\cos[\aleph s + C]\,\mathbf{e}_1 + \sin\varphi\sin[\aleph s + C]\,\mathbf{e}_2 + \cos\varphi\mathbf{e}_3. \tag{3.13}$$

Using (2.1) in (3.13), we obtain

$$\mathbf{T} = (\sin\varphi\cos[\aleph s + C] - \sin\varphi\sin[\aleph s + C], z\sin\varphi\sin[\aleph s + C], z\cos\varphi). \tag{3.14}$$

By direct calculations we have

$$\begin{aligned} \frac{dx}{ds} &= \sin\varphi\cos[\aleph s + C] - \sin\varphi\sin[\aleph s + C], \\ \frac{dy}{ds} &= z\sin\varphi\sin[\aleph s + C], \\ \frac{dz}{ds} &= z\cos\varphi. \end{aligned}$$

Firstly, we have

$$z(s) = \wp_1 e^{\cos\varphi s},$$

where $\wp_1$ is constant of integration.

$$x(s) = \frac{1}{\aleph}\sin\varphi\sin[\aleph s + C] + \frac{1}{\aleph}\sin\varphi\cos[\aleph s + C] + \wp_2,$$

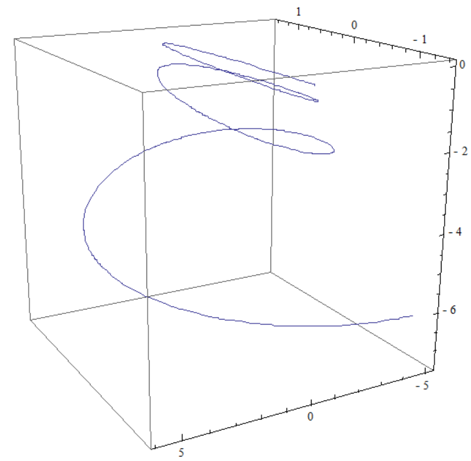
where $\wp_2$ is constant of integration.

Moreover, above equations, imply

$$y(s) = \frac{1}{\aleph^2 + \cos^2\varphi}\sin\varphi\wp_1 e^{\cos\varphi s}(-\aleph\cos[\aleph s + C] + \cos\varphi\sin[\aleph s + C]),$$

which proves our assertion.

We can use Mathematica in above theorem, yields



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