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# DARBOUX VECTORS OF GENERAL HELICES IN THE SOL SPACE

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ABSTRACT. In this paper, we study Darboux rotation axis for general helices in the Sol space. We obtain equation of Darboux vectors of general helices in the Sol space.

## 1. Introduction

The object moves along the curve, let its intrinsic coordinate system keep itself aligned with the curve's Frenet frame. As it does so, the object's motion will be described by two vectors: a translation vector, and a rotation vector $\omega$, which is an areal velocity vector: the Darboux vector.

Note that this rotation is kinematic, rather than physical, because usually when a rigid object moves freely in space its rotation is independent of its translation. The exception would be if the object's rotation is physically constrained to align itself with the object's translation, as is the case with the cart of a roller coaster.

In this paper, we study Darboux rotation axis for general helices in the Sol space. We obtain equation of Darboux vector of general helices in the Sol space.

## 2. Riemannian Structure of Sol Space $\mathfrak{Sol}^3$

Sol space, one of Thurston's eight 3-dimensional geometries, can be viewed as $\mathbb{R}^3$ provided with Riemannian metric

$$g_{\mathfrak{Sol}^3} = e^{2z}dx^2 + e^{-2z}dy^2 + dz^2, \tag{2.1}$$

where $(x, y, z)$ are the standard coordinates in $\mathbb{R}^3$.

Note that the Sol metric can also be written as:

$$g_{\mathfrak{Sol}^3} = \sum_{i=1}^{3} \omega^i \otimes \omega^i, \tag{2.2}$$

where

$$\omega^1 = e^z dx, \quad \omega^2 = e^{-z}dy, \quad \omega^3 = dz, \tag{2.3}$$

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and the orthonormal basis dual to the 1-forms is

$$\mathbf{e}_1 = e^{-z}\frac{\partial}{\partial x}, \quad \mathbf{e}_2 = e^{z}\frac{\partial}{\partial y}, \quad \mathbf{e}_3 = \frac{\partial}{\partial z}. \tag{2.4}$$

**Proposition 2.1**. *For the covariant derivatives of the Levi-Civita connection of the left-invariant metric $g_{\mathfrak{Sol}^3}$, defined above the following is true:*

$$\nabla = \begin{pmatrix} -\mathbf{e}_3 & 0 & \mathbf{e}_1 \\ 0 & \mathbf{e}_3 & -\mathbf{e}_2 \\ 0 & 0 & 0 \end{pmatrix}, \tag{2.5}$$

*where the $(i,j)$-element in the table above equals $\nabla_{\mathbf{e}_i}\mathbf{e}_j$ for our basis*

$$\{\mathbf{e}_k, k = 1,2,3\} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}.$$

Lie brackets can be easily computed as:

$$[\mathbf{e}_1, \mathbf{e}_2] = 0, \quad [\mathbf{e}_2, \mathbf{e}_3] = -\mathbf{e}_2, \quad [\mathbf{e}_1, \mathbf{e}_3] = \mathbf{e}_1.$$

The isometry group of $\mathfrak{Sol}^3$ has dimension 3. The connected component of the identity is generated by the following three families of isometries:

$$\begin{aligned} (x,y,z) &\to (x+c, y, z), \\ (x,y,z) &\to (x, y+c, z), \\ (x,y,z) &\to \left(e^{-c}x, e^{c}y, z+c\right). \end{aligned}$$

## 3. General Helices in Sol Space $\mathfrak{Sol}^3$

Assume that $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ be the Frenet frame field along $\gamma$. Then, the Frenet frame satisfies the following Frenet–Serret equations:

$$\begin{aligned} \nabla_{\mathbf{T}}\mathbf{T} &= \kappa\mathbf{N}, \\ \nabla_{\mathbf{T}}\mathbf{N} &= -\kappa\mathbf{T} + \tau\mathbf{B}, \\ \nabla_{\mathbf{T}}\mathbf{B} &= -\tau\mathbf{N}, \end{aligned} \tag{3.1}$$

where $\kappa$ is the curvature of $\gamma$ and $\tau$ its torsion and

$$\begin{aligned} &g_{\mathfrak{Sol}^3}(\mathbf{T}, \mathbf{T}) = 1, \; g_{\mathfrak{Sol}^3}(\mathbf{N}, \mathbf{N}) = 1, \; g_{\mathfrak{Sol}^3}(\mathbf{B}, \mathbf{B}) = 1, \\ &g_{\mathfrak{Sol}^3}(\mathbf{T}, \mathbf{N}) = g_{\mathfrak{Sol}^3}(\mathbf{T}, \mathbf{B}) = g_{\mathfrak{Sol}^3}(\mathbf{N}, \mathbf{B}) = 0. \end{aligned} \tag{3.2}$$

With respect to the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, we can write

$$\begin{aligned} \mathbf{T} &= T_1\mathbf{e}_1 + T_2\mathbf{e}_2 + T_3\mathbf{e}_3, \\ \mathbf{N} &= N_1\mathbf{e}_1 + N_2\mathbf{e}_2 + N_3\mathbf{e}_3, \\ \mathbf{B} &= \mathbf{T} \times \mathbf{N} = B_1\mathbf{e}_1 + B_2\mathbf{e}_2 + B_3\mathbf{e}_3. \end{aligned} \tag{3.3}$$

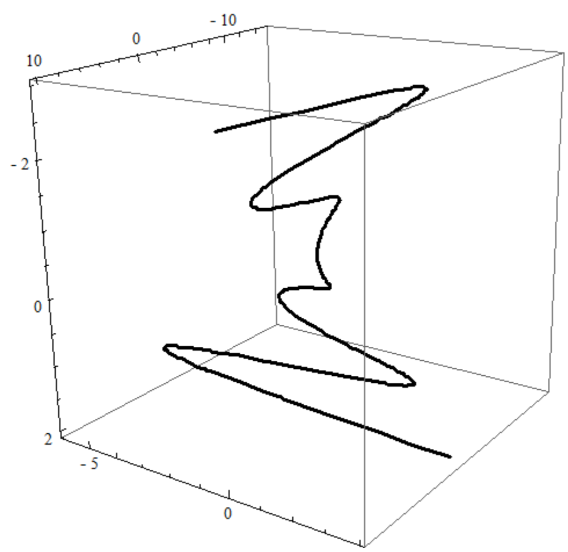
**Theorem 3.1.** ([15]) *Let* $\gamma : I \longrightarrow \mathfrak{Sol}^3$ *be a unit speed non-geodesic general helix. Then, the parametric equations of* $\gamma$ *are*

$$
\begin{aligned}
x(s) &= \frac{\sin \mathfrak{P} e^{-\cos \mathfrak{P} s - \mathfrak{C}_3}}{\mathfrak{C}_1^2 + \cos^2 \mathfrak{P}} \left[ -\cos \mathfrak{P} \cos [\mathfrak{C}_1 s + \mathfrak{C}_2] + \mathfrak{C}_1 \sin [\mathfrak{C}_1 s + \mathfrak{C}_2] \right] + \mathfrak{C}_4, \\
y(s) &= \frac{\sin \mathfrak{P} e^{\cos \mathfrak{P} s + \mathfrak{C}_3}}{\mathfrak{C}_1^2 + \cos^2 \mathfrak{P}} \left[ -\mathfrak{C}_1 \cos [\mathfrak{C}_1 s + \mathfrak{C}_2] + \cos \mathfrak{P} \sin [\mathfrak{C}_1 s + \mathfrak{C}_2] \right] + \mathfrak{C}_5, \\
z(s) &= \cos \mathfrak{P} s + \mathfrak{C}_3,
\end{aligned} \tag{3.4}
$$

*where* $\mathfrak{C}_1, \mathfrak{C}_2, \mathfrak{C}_3, \mathfrak{C}_4, \mathfrak{C}_5$ *are constants of integration.*

The obtained parametric equations for Eq. (3.4) is illustrated in Fig. 1:

We can use Mathematica in Theorem 3.1, yields




## 4. Darboux Rotation Axis of General Helices in Sol Space $\mathfrak{Sol}^3$

Using Frenet equations form a rotation motion with Darboux vector,

$$\mathbf{D} = \tau \mathbf{T} + \kappa \mathbf{B}. \tag{4.1}$$

From above equation, momentum rotation vector is expressed as follows:

$$
\begin{aligned}
\nabla_{\mathbf{T}} \mathbf{T} &= \mathbf{D} \times \mathbf{T}, \\
\nabla_{\mathbf{T}} \mathbf{N} &= \mathbf{D} \times \mathbf{N}, \\
\nabla_{\mathbf{T}} \mathbf{B} &= \mathbf{D} \times \mathbf{B}.
\end{aligned}
$$

Darboux rotation of Frenet frame can be separated into two rotation motions: $\mathbf{T}$ tangent vector rotates with a $\kappa$ angular speed round $\mathbf{B}$ binormal vector, that is

$$\nabla_{\mathbf{T}} \mathbf{T} = (\kappa \mathbf{B}) \times \mathbf{T}$$

and $\mathbf{B}$ binormal vector rotates with a $\tau$ angular speed round $\mathbf{T}$ tangent vector, that is

$$\nabla_{\mathbf{T}}\mathbf{B} = (\tau \mathbf{T}) \times \mathbf{B}.$$

**Lemma 4.1.** $\mathbf{D}$ *vector rotates with zero angular speed round* $\mathbf{N}$ *principal normal for general helix in the Sol space.*

**Proof.** We assume that $\mathbf{D}$ vector rotates round $\mathbf{N}$ principal normal of $\gamma$.
So, by differentiating of the formula (4.1), we get

$$\nabla_{\mathbf{T}}\mathbf{D} = \tau'\mathbf{T} + \kappa'\mathbf{B}.$$

Hence we put

$$\tilde{\mathbf{D}} = \frac{\mathbf{D}}{|g(\mathbf{D},\mathbf{D})|^{\frac{1}{2}}}$$

Since Theorem 3.2, we immediately arrive at

$$\nabla_{\mathbf{T}}\tilde{\mathbf{D}} = 0.$$

In terms of above equation, we may give:

$$g\left(\nabla_{\mathbf{T}}\tilde{\mathbf{D}}, \nabla_{\mathbf{T}}\tilde{\mathbf{D}}\right) = 0.$$

This concludes the proof of Lemma.

Hence, we have the following theorem.

**Theorem 4.2.** *Let* $\gamma : I \longrightarrow \mathfrak{Sol}^3$ *is a non geodesic biharmonic helix in the Sol space. Then, Darboux vector of* $\gamma$ *is constant vector.*

**Proof.** Using Lemma 4.1, we immediately arrive at $\tilde{\mathbf{D}}$ is constant vector.

**Theorem 4.3.** *Let* $\gamma : I \longrightarrow \mathfrak{Sol}^3$ *is a non geodesic general helix in the Sol space. Then, the equation of Darboux vector of* $\gamma$ *is*

$$\begin{aligned}
\mathbf{D}=&[\tau \sin\mathfrak{P}\cos[\mathfrak{C}_1 s+\mathfrak{C}_2] + \sin\mathfrak{P}\sin[\mathfrak{C}_1 s+\mathfrak{C}_2][\sin^2\mathfrak{P}\sin^2[\mathfrak{C}_1 s+\mathfrak{C}_2] - \sin^2\mathfrak{P}\cos^2[\mathfrak{C}_1 s+\mathfrak{C}_2]\\
&-\cos\mathfrak{P}[\frac{1}{\mathfrak{C}_1}\sin\mathfrak{P}\cos[\mathfrak{C}_1 s+\mathfrak{C}_2] - \cos\mathfrak{P}\sin\mathfrak{P}\sin[\mathfrak{C}_1 s+\mathfrak{C}_2]]]\mathbf{e}_1\\
&-[\tau\sin\mathfrak{P}\sin[\mathfrak{C}_1 s+\mathfrak{C}_2] + \sin\mathfrak{P}\cos[\mathfrak{C}_1 s+\mathfrak{C}_2][\sin^2\mathfrak{P}\sin^2[\mathfrak{C}_1 s+\mathfrak{C}_2] - \sin^2\mathfrak{P}\cos^2[\mathfrak{C}_1 s+\mathfrak{C}_2]\\
&-\cos\mathfrak{P}[-\frac{1}{\mathfrak{C}_1}\sin\mathfrak{P}\sin[\mathfrak{C}_1 s+\mathfrak{C}_2] + \cos\mathfrak{P}\sin\mathfrak{P}\cos[\mathfrak{C}_1 s+\mathfrak{C}_2]]]\mathbf{e}_2\\
&+[\tau\cos\mathfrak{P}+\sin\mathfrak{P}\cos[\mathfrak{C}_1 s+\mathfrak{C}_2][\frac{1}{\mathfrak{C}_1}\sin\mathfrak{P}\cos[\mathfrak{C}_1 s+\mathfrak{C}_2] - \cos\mathfrak{P}\sin\mathfrak{P}\sin[\mathfrak{C}_1 s+\mathfrak{C}_2]\\
&-\sin\mathfrak{P}\sin[\mathfrak{C}_1 s+\mathfrak{C}_2][-\frac{1}{\mathfrak{C}_1}\sin\mathfrak{P}\sin[\mathfrak{C}_1 s+\mathfrak{C}_2] + \cos\mathfrak{P}\sin\mathfrak{P}\cos[\mathfrak{C}_1 s+\mathfrak{C}_2]]]\mathbf{e}_3,
\end{aligned} \tag{4.4}$$

**Proof.** Using Theorem 3.1, the tangent vector of $\gamma$ can be written in the following form:

$$\mathbf{T} = \sin\mathfrak{P}\cos[\mathfrak{C}_1 s+\mathfrak{C}_2]\,\mathbf{e}_1 + \sin\mathfrak{P}\sin[\mathfrak{C}_1 s+\mathfrak{C}_2]\,\mathbf{e}_2 + \cos\mathfrak{P}\mathbf{e}_3.$$

Using first equation of Eq.(3.3), we have

$$\nabla_{\mathbf{T}}\mathbf{T} = (T_1' + T_1T_3)\,\mathbf{e}_1 + (T_2' - T_2T_3)\,\mathbf{e}_2 + \left(T_3' - T_1^2 + T_2^2\right)\mathbf{e}_3.$$

By the use of Frenet formulas and above equation, we get

$$\begin{aligned}\mathbf{N} = &\frac{1}{\kappa}[-\frac{1}{\mathfrak{C}_1}\sin\mathfrak{P}\sin[\mathfrak{C}_1 s+\mathfrak{C}_2]+\cos\mathfrak{P}\sin\mathfrak{P}\cos[\mathfrak{C}_1 s+\mathfrak{C}_2]]\mathbf{e}_1\\ &+\frac{1}{\kappa}[\frac{1}{\mathfrak{C}_1}\sin\mathfrak{P}\cos[\mathfrak{C}_1 s+\mathfrak{C}_2]-\cos\mathfrak{P}\sin\mathfrak{P}\sin[\mathfrak{C}_1 s+\mathfrak{C}_2]]\mathbf{e}_2\\ &+\frac{1}{\kappa}[\sin^2\mathfrak{P}\sin^2[\mathfrak{C}_1 s+\mathfrak{C}_2]-\sin^2\mathfrak{P}\cos^2[\mathfrak{C}_1 s+\mathfrak{C}_2]]\mathbf{e}_3.\end{aligned}\tag{4.3}$$

On the other hand, we immediately arrive at

$$\begin{aligned}\mathbf{B}=&[\frac{1}{\kappa}\sin\mathfrak{P}\sin[\mathfrak{C}_1 s+\mathfrak{C}_2]\,[\sin^2\mathfrak{P}\sin^2[\mathfrak{C}_1 s+\mathfrak{C}_2]-\sin^2\mathfrak{P}\cos^2[\mathfrak{C}_1 s+\mathfrak{C}_2]\\ &-\frac{1}{\kappa}\cos\mathfrak{P}[\frac{1}{\mathfrak{C}_1}\sin\mathfrak{P}\cos[\mathfrak{C}_1 s+\mathfrak{C}_2]-\cos\mathfrak{P}\sin\mathfrak{P}\sin[\mathfrak{C}_1 s+\mathfrak{C}_2]]]\mathbf{e}_1\\ &-[\frac{1}{\kappa}\sin\mathfrak{P}\cos[\mathfrak{C}_1 s+\mathfrak{C}_2]\,[\sin^2\mathfrak{P}\sin^2[\mathfrak{C}_1 s+\mathfrak{C}_2]-\sin^2\mathfrak{P}\cos^2[\mathfrak{C}_1 s+\mathfrak{C}_2]\\ &-\frac{1}{\kappa}\cos\mathfrak{P}[-\frac{1}{\mathfrak{C}_1}\sin\mathfrak{P}\sin[\mathfrak{C}_1 s+\mathfrak{C}_2]+\cos\mathfrak{P}\sin\mathfrak{P}\cos[\mathfrak{C}_1 s+\mathfrak{C}_2]]]\mathbf{e}_2\\ &+[\frac{1}{\kappa}\sin\mathfrak{P}\cos[\mathfrak{C}_1 s+\mathfrak{C}_2]\,[\frac{1}{\mathfrak{C}_1}\sin\mathfrak{P}\cos[\mathfrak{C}_1 s+\mathfrak{C}_2]-\cos\mathfrak{P}\sin\mathfrak{P}\sin[\mathfrak{C}_1 s+\mathfrak{C}_2]\\ &-\frac{1}{\kappa}\sin\mathfrak{P}\sin[\mathfrak{C}_1 s+\mathfrak{C}_2]\,[-\frac{1}{\mathfrak{C}_1}\sin\mathfrak{P}\sin[\mathfrak{C}_1 s+\mathfrak{C}_2]+\cos\mathfrak{P}\sin\mathfrak{P}\cos[\mathfrak{C}_1 s+\mathfrak{C}_2]]]\mathbf{e}_3.\end{aligned}\tag{4.4}$$

If we substitute the equations (4.3) and (4.4) in the equation (4.1), we have (4.2), which it completes the proof.

We put

$$\mathbf{E}=\frac{\mathbf{D}}{\|\mathbf{D}\|}.$$

**Corollary 4.4.** *Let* $\gamma : I \longrightarrow \mathfrak{Sol}^3$ *is a non geodesic general helix in the Sol space. Then,* $\mathbf{E}$ *is constant vector.*

According to the second Frenet formula, we have

$$\nabla_{\mathbf{T}}\mathbf{N} = |g(\mathbf{D},\mathbf{D})|^{\frac{1}{2}}(\mathbf{E}\times\mathbf{N}).$$

Furthermore, we put

$$\mathbf{U}=\mathbf{E}\times\mathbf{N}.$$

On the other hand, from above corollary we have

$$\nabla_{\mathbf{T}}\mathbf{U} = |g(\mathbf{D},\mathbf{D})|^{\frac{1}{2}}\mathbf{N}$$

Since,

$$\begin{aligned}&\nabla_{\mathbf{T}}\mathbf{E}=0,\\ &\nabla_{\mathbf{T}}\mathbf{N} = |g(\mathbf{D},\mathbf{D})|^{\frac{1}{2}}(\mathbf{E}\times\mathbf{N}),\\ &\nabla_{\mathbf{T}}\mathbf{U}=|g(\mathbf{D},\mathbf{D})|^{\frac{1}{2}}\mathbf{N}.\end{aligned}$$

Thus, the vectors $\mathbf{N}$, $\mathbf{E}\times\mathbf{N}$, $\mathbf{E}$ define a rotation motion together the rotation vector,

$$\mathbf{D}_1 = |g(\mathbf{D},\mathbf{D})|^{\frac{1}{2}}\,\mathbf{E}.$$

**Corollary 4.5.** *Let $\gamma : I \longrightarrow \mathfrak{Sol}^3$ is a non geodesic general helix in the Sol space. Then, momentum rotation vector is expressed as follows:*

$$\nabla_{\mathbf{T}}\mathbf{N} = \mathbf{D}_1\times\mathbf{N},$$
$$\nabla_{\mathbf{T}}\mathbf{U} = \mathbf{D}_1\times\mathbf{U}.$$

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