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NEW TYPE OF INEXTENSIBLE FLOWS OF TIMELIKE CURVES IN MINKOWSKI SPACE-TIME $\mathbb{E}_1^4$

SELÇUK BAŞ, TALAT KÖRPINAR AND ESSIN TURHAN

Abstract. In this paper, we study inextensible flows of timelike curves in $\mathbb{E}_1^4$. Necessary and sufficient conditions for an inextensible flow are expressed as a partial differential equation involving the curvature.

1. Introduction

Construction of fluid flows constitutes an active research field with a high industrial impact. Corresponding real-world measurements in concrete scenarios complement numerical results from direct simulations of the Navier-Stokes equation, particularly in the case of turbulent flows, and for the understanding of the complex spatio-temporal evolution of instationary flow phenomena. More and more advanced imaging devices (lasers, highspeed cameras, control logic, etc.) are currently developed that allow to record fully timeresolved image sequences of fluid flows at high resolutions. As a consequence, there is a need for advanced algorithms for the analysis of such data, to provide the basis for a subsequent pattern analysis, and with abundant applications across various areas.

In this paper, we study inextensible flows of timelike curves in $\mathbb{E}_1^4$. We research necessary and sufficient conditions for an inelastic curve flow are expressed as a partial differential equation involving the curvature.

2. Preliminaries

To meet the requirements in the next sections, the basic elements of the theory of curves in Minkowski space-time $\mathbb{E}_1^4$ are briefly presented in this section.

Minkowski space-time $\mathbb{E}_1^4$ is a usual vector space provided with the standart metric given by

$$\langle , \rangle = -dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2, \tag{2.1}$$

where (x_1, x_2, x_3, x_4) is a rectangular coordinate system in $\mathbb{E}_1^4$.

Since $\langle , \rangle$ is an indefinite metric, recall that a $v \in \mathbb{E}_1^4$ can have one of the three causal characters; it can be spacelike if $\langle v, v\rangle > 0$ or $v = 0$, timelike if $\langle v, v\rangle < 0$,

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and null (ligthlike) if $\langle v, v\rangle = 0$ and $v \neq 0$. Similarly, an arbitrary curve $\gamma = \gamma(t)$ in $\mathbb{E}_1^4$ can locally be spacelike, timelike, or null (lightlike) if all of its velocity vectors $\gamma'(t)$ are, respectively, spacelike, timelike, or null. The norm of $v \in \mathbb{E}_1^4$ is given by $\|v\| = \sqrt{|\langle v, v\rangle|}$. If $\|\gamma'(s)\| = \sqrt{|\langle \gamma'(s), \gamma'(t)\rangle|} \neq 0$ for all $t \in I$, then γ is a regular curve in $\mathbb{E}_1^4$. A timelike (spacelike) regular curve γ is parameterized by arc-length parameter s which is given by $\gamma : I \to \mathbb{E}_1^4$, then the tangent vector $\gamma'(s)$ along γ has unit length, that is, $\langle \gamma'(s), \gamma'(s)\rangle = -1, (\langle \gamma'(s), \gamma'(s)\rangle = 1)$ for all $s \in I$.

Hereafter, curves considered are timelike and regular C^∞ curves in $\mathbb{E}_1^4$. Let $\mathbf{T}(s) = \gamma'(s)$ for all $s \in I$; then the vector field $\mathbf{T}(s)$ is timelike and it is called timelike unit tangent vector field on γ.

The timelike curve γ is called special timelike Frenet curve if there exist three smooth functions k_1, k_2, k_3 on γ and smooth non null frame field $\{\mathbf{T}, \mathbf{N}, \mathbf{B}_1, \mathbf{B}_2\}$ along the curve γ. Also, the functions k_1, k_2 and k_3 are called the first, the second, and the third curvature function on γ, respectively. For the C^∞ special timelike Frenet curve γ, the following Frenet formula is

$$\begin{bmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}_1' \\ \mathbf{B}_2' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ k_1 & 0 & k_2 & 0 \\ 0 & -k_2 & 0 & k_3 \\ 0 & 0 & -k_3 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix}, \tag{2.2}$$

see [2,13].

Here, due to characters of Frenet vectors of the timelike curve, $\mathbf{T}, \mathbf{N}, \mathbf{B}_1$ and $\mathbf{B}_2$ are mutually orthogonal vector fields satisfying equations

$$\langle \mathbf{T}, \mathbf{T}\rangle = -1, \quad \langle \mathbf{N}, \mathbf{N}\rangle = \langle \mathbf{B}_1, \mathbf{B}_1\rangle = \langle \mathbf{B}_2, \mathbf{B}_2\rangle = 1. \tag{2.3}$$

For $s \in I$, the nonnull frame field $\{\mathbf{T}, \mathbf{N}, \mathbf{B}_1, \mathbf{B}_2\}$ and curvature functions k_1, k_2 and k_3 are determined as follows:

$$\begin{aligned} \text{1st step } \mathbf{T}(s) &= c'(s), \\ \text{2nd step } k_1(s) &= \|\mathbf{T}(s)\| > 0, \\ \mathbf{N}(s) &= \frac{1}{k_1(s)}\mathbf{T}'(s), \\ \text{3rd step } k_1(s) &= \|\mathbf{N}'(s) - k_1(s)\mathbf{T}(s)\| > 0, \\ \mathbf{B}_1(s) &= \frac{1}{k_2(s)}(\mathbf{N}'(s) - k_1(s)\mathbf{T}(s)), \\ \text{4th step } \mathbf{B}_2(s) &= \delta\frac{1}{\|\mathbf{B}_1'(s) + k_2(s)\mathbf{N}(s)\|}(\mathbf{B}_1'(s) + k_2(s)\mathbf{N}(s)), \end{aligned} \tag{2.4}$$

where δ is determined by the fact that orthonormal frame field $\{\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}_1(s), \mathbf{B}_2(s)\}$ is of positive orientation. The function k_3 is determined by

$$k_3(t) = \langle \mathbf{B}_1'(s), \mathbf{B}_2(s)\rangle \neq 0. \tag{2.5}$$

So, the function k_3 never vanishes.

In order to make sure that the curve γ is a special timelike Frenet curve, above steps must be checked, from 1st step to 4th step, for $t \in I$.

Let $\{\mathbf{T}, \mathbf{N}, \mathbf{B}_1, \mathbf{B}_2\}$ be the moving Frenet frame along a unit speed timelike curve γ in $\mathbb{E}_1^4$, consisting of the tangent, the principal normal, the first binormal, and the second binormalvector field, respectively. Since γ is a timelike curve, its Frenet frame contains only nonnull vector fields.

3. Inextensible Flows of Timelike Curves in $\mathbb{E}_1^4$

Throughout this article, we assume that $\gamma : [0, l] \times [0, \omega] \to \mathbb{E}_1^4$ is a one parameter family of smooth timelike curves in Minkowski space $\mathbb{E}_1^4$, where l is the arclength of the initial curve. Let u be the curve parametrization variable, $0 \leq u \leq l$.

The arclength of γ is given by

$$s(u) = \int_0^u \left| \frac{\partial \gamma}{\partial u} \right| du, \tag{3.1}$$

where

$$\left| \frac{\partial \gamma}{\partial u} \right| = \left| \left\langle \frac{\partial \gamma}{\partial u}, \frac{\partial \gamma}{\partial u} \right\rangle \right|^{\frac{1}{2}}. \tag{3.2}$$

The operator $\frac{\partial}{\partial s}$ is given in terms of u by

$$\frac{\partial}{\partial s} = \frac{1}{\nu} \frac{\partial}{\partial u},$$

where $v = \left| \frac{\partial \gamma}{\partial u} \right|$. The arclength parameter is $ds = v du$.

Any flow of γ can be represented as

$$\frac{\partial \gamma}{\partial t} = \mathfrak{f}_1 \mathbf{T} + \mathfrak{f}_2 \mathbf{N} + \mathfrak{f}_3 \mathbf{B}_1 + \mathfrak{f}_4 \mathbf{B}_2. \tag{3.3}$$

Letting the arclength variation be

$$s(u, t) = \int_0^u v du.$$

In the Minkowski space the requirement that the curve not be subject to any elongation or compression can be expressed by the condition

$$\frac{\partial}{\partial t} s(u, t) = \int_0^u \frac{\partial v}{\partial t} du = 0, \tag{3.4}$$

for all $u \in [0, l]$.

Definition 3.1. *A curve evolution $\gamma(u, t)$ and its flow $\frac{\partial \gamma}{\partial t}$ in $\mathbb{E}_1^4$ are said to be inextensible if*

$$\frac{\partial}{\partial t} \left| \frac{\partial \gamma}{\partial u} \right| = 0.$$

Lemma 3.2. Let $\frac{\partial \gamma}{\partial t} = \mathfrak{f}_1 \mathbf{T} + \mathfrak{f}_2 \mathbf{N} + \mathfrak{f}_3 \mathbf{B}_1 + \mathfrak{f}_4 \mathbf{B}_2$ *be a smooth flow of the timelike curve* γ *in* $\mathbb{E}_1^4$*. The flow is inextensible if and only if*

$$\frac{\partial v}{\partial t} = -\frac{\partial \mathfrak{f}_1}{\partial u} - \mathfrak{f}_2 v k_1. \tag{3.5}$$

Proof. Suppose that $\frac{\partial \gamma}{\partial t}$ be a smooth flow of the timelike curve γ. Using definition of γ, we have

$$v^2 = \left\langle \frac{\partial \gamma}{\partial u}, \frac{\partial \gamma}{\partial u} \right\rangle. \tag{3.6}$$

$\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial t}$ commute since and are independent coordinates. So, by differentiating of the formula (3.6), we get

$$2v \frac{\partial v}{\partial t} = \frac{\partial}{\partial t} \left\langle \frac{\partial \gamma}{\partial u}, \frac{\partial \gamma}{\partial u} \right\rangle.$$

On the other hand, changing $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial t}$, we have

$$v \frac{\partial v}{\partial t} = \left\langle \frac{\partial \gamma}{\partial u}, \frac{\partial}{\partial u} (\frac{\partial \gamma}{\partial t}) \right\rangle.$$

From (3.3), we obtain

$$v \frac{\partial v}{\partial t} = \left\langle \frac{\partial \gamma}{\partial u}, \frac{\partial}{\partial u} \left(\mathfrak{f}_1 \mathbf{T} + \mathfrak{f}_2 \mathbf{N} + \mathfrak{f}_3 \mathbf{B}_1 + \mathfrak{f}_4 \mathbf{B}_2 \right) \right\rangle.$$

By the formula of the Frenet, we have

$$\begin{aligned} \frac{\partial v}{\partial t} =&< \mathbf{T}, (\frac{\partial \mathfrak{f}_1}{\partial u} + \mathfrak{f}_2 v k_1) \mathbf{T} + (\mathfrak{f}_1 v k_1 + \frac{\partial \mathfrak{f}_2}{\partial u} - \mathfrak{f}_3 v k_2) \mathbf{N} \\ &+ (\mathfrak{f}_2 v k_2 + \frac{\partial \mathfrak{f}_3}{\partial u} - \mathfrak{f}_4 k_3) \mathbf{B}_1 + (\mathfrak{f}_3 v k_3 + \frac{\partial \mathfrak{f}_4}{\partial u}) \mathbf{B}_2 > . \end{aligned}$$

Making necessary calculations from above equation, we have (3.5), which proves the lemma.

Theorem 3.3. *Let* $\frac{\partial \gamma}{\partial t} = \mathfrak{f}_1 \mathbf{T} + \mathfrak{f}_2 \mathbf{N} + \mathfrak{f}_3 \mathbf{B}_1 + \mathfrak{f}_4 \mathbf{B}_2$ *be a smooth flow of the timelike curve* γ *in* $\mathbb{E}_1^4$*. The flow is inextensible if and only if*

$$\frac{\partial \mathfrak{f}_1}{\partial u} = -\mathfrak{f}_2 v k_1. \tag{3.7}$$

Proof. Now let $\frac{\partial \gamma}{\partial t}$ be extensible. From (3.4), we have

$$\frac{\partial}{\partial t} s(u,t) = \int_0^u \frac{\partial v}{\partial t} du = \int_0^u (\frac{\partial \mathfrak{f}_1}{\partial u} + \mathfrak{f}_2 v k_1) du = 0. \tag{3.8}$$

Substituting (3.5) in (3.8) complete the proof of the theorem.

We now restrict ourselves to arc length parametrized curves. That is, $v = 1$ and the local coordinate u corresponds to the curve arc length s. We require the following lemma.

Lemma 3.4. *Let* $\frac{\partial \gamma}{\partial t} = \mathfrak{f}_1 \mathbf{T} + \mathfrak{f}_2 \mathbf{N} + \mathfrak{f}_3 \mathbf{B}_1 + \mathfrak{f}_4 \mathbf{B}_2$ *be a smooth flow of the timelike curve* γ *in* $\mathbb{E}_1^4$. *Then,*

$$\frac{\partial \mathbf{T}}{\partial t} = (\mathfrak{f}_1 k_1 + \frac{\partial \mathfrak{f}_2}{\partial s} - \mathfrak{f}_3 k_2)\mathbf{N} + (\mathfrak{f}_2 k_2 + \frac{\partial \mathfrak{f}_3}{\partial s} - \mathfrak{f}_4 k_3)\mathbf{B}_1 + (\mathfrak{f}_3 k_3 + \frac{\partial \mathfrak{f}_4}{\partial s})\mathbf{B}_2, \tag{3.9}$$

$$\frac{\partial \mathbf{N}}{\partial t} = \left(\mathfrak{f}_1 k_1 + \frac{\partial \mathfrak{f}_2}{\partial s} - \mathfrak{f}_3 k_2\right)\mathbf{T} + \psi_1 \mathbf{B}_1 + \psi_2 \mathbf{B}_2, \tag{3.10}$$

$$\frac{\partial \mathbf{B}_1}{\partial t} = \left(\mathfrak{f}_2 k_2 + \frac{\partial \mathfrak{f}_3}{\partial s} - \mathfrak{f}_4 k_3\right)\mathbf{T} - \psi_1 \mathbf{N} + \psi_3 \mathbf{B}_2, \tag{3.11}$$

$$\frac{\partial \mathbf{B}_2}{\partial t} = \left(\mathfrak{f}_3 k_3 + \frac{\partial \mathfrak{f}_4}{\partial s}\right)\mathbf{T} - \psi_2 \mathbf{N} - \psi_3 \mathbf{B}_1, \tag{3.12}$$

where

$$\psi_1 = \left\langle \frac{\partial \mathbf{N}}{\partial t}, \mathbf{B}_1 \right\rangle, \ \psi_2 = \left\langle \frac{\partial \mathbf{N}}{\partial t}, \mathbf{B}_2 \right\rangle, \ \psi_3 = \left\langle \mathbf{B}_2, \frac{\partial \mathbf{B}_1}{\partial t} \right\rangle.$$

Proof. Under the assumption, we have

$$\frac{\partial \mathbf{T}}{\partial t} = \frac{\partial}{\partial t}\frac{\partial \gamma}{\partial s} = \frac{\partial}{\partial s}(\mathfrak{f}_1 \mathbf{T} + \mathfrak{f}_2 \mathbf{N} + \mathfrak{f}_3 \mathbf{B}_1 + \mathfrak{f}_4 \mathbf{B}_2).$$

Thus, it is seen that

$$\frac{\partial \mathbf{T}}{\partial t} = (\frac{\partial \mathfrak{f}_1}{\partial s} + \mathfrak{f}_2 k_1)\mathbf{T} + (\mathfrak{f}_1 k_1 + \frac{\partial \mathfrak{f}_2}{\partial s} - \mathfrak{f}_3 k_2)\mathbf{N} + (\mathfrak{f}_2 k_2 + \frac{\partial \mathfrak{f}_3}{\partial s} - \mathfrak{f}_4 k_3)\mathbf{B}_1 + (\mathfrak{f}_3 k_3 + \frac{\partial \mathfrak{f}_4}{\partial s})\mathbf{B}_2. \tag{3.13}$$

Substituting (3.7) in (3.13), we get

$$\frac{\partial \mathbf{T}}{\partial t} = (\mathfrak{f}_1 k_1 + \frac{\partial \mathfrak{f}_2}{\partial s} - \mathfrak{f}_3 k_2)\mathbf{N} + (\mathfrak{f}_2 k_2 + \frac{\partial \mathfrak{f}_3}{\partial s} - \mathfrak{f}_4 k_3)\mathbf{B}_1 + (\mathfrak{f}_3 k_3 + \frac{\partial \mathfrak{f}_4}{\partial s})\mathbf{B}_2.$$

The differentiation of the Frenet frame with respect to t is

$$\begin{aligned}
0 &= \frac{\partial}{\partial t}\langle \mathbf{T},\mathbf{N}\rangle = \left\langle \frac{\partial \mathbf{T}}{\partial t},\mathbf{N}\right\rangle + \left\langle \mathbf{T},\frac{\partial \mathbf{N}}{\partial t}\right\rangle \\
&= \mathfrak{f}_1 k_1 + \frac{\partial \mathfrak{f}_2}{\partial s} - \mathfrak{f}_3 k_2 + \left\langle \mathbf{T},\frac{\partial \mathbf{N}}{\partial t}\right\rangle, \\
0 &= \frac{\partial}{\partial t}\langle \mathbf{T},\mathbf{B}_1\rangle = \left\langle \frac{\partial \mathbf{T}}{\partial t},\mathbf{B}_1\right\rangle + \left\langle \mathbf{T},\frac{\partial \mathbf{B}_1}{\partial t}\right\rangle \\
&= \mathfrak{f}_2 k_2 + \frac{\partial \mathfrak{f}_3}{\partial s} - \mathfrak{f}_4 k_3 + \left\langle \mathbf{T},\frac{\partial \mathbf{B}_1}{\partial t}\right\rangle, \\
0 &= \frac{\partial}{\partial t}\langle \mathbf{T},\mathbf{B}_2\rangle = \left\langle \frac{\partial \mathbf{T}}{\partial t},\mathbf{B}_2\right\rangle + \left\langle \mathbf{T},\frac{\partial \mathbf{B}_2}{\partial t}\right\rangle \\
&= \mathfrak{f}_3 k_3 + \frac{\partial \mathfrak{f}_4}{\partial s} + \left\langle \mathbf{T},\frac{\partial \mathbf{B}_2}{\partial t}\right\rangle, \\
0 &= \frac{\partial}{\partial t}\langle \mathbf{N},\mathbf{B}_1\rangle = \left\langle \frac{\partial \mathbf{N}}{\partial t},\mathbf{B}_1\right\rangle + \left\langle \mathbf{N},\frac{\partial \mathbf{B}_1}{\partial t}\right\rangle \\
&= \psi_1 + \left\langle \mathbf{N},\frac{\partial \mathbf{B}_1}{\partial t}\right\rangle, \\
0 &= \frac{\partial}{\partial t}\langle \mathbf{N},\mathbf{B}_2\rangle = \left\langle \frac{\partial \mathbf{N}}{\partial t},\mathbf{B}_2\right\rangle + \left\langle \mathbf{N},\frac{\partial \mathbf{B}_2}{\partial t}\right\rangle \\
&= \psi_2 + \left\langle \mathbf{N},\frac{\partial \mathbf{B}_2}{\partial t}\right\rangle, \\
0 &= \frac{\partial}{\partial t}\langle \mathbf{B}_1,\mathbf{B}_2\rangle = \left\langle \frac{\partial \mathbf{B}_1}{\partial t},\mathbf{B}_2\right\rangle + \left\langle \mathbf{B}_1,\frac{\partial \mathbf{B}_2}{\partial t}\right\rangle \\
&= \psi_3 + \left\langle \mathbf{B}_2,\frac{\partial \mathbf{B}_1}{\partial t}\right\rangle.
\end{aligned}$$

From the above and using

$$\left\langle \frac{\partial \mathbf{N}}{\partial t},\mathbf{N}\right\rangle = \left\langle \frac{\partial \mathbf{B}_1}{\partial t},\mathbf{B}_1\right\rangle = \left\langle \frac{\partial \mathbf{B}_2}{\partial t},\mathbf{B}_2\right\rangle = 0,$$

we obtain

$$\begin{aligned}
\frac{\partial \mathbf{N}}{\partial t} &= \left(\mathfrak{f}_1 k_1 + \frac{\partial \mathfrak{f}_2}{\partial s} - \mathfrak{f}_3 k_2\right)\mathbf{T} + \psi_1 \mathbf{B}_1 + \psi_2 \mathbf{B}_2, \\
\frac{\partial \mathbf{B}_1}{\partial t} &= \left(\mathfrak{f}_2 k_2 + \frac{\partial \mathfrak{f}_3}{\partial s} - \mathfrak{f}_4 k_3\right)\mathbf{T} - \psi_1 \mathbf{N} + \psi_3 \mathbf{B}_2, \\
\frac{\partial \mathbf{B}_2}{\partial t} &= \left(\mathfrak{f}_3 k_3 + \frac{\partial \mathfrak{f}_4}{\partial s}\right)\mathbf{T} - \psi_2 \mathbf{N} - \psi_3 \mathbf{B}_1,
\end{aligned}$$

where

$$\psi_1 = \left\langle \frac{\partial \mathbf{N}}{\partial t},\mathbf{B}_1\right\rangle,\ \psi_2 = \left\langle \frac{\partial \mathbf{N}}{\partial t},\mathbf{B}_2\right\rangle,\ \psi_3 = \left\langle \mathbf{B}_2,\frac{\partial \mathbf{B}_1}{\partial t}\right\rangle.$$

The following theorem states the conditions on the curvature and torsion for the curve flow $\gamma(s,t)$ to be inextensible.

Theorem 3.5. *Let* $\frac{\partial \gamma}{\partial t} = \mathfrak{f}_1 \mathbf{T} + \mathfrak{f}_2 \mathbf{N} + \mathfrak{f}_3 \mathbf{B}_1 + \mathfrak{f}_4 \mathbf{B}_2$ *be a smooth flow of the timelike curve* γ*. Then, the following system of partial differential equations holds:*

$$\frac{\partial k_1}{\partial t} = \frac{\partial}{\partial s}(\mathfrak{f}_1 k_1) + \frac{\partial^2 \mathfrak{f}_2}{\partial s^2} - \frac{\partial}{\partial s}(\mathfrak{f}_3 k_2) - \mathfrak{f}_2 k_2^2 - \frac{\partial \mathfrak{f}_3}{\partial s} k_2 + \mathfrak{f}_4 k_2 k_3.$$

Proof. From our assumption, we have

$$\begin{aligned}\frac{\partial}{\partial s}\frac{\partial \mathbf{T}}{\partial t} &= \left(\frac{\partial}{\partial s}(\mathfrak{f}_1 k_1) + \frac{\partial^2 \mathfrak{f}_2}{\partial s^2} - \frac{\partial}{\partial s}(\mathfrak{f}_3 k_2)\right)\mathbf{N} + \left(\mathfrak{f}_1 k_1 + \frac{\partial \mathfrak{f}_2}{\partial s} - \mathfrak{f}_3 k_2\right)(k_1 \mathbf{T} + k_2 \mathbf{B}_1) \\ &+ \left(\frac{\partial}{\partial s}(\mathfrak{f}_2 k_2) + \frac{\partial^2 \mathfrak{f}_3}{\partial s^2} - \frac{\partial}{\partial s}(\mathfrak{f}_4 k_3)\right)\mathbf{B}_1 + \left(\mathfrak{f}_2 k_2 + \frac{\partial \mathfrak{f}_3}{\partial s} - \mathfrak{f}_4 k_3\right)(-k_2 \mathbf{N} + k_3 \mathbf{B}_2) \\ &+ \left(\frac{\partial}{\partial s}(\mathfrak{f}_3 k_3) + \frac{\partial^2 \mathfrak{f}_4}{\partial s^2}\right)\mathbf{B}_2 + \left(\mathfrak{f}_3 k_3 + \frac{\partial \mathfrak{f}_4}{\partial s}\right)(-k_3 \mathbf{B}_1).\end{aligned}$$

Also, we get

$$\begin{aligned}\frac{\partial}{\partial s}\frac{\partial \mathbf{T}}{\partial t} &= \left(\mathfrak{f}_1 k_1^2 + \frac{\partial \mathfrak{f}_2}{\partial s} k_1 - \mathfrak{f}_3 k_2 k_1\right)\mathbf{T} \\ &+ \left(\frac{\partial}{\partial s}(\mathfrak{f}_1 k_1) + \frac{\partial^2 \mathfrak{f}_2}{\partial s^2} - \frac{\partial}{\partial s}(\mathfrak{f}_3 k_2) - \mathfrak{f}_2 k_2^2 - \frac{\partial \mathfrak{f}_3}{\partial s} k_2 + \mathfrak{f}_4 k_2 k_3\right)\mathbf{N} \\ &+ \left(\frac{\partial}{\partial s}(\mathfrak{f}_2 k_2) + \frac{\partial^2 \mathfrak{f}_3}{\partial s^2} - \frac{\partial}{\partial s}(\mathfrak{f}_4 k_3) + \mathfrak{f}_1 k_1 k_2 + \frac{\partial \mathfrak{f}_2}{\partial s} k_2 - \mathfrak{f}_3 k_2^2 - \mathfrak{f}_3 k_3^2 - \frac{\partial \mathfrak{f}_4}{\partial s} k_3\right)\mathbf{B}_1 \\ &+ \left(\frac{\partial}{\partial s}(\mathfrak{f}_3 k_3) + \frac{\partial^2 \mathfrak{f}_4}{\partial s^2} + \mathfrak{f}_2 k_2 k_3 + \frac{\partial \mathfrak{f}_3}{\partial s} k_3 - \mathfrak{f}_4 k_3^2\right)\mathbf{B}_2.\end{aligned} \tag{3.14}$$

On the other hand, from Frenet frame we have

$$\frac{\partial}{\partial t}\frac{\partial \mathbf{T}}{\partial s} = \frac{\partial k_1}{\partial t}\mathbf{N} + k_1\left(\mathfrak{f}_1 k_1 + \frac{\partial \mathfrak{f}_2}{\partial s} - \mathfrak{f}_3 k_2\right)\mathbf{T} + k_1 \psi_1 \mathbf{B}_1 + k_1 \psi_2 \mathbf{B}_2. \tag{3.15}$$

Hence from (3.14) and (3.15), we get

$$\frac{\partial k_1}{\partial t} = \frac{\partial}{\partial s}(\mathfrak{f}_1 k_1) + \frac{\partial^2 \mathfrak{f}_2}{\partial s^2} - \frac{\partial}{\partial s}(\mathfrak{f}_3 k_2) - \mathfrak{f}_2 k_2^2 - \frac{\partial \mathfrak{f}_3}{\partial s} k_2 + \mathfrak{f}_4 k_2 k_3.$$

Theorem 3.6. *Let* $\frac{\partial \gamma}{\partial t} = \mathfrak{f}_1 \mathbf{T} + \mathfrak{f}_2 \mathbf{N} + \mathfrak{f}_3 \mathbf{B}_1 + \mathfrak{f}_4 \mathbf{B}_2$ *be a smooth flow of the timelike curve* γ*. Then, the following system of partial differential equations holds:*

$$\begin{aligned}\frac{\partial \mathfrak{f}_2}{\partial s} - \frac{\partial k_2}{\partial t} + \frac{\partial^2 \mathfrak{f}_3}{\partial s^2} - \frac{\partial \mathfrak{f}_4}{\partial s} k_3 &= -\mathfrak{f}_1 k_1 k_2 + \mathfrak{f}_3 k_2^2 + \mathfrak{f}_3 k_3^2 - \frac{\partial}{\partial s}(\mathfrak{f}_2 k_2) - \frac{\partial}{\partial s}(\mathfrak{f}_4 k_3) - \psi_1 k_1, \\ \frac{\partial k_3}{\partial t} &= \psi_2 k_2 + \frac{\partial \psi_3}{\partial s}, \\ \frac{\partial \psi_1}{\partial s} - \frac{\partial k_2}{\partial t} &= \mathfrak{f}_2 k_1 k_2 + \frac{\partial \mathfrak{f}_3}{\partial s} k_1 - \mathfrak{f}_4 k_1 k_3 + -\psi_2 k_3,\end{aligned}$$

where

$$\psi_1 = \left\langle \frac{\partial \mathbf{N}}{\partial t}, \mathbf{B}_1 \right\rangle,\ \psi_2 = \left\langle \frac{\partial \mathbf{N}}{\partial t}, \mathbf{B}_2 \right\rangle,\ \psi_3 = \left\langle \mathbf{B}_2, \frac{\partial \mathbf{B}_1}{\partial t} \right\rangle.$$

Proof. Similarly, we have

$$\frac{\partial}{\partial s}\frac{\partial \mathbf{B}_1}{\partial t}=\frac{\partial}{\partial s}\left[\left(\mathfrak{f}_2k_2+\frac{\partial \mathfrak{f}_3}{\partial s}-\mathfrak{f}_4k_3\right)\mathbf{T}-\psi_1\mathbf{N}+\psi_3\mathbf{B}_2\right]$$

From Frenet formulas, we have

$$\begin{aligned}\frac{\partial}{\partial s}\frac{\partial \mathbf{B}_1}{\partial t}&=\left(\frac{\partial}{\partial s}(\mathfrak{f}_2k_2)+\frac{\partial^2 \mathfrak{f}_3}{\partial s^2}+\frac{\partial}{\partial s}(\mathfrak{f}_4k_3)-\psi_1k_1\right)\mathbf{T}\\&+\left(\mathfrak{f}_2k_1k_2+k_1\frac{\partial \mathfrak{f}_3}{\partial s}-\mathfrak{f}_4k_1k_3-\frac{\partial \psi_1}{\partial s}\right)\mathbf{N}\\&+\left(-\psi_3k_3-\psi_1k_2\right)\mathbf{B}_1+\frac{\partial \psi_3}{\partial s}\mathbf{B}_2.\end{aligned}\tag{3.16}$$

Also,

$$\begin{aligned}\frac{\partial}{\partial t}\frac{\partial \mathbf{B}_1}{\partial s}&=\frac{\partial}{\partial t}\left(-k_2\mathbf{N}+k_3\mathbf{B}_2\right)\\&=\left(-\frac{\partial k_2}{\partial t}+\mathfrak{f}_1k_1k_2+\frac{\partial \mathfrak{f}_2}{\partial s}-\mathfrak{f}_3k_2^2-\left(\mathfrak{f}_3k_3^2+\frac{\partial \mathfrak{f}_4}{\partial s}k_3\right)\right)\mathbf{T}\\&-\psi_2k_3\mathbf{N}+\left(-\psi_1k_2-\psi_3k_3\right)\mathbf{B}_1+\left(\psi_2k_2+\frac{\partial k_3}{\partial t}\right)\mathbf{B}_2.\end{aligned}\tag{3.17}$$

Hence from (3.16) and (3.17)

$$-\frac{\partial k_2}{\partial t}+\frac{\partial \mathfrak{f}_2}{\partial s}+\frac{\partial^2 \mathfrak{f}_3}{\partial s^2}-\frac{\partial \mathfrak{f}_4}{\partial s}k_3=-\mathfrak{f}_1k_1k_2+\mathfrak{f}_3k_2^2+\mathfrak{f}_3k_3^2-\frac{\partial}{\partial s}(\mathfrak{f}_2k_2)-\frac{\partial}{\partial s}(\mathfrak{f}_4k_3)-\psi_1k_1.$$

Similarly, we have

$$\frac{\partial}{\partial s}\frac{\partial \mathbf{B}_2}{\partial t}=\frac{\partial}{\partial s}\left[\left(\mathfrak{f}_3k_3+\frac{\partial \mathfrak{f}_4}{\partial s}\right)\mathbf{T}-\psi_2\mathbf{N}-\psi_3\mathbf{B}_1\right],$$

which implies that

$$\begin{aligned}\frac{\partial}{\partial s}\frac{\partial \mathbf{B}_2}{\partial t}&=\left(\frac{\partial}{\partial s}(\mathfrak{f}_3k_3)+\frac{\partial^2 \mathfrak{f}_4}{\partial s^2}-\psi_2k_1\right)\mathbf{T}+\left(\mathfrak{f}_3k_1k_3+\frac{\partial \mathfrak{f}_4}{\partial s}k_1+\psi_3k_2-\frac{\partial \psi_2}{\partial s}\right)\mathbf{N}\\&+(-\psi_2k_2-\frac{\partial \psi_3}{\partial s})\mathbf{B}_1-\psi_3k_3\mathbf{B}_2.\end{aligned}\tag{3.18}$$

Also,

$$\begin{aligned}\frac{\partial}{\partial t}\frac{\partial \mathbf{B}_2}{\partial s}&=-\frac{\partial}{\partial t}(k_3\mathbf{B}_1)\\&=-\frac{\partial k_3}{\partial t}\mathbf{B}_1-k_3\left(\left(\mathfrak{f}_2k_2+\frac{\partial \mathfrak{f}_3}{\partial s}-\mathfrak{f}_4k_3\right)\mathbf{T}-\psi_1\mathbf{N}+\psi_3\mathbf{B}_2\right).\end{aligned}$$

Combining this with (3.18) gives

$$\frac{\partial k_3}{\partial t}=\psi_2k_2+\frac{\partial \psi_3}{\partial s}.$$

Similarly, we have

$$\frac{\partial}{\partial s}\frac{\partial \mathbf{N}}{\partial t}=\frac{\partial}{\partial s}\left[\left(\mathfrak{f}_1k_1+\frac{\partial \mathfrak{f}_2}{\partial s}-\mathfrak{f}_3k_2\right)\mathbf{T}+\psi_1\mathbf{B}_1+\psi_2\mathbf{B}_2\right],$$

which implies that

$$\frac{\partial}{\partial s}\frac{\partial \mathbf{N}}{\partial t}=\left(\frac{\partial}{\partial s}(\mathfrak{f}_1k_1)+\frac{\partial^2\mathfrak{f}_2}{\partial s^2}-\frac{\partial}{\partial s}(\mathfrak{f}_3k_2)\right)\mathbf{T}-(\psi_1k_2)\,\mathbf{N}$$
$$+\left(\frac{\partial\psi_1}{\partial s}-\psi_2k_3\right)\mathbf{B}_1+\left(\frac{\partial\psi_2}{\partial s}+\psi_1k_3\right)\mathbf{B}_2.$$

Also,

$$\frac{\partial}{\partial t}\frac{\partial \mathbf{N}}{\partial s}=\frac{\partial}{\partial t}(k_1\mathbf{T}+k_2\mathbf{B}_1)$$
$$=\left(\frac{\partial k_1}{\partial t}+\mathfrak{f}_2k_2^2+\frac{\partial\mathfrak{f}_3}{\partial s}k_2-\mathfrak{f}_4k_2k_3\right)\mathbf{T}+\left(\mathfrak{f}_1k_1^2+\frac{\partial\mathfrak{f}_2}{\partial s}k_1-\mathfrak{f}_3k_1k_2-\psi_1k_2\right)\mathbf{N}$$
$$+\left(\mathfrak{f}_2k_1k_2+\frac{\partial\mathfrak{f}_3}{\partial s}k_1-\mathfrak{f}_4k_1k_3+\frac{\partial k_2}{\partial t}\right)\mathbf{B}_1+(\psi_3k_2)\,\mathbf{B}_2. \tag{3.19}$$

Combining this with (3.18) gives

$$\frac{\partial\psi_1}{\partial s}-\frac{\partial k_2}{\partial t}=\mathfrak{f}_2k_1k_2+\frac{\partial\mathfrak{f}_3}{\partial s}k_1-\mathfrak{f}_4k_1k_3+-\psi_2k_3.$$

In the light of Theorem 3.6, we express the following corollaries without proofs:

Corollary 3.7.

$$\frac{\partial\psi_2}{\partial s}+\psi_1=\mathfrak{f}_3k_1k_3+\frac{\partial\mathfrak{f}_4}{\partial s}k_1+\psi_3k_2,$$

where

$$\psi_1=\left\langle\frac{\partial\mathbf{N}}{\partial t},\mathbf{B}_1\right\rangle,\ \psi_2=\left\langle\frac{\partial\mathbf{N}}{\partial t},\mathbf{B}_2\right\rangle,\ \psi_3=\left\langle\mathbf{B}_2,\frac{\partial\mathbf{B}_1}{\partial t}\right\rangle.$$

Corollary 3.8.

$$-\frac{\partial}{\partial s}(\mathfrak{f}_3k_3)-\frac{\partial^2\mathfrak{f}_4}{\partial s^2}-\psi_2k_1=\mathfrak{f}_2k_2k_3+\frac{\partial\mathfrak{f}_3}{\partial s}k_3-\mathfrak{f}_4k_3^2,$$

where

$$\psi_1=\left\langle\frac{\partial\mathbf{N}}{\partial t},\mathbf{B}_1\right\rangle,\ \psi_2=\left\langle\frac{\partial\mathbf{N}}{\partial t},\mathbf{B}_2\right\rangle,\ \psi_3=\left\langle\mathbf{B}_2,\frac{\partial\mathbf{B}_1}{\partial t}\right\rangle.$$

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Firat University, Department of Mathematics,23119, Elaziğ, Turkey
E-mail address: selcukbas79@gmail.com, talatkorpinar@gmail.com, essin.turhan@gmail.com