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TCHEBYSHEF NET ON $\mathcal{B}$-TANGENT DEVELOPABLE SURFACES OF SPACELIKE BIHARMONIC NEW TYPE $\mathcal{B}$-SLANT HELICES IN $\mathcal{H}^3$

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Abstract. In this paper, we construct Tchebyshef net of tangent developable surfaces of biharmonic spacelike new type $\mathcal{B}$-slant helices according to Bishop frame in the Lorentzian Heisenberg group $\mathcal{H}^3$. We give necessary and sufficient conditions for new type $\mathcal{B}$-slant helices to be biharmonic. We characterize Tchebyshef net of $\mathcal{B}$-tangent developable surfaces in the Lorentzian Heisenberg group $\mathcal{H}^3$.

1. Introduction

A number of new results in web geometry have been obtained in the last twenty years. Germs of webs defined by few foliations in general position are far from being interesting. Basic results from differential calculus imply that the theory is locally trivial. As soon as the number of foliations surpasses the dimension of the ambient manifold this is no longer true. The discovery in the last years of the 1920 decade of the curvature for 3-webs on surfaces is considered as the birth of web geometry.

On the other hand, Jiang derived the first and the second variation formula for the bienergy in [6], showing that the Euler–Lagrange equation associated to E_2 is

$$\begin{aligned} \tau_2(f) &= -\mathcal{J}^f(\tau(f)) = -\Delta\tau(f) - \operatorname{trace}R^N(df,\tau(f))\,df \\ &= 0, \end{aligned} \tag{1.4}$$

where $\mathcal{J}^f$ is the Jacobi operator of f and $\tau(f) = \operatorname{trace}\nabla df$. The equation $\tau_2(f) = 0$ is called the biharmonic equation. Since $\mathcal{J}^f$ is linear, any harmonic map is biharmonic. Therefore, we are interested in proper biharmonic maps, that is non-harmonic biharmonic maps.

This study is organised as follows: Firstly, we construct Tchebyshef net of tangent developable surfaces of biharmonic spacelike new type $\mathcal{B}$-slant helices according to Bishop frame in the Lorentzian Heisenberg group $\mathcal{H}^3$. Secondly, we give necessary and sufficient conditions for new type $\mathcal{B}$-slant helices to be biharmonic. Finally, we characterize Tchebyshef net of $\mathcal{B}$-tangent developable surfaces in the Lorentzian Heisenberg group $\mathcal{H}^3$.

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2. The Lorentzian Heisenberg Group $\mathcal{H}^3$

The Heisenberg group Heis^3 is a Lie group which is diffeomorphic to $\mathbb{R}^3$ and the group operation is defined as

$$(x,y,z)*(\overline{x},\overline{y},\overline{z})=(x+\overline{x},y+\overline{y},z+\overline{z}-\frac{1}{2}\overline{x}y+\frac{1}{2}x\overline{y}).$$

The identity of the group is $(0,0,0)$ and the inverse of (x,y,z) is given by $(-x,-y,-z)$. The left-invariant Lorentz metric on Heis^3 is

$$g=-dx^2+dy^2+(xdy+dz)^2.$$

The following set of left-invariant vector fields forms an orthonormal basis for the corresponding Lie algebra:

$$\left\{\mathbf{e}_1=\frac{\partial}{\partial z},\ \mathbf{e}_2=\frac{\partial}{\partial y}-x\frac{\partial}{\partial z},\ \mathbf{e}_3=\frac{\partial}{\partial x}\right\}. \tag{1}$$

The characterising properties of this algebra are the following commutation relations, [15]:

$$g(\mathbf{e}_1,\mathbf{e}_1)=g(\mathbf{e}_2,\mathbf{e}_2)=1,\quad g(\mathbf{e}_3,\mathbf{e}_3)=-1.$$

Proposition 2.1. *For the covariant derivatives of the Levi-Civita connection of the left-invariant metric g, defined above the following is true:*

$$\nabla=\frac{1}{2}\begin{pmatrix} 0 & \mathbf{e}_3 & \mathbf{e}_2 \\ \mathbf{e}_3 & 0 & \mathbf{e}_1 \\ \mathbf{e}_2 & -\mathbf{e}_1 & 0 \end{pmatrix}, \tag{2}$$

where the (i,j)-element in the table above equals $\nabla_{\mathbf{e}_i}\mathbf{e}_j$ for our basis

$$\{\mathbf{e}_k,k=1,2,3\}.$$

3. Spacelike Biharmonic New Type $\mathcal{B}$–Slant Helices with Bishop Frame In The Lorentzian Heisenberg Group $\mathcal{H}^3$

Let $\gamma:I\longrightarrow\mathcal{H}^3$ be a non geodesic spacelike curve on the Lorentzian Heisenberg group $\mathcal{H}^3$ parametrized by arc length. Let $\{\mathbf{t},\mathbf{n},\mathbf{b}\}$ be the Frenet frame fields tangent to the Lorentzian Heisenberg group $\mathcal{H}^3$ along γ defined as follows:

$\mathbf{t}$ is the unit vector field γ' tangent to γ, $\mathbf{n}$ is the unit vector field in the direction of $\nabla_{\mathbf{t}}\mathbf{t}$ (normal to γ), and $\mathbf{b}$ is chosen so that $\{\mathbf{t},\mathbf{n},\mathbf{b}\}$ is a positively oriented orthonormal basis. Then, we have the following Frenet formulas:

$$\begin{aligned}\nabla_{\mathbf{t}}\mathbf{t}&=\kappa\mathbf{n},\\ \nabla_{\mathbf{t}}\mathbf{n}&=\kappa\mathbf{t}+\tau\mathbf{b},\\ \nabla_{\mathbf{T}}\mathbf{B}&=\tau\mathbf{n},\end{aligned} \tag{3}$$

where κ is the curvature of γ and τ is its torsion and

$$\begin{aligned}&g(\mathbf{t},\mathbf{t})=1,\ g(\mathbf{n},\mathbf{n})=-1,\ g(\mathbf{b},\mathbf{b})=1,\\ &g(\mathbf{t},\mathbf{n})=g(\mathbf{t},\mathbf{b})=g(\mathbf{n},\mathbf{b})=0.\end{aligned}$$

In the rest of the paper, we suppose everywhere $\kappa\neq0$ and $\tau\neq0$.

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative. The Bishop frame is expressed as

$$\begin{aligned} \nabla_{\mathbf{t}}\mathbf{t} &= k_1\mathbf{m}_1 - k_2\mathbf{m}_2, \\ \nabla_{\mathbf{t}}\mathbf{m}_1 &= k_1\mathbf{t}, \\ \nabla_{\mathbf{t}}\mathbf{m}_2 &= k_2\mathbf{t}, \end{aligned} \tag{4}$$

where

$$g(\mathbf{t},\mathbf{t}) = 1,\ g(\mathbf{m}_1,\mathbf{m}_1) = -1,\ g(\mathbf{m}_2,\mathbf{m}_2) = 1,$$
$$g(\mathbf{t},\mathbf{m}_1) = g(\mathbf{t},\mathbf{m}_2) = g(\mathbf{m}_1,\mathbf{m}_2) = 0.$$

Here, we shall call the set $\{\mathbf{t},\mathbf{m}_1,\mathbf{m}_2\}$ as Bishop trihedra, k_1 and k_2 as Bishop curvatures.

Also, $\tau(s) = \psi'(s)$ and $\kappa(s) = \sqrt{|k_2^2 - k_1^2|}$. Thus, Bishop curvatures are defined by

$$\begin{aligned} k_1 &= \kappa(s)\sinh\psi(s), \\ k_2 &= \kappa(s)\cosh\psi(s). \end{aligned}$$

With respect to the orthonormal basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ we can write

$$\begin{aligned} \mathbf{t} &= t^1\mathbf{e}_1 + t^2\mathbf{e}_2 + t^3\mathbf{e}_3, \\ \mathbf{m}_1 &= m_1^1\mathbf{e}_1 + m_1^2\mathbf{e}_2 + m_1^3\mathbf{e}_3, \\ \mathbf{m}_2 &= m_2^1\mathbf{e}_1 + m_2^2\mathbf{e}_2 + m_2^3\mathbf{e}_3. \end{aligned} \tag{5}$$

Theorem 3.1. *$\gamma : I \longrightarrow \mathcal{H}^3$ is a spacelike biharmonic curve with Bishop frame if and only if*

$$\begin{aligned} k_1^2 - k_2^2 &= \text{constant} = C \neq 0, \\ k_1'' + \left[k_1^2 - k_2^2\right]k_1 &= -k_1\left[1 + \left(m_2^1\right)^2\right] + k_2 m_1^1 m_2^1, \\ k_2'' + \left[k_1^2 - k_2^2\right]k_2 &= -k_1 m_1^1 m_2^1 - k_2\left[-1 + \left(m_1^1\right)^2\right]. \end{aligned} \tag{6}$$

The condition is not altered by reparametrization, so without loss of generality we may assume that slant helices have unit speed. The slant helices can be identified by a simple condition on natural curvatures.

To separate a spacelike new type slant helix according to Bishop frame from that of Frenet- Serret frame, in the rest of the paper, we shall use notation for the curve defined above as spacelike new type $\mathcal{B}$-slant helix.

Theorem 3.2. *Let $\gamma : I \longrightarrow \mathcal{H}^3$ be a unit speed spacelike curve with non-zero natural curvatures. Then γ is a new type $\mathcal{B}$-slant helix if and only if*

$$k_2 = k_1 \coth \mathfrak{Q}. \tag{7}$$

Corollary 3.3. *$\gamma : I \longrightarrow \mathcal{H}^3$ is spacelike biharmonic new type $\mathcal{B}-$slant helix if and only if*

$$\begin{aligned} k_1 &= \text{constant} \neq 0, \\ k_2 &= \text{constant} \neq 0, \\ [k_1^2 - k_2^2 + 1 + \left(m_2^1\right)^2] &= \coth \mathfrak{Q} m_1^1 m_2^1, \\ [k_1^2 - k_2^2 - 1 + \left(m_1^1\right)^2] &= -\coth \mathfrak{Q} m_1^1 m_2^1. \end{aligned} \tag{8}$$

4. Tchebyshef Net of $\mathcal{B}-$Tangent Developable Surfaces of Spacelike Biharmonic New Type $\mathcal{B}-$Slant Helices with Bishop Frame In The Lorentzian Heisenberg Group $\mathcal{H}^3$

For each fixed $(s_0, t_0) \in U \subset \mathbb{R}^2$, the curves

$$s \to \mathcal{O}_{new}(s, t_0) \text{ and } u \to \mathcal{O}_{new}(s_0, t)$$

are called the coordinate curves of $\mathcal{O}_{new}$. We say that the coordinate curves of $\mathcal{O}_{new}$ constitute a Tchebyshef net if the lengths of the opposite sides of any quadrilateral formed by them are equal.

By definition, $\mathcal{O}_{new}$ defines a Tchebyshef net iff for every two points $(s_0, t_0), (s_1, t_1) \in U$

$$\int_{t_0}^{t} \left| g(\frac{\partial}{\partial s}\mathcal{O}_{new}(u, u_0), \frac{\partial}{\partial s}\mathcal{O}_{new}(u, u_0)) \right|^{\frac{1}{2}} du = \int_{t_0}^{t} \left| g(\frac{\partial}{\partial s}\mathcal{O}_{new}(u, t_1), \frac{\partial}{\partial s}\mathcal{O}_{new}(u, t_1)) \right|^{\frac{1}{2}} du,$$

$$\int_{s_0}^{s} \left| g(\frac{\partial}{\partial t}\mathcal{O}_{new}(s_0, u), \frac{\partial}{\partial t}\mathcal{O}_{new}(s_0, u)) \right|^{\frac{1}{2}} du = \int_{s_0}^{s} \left| g(\frac{\partial}{\partial t}\mathcal{O}_{new}(s_1, u), \frac{\partial}{\partial t}\mathcal{O}_{new}(s_1, u)) \right|^{\frac{1}{2}} du.$$

By the Fundamental Theorem of Calculus, this is equivalent to

$$\left| g(\frac{\partial}{\partial s}\mathcal{O}_{new}(s, u_0), \frac{\partial}{\partial s}\mathcal{O}_{new}(s, u_0)) \right|^{\frac{1}{2}} = \left| g(\frac{\partial}{\partial s}\mathcal{O}_{new}(s, u_1), \frac{\partial}{\partial s}\mathcal{O}_{new}(s, u_1)) \right|^{\frac{1}{2}},$$

$$\left| g(\frac{\partial}{\partial t}\mathcal{O}_{new}(s_0, t), \frac{\partial}{\partial t}\mathcal{O}_{new}(s_0, t)) \right|^{\frac{1}{2}} = \left| g(\frac{\partial}{\partial t}\mathcal{O}_{new}(s_1, t), \frac{\partial}{\partial t}\mathcal{O}_{new}(s_1, t)) \right|^{\frac{1}{2}},$$

for all $(s_0, t_0), (s_1, t_1) \in U$. Since

$$E = g(\frac{\partial}{\partial s}\mathcal{R}(s, t_0), \frac{\partial}{\partial s}\mathcal{R}(s, t_0)), G = g(\frac{\partial}{\partial t}\mathcal{R}(s_0, t), \frac{\partial}{\partial t}\mathcal{R}(s_0, t)) \tag{9}$$

it follows that above equations holds iff E is independent of t and G is independent of s, i.e., iff

$$\frac{\partial E}{\partial t} = \frac{\partial G}{\partial s} = 0. \tag{10}$$

The purpose of this section is to study $\mathcal{B}-$tangent developable of biharmonic spacelike new type $\mathcal{B}-$slant helix in $\mathcal{H}^3$.

The $\mathcal{B}-$tangent developable of γ is a ruled surface

$$\mathcal{O}_{new}(s, t) = \gamma(s) + t\gamma'(s). \tag{11}$$

Theorem 4.1. *Let $\mathcal{O}_{new}$ be one-parameter family of the $\mathcal{B}-$tangent developable surface of a unit speed non-geodesic biharmonic new type $\mathcal{B}-$slant helix. $\mathcal{O}_{new}(s,t)$ defines a Tchebyshef net if and only if*

$$
\begin{aligned}
&-\frac{\partial}{\partial t}(\sin\mathfrak{Q}-tk_2\cos\mathfrak{Q})^2\\
&=\frac{\partial}{\partial t}(\cos\mathfrak{Q}\cosh[\mathcal{C}_0s+\mathcal{C}_1]+tk_1\sinh[\mathcal{C}_0s+\mathcal{C}_1]\\
&-tk_2\sin\mathfrak{Q}\cosh[\mathcal{C}_0s+\mathcal{C}_1])^2\\
&+\frac{\partial}{\partial t}(\cos\mathfrak{Q}\sinh[\mathcal{C}_0s+\mathcal{C}_1]+tk_1\cosh[\mathcal{C}_0s+\mathcal{C}_1]\\
&-tk_2\sin\mathfrak{Q}\sinh[\mathcal{C}_0s+\mathcal{C}_1])^2,
\end{aligned}\tag{12}
$$

where $\mathcal{C}_0$,$\mathcal{C}_1$ are constants of integration and

$$\mathcal{C}_0=\frac{\sqrt{k_2^2-k_1^2}}{\cos\mathfrak{Q}}-\sin\mathfrak{Q}.$$

Proof. From our assumption, we get the following equation

$$\mathbf{m}_2=\cos\mathfrak{Q}\mathbf{e}_1+\sin\mathfrak{Q}\cosh[\mathcal{C}_0s+\mathcal{C}_1]\mathbf{e}_2+\sin\mathfrak{Q}\sinh[\mathcal{C}_0s+\mathcal{C}_1]\mathbf{e}_3. \tag{13}$$

where $\mathcal{C}_0,\mathcal{C}_1$ are smooth functions of time.

On the other hand, using Bishop formulas Eq.(4) and Eq.(1), we have

$$\mathbf{m}_1=\sinh[\mathcal{C}_0s+\mathcal{C}_1]\mathbf{e}_2+\cosh[\mathcal{C}_0s+\mathcal{C}_1]\mathbf{e}_3. \tag{14}$$

Using above equation and Eq.(13), we get

$$\mathbf{t}=\sin\mathfrak{Q}\mathbf{e}_1+\cos\mathfrak{Q}\cosh[\mathcal{C}_0s+\mathcal{C}_1]\mathbf{e}_2+\cos\mathfrak{Q}\sinh[\mathcal{C}_0s+\mathcal{C}_1]\mathbf{e}_3. \tag{15}$$

Furthermore, we have the natural frame $\{(\mathcal{O}_{new})_s,(\mathcal{O}_{new})_t\}$ given by

$$
\begin{aligned}
(\mathcal{O}_{new})_s&=(\sin\mathfrak{Q}-tk_2\cos\mathfrak{Q})\mathbf{e}_1+(\cos\mathfrak{Q}\cosh[\mathcal{C}_0s+\mathcal{C}_1]\\
&+tk_1\sinh[\mathcal{C}_0s+\mathcal{C}_1]-tk_2\sin\mathfrak{Q}\cosh[\mathcal{C}_0s+\mathcal{C}_1])\mathbf{e}_2\\
&+(\cos\mathfrak{Q}\sinh[\mathcal{C}_0s+\mathcal{C}_1]+tk_1\cosh[\mathcal{C}_0s+\mathcal{C}_1]\\
&-tk_2\sin\mathfrak{Q}\sinh[\mathcal{C}_0s+\mathcal{C}_1])\mathbf{e}_3,
\end{aligned}
$$

and

$$(\mathcal{O}_{new})_t=\sin\mathfrak{Q}\mathbf{e}_1+\cos\mathfrak{Q}\cosh[\mathcal{C}_0s+\mathcal{C}_1]\mathbf{e}_2+\cos\mathfrak{Q}\sinh[\mathcal{C}_0s+\mathcal{C}_1]\mathbf{e}_3.$$

The components of the first fundamental form are

$$
\begin{aligned}
\frac{\partial\mathbf{E}}{\partial t}&=\frac{\partial}{\partial t}g((\mathcal{O}_{new})_s,(\mathcal{O}_{new})_s)=\frac{\partial}{\partial t}(\sin\mathfrak{Q}-tk_2\cos\mathfrak{Q})^2\\
&+\frac{\partial}{\partial t}(\cos\mathfrak{Q}\cosh[\mathcal{C}_0s+\mathcal{C}_1]+tk_1\sinh[\mathcal{C}_0s+\mathcal{C}_1]\\
&-tk_2\sin\mathfrak{Q}\cosh[\mathcal{C}_0s+\mathcal{C}_1])^2\\
&+\frac{\partial}{\partial t}(\cos\mathfrak{Q}\sinh[\mathcal{C}_0s+\mathcal{C}_1]+tk_1\cosh[\mathcal{C}_0s+\mathcal{C}_1]\\
&-tk_2\sin\mathfrak{Q}\sinh[\mathcal{C}_0s+\mathcal{C}_1])^2,
\end{aligned}
$$

$$\frac{\partial\mathbf{G}}{\partial s}=0.$$

From Theorem 4.1, we get the following result.

Corollary 4.2. *Let $\mathcal{O}_{new}$ be one-parameter family of the $\mathcal{B}-$tangent developable surface of a unit speed non-geodesic biharmonic new type $\mathcal{B}-$slant helix. If $\mathcal{O}_{new}(s,t)$ defines a Tchebyshef net, then the coordinate curves of $\mathcal{O}_{new}$ are orthogonal.*

Theorem 4.3. *Let $\mathcal{O}_{new}$ be $\mathcal{B}-$tangent developable of a unit speed non-geodesic biharmonic spacelike new type $\mathcal{B}-$slant helix. Then, the parametric equations of $\mathcal{B}-$tangent developable are*

$$\begin{aligned} \boldsymbol{x}_{\mathcal{O}_{new}}(s,t) &= \frac{1}{\mathcal{C}_0}\cos\mathfrak{Q}\cosh[\mathcal{C}_0 s+\mathcal{C}_1]+t\cos\mathfrak{Q}\sinh[\mathcal{C}_0 s+\mathcal{C}_1]+\mathcal{C}_2, \\ \boldsymbol{y}_{\mathcal{O}_{new}}(s,t) &= \frac{1}{\mathcal{C}_0}\cos\mathfrak{Q}\sinh[\mathcal{C}_0 s+\mathcal{C}_1]+t\cos\mathfrak{Q}\cosh[\mathcal{C}_0 s+\mathcal{C}_1]+\mathcal{C}_3, \\ \boldsymbol{z}_{\mathcal{O}_{new}}(s,t) &= \sin\mathfrak{Q}s-\frac{\mathcal{C}_2}{\mathcal{C}_0}\cos\mathfrak{Q}\sinh[\mathcal{C}_0 s+\mathcal{C}_1] \\ &\quad -\frac{1}{4\mathcal{C}_0}\cos^2\mathfrak{Q}(2[\mathcal{C}_0 s+\mathcal{C}_1]+\sinh 2[\mathcal{C}_0 s+\mathcal{C}_1])+t\sin\mathfrak{Q} \\ &\quad -t(\frac{1}{\mathcal{C}_0}\cos\mathfrak{Q}\cosh[\mathcal{C}_0 s+\mathcal{C}_1]+\mathcal{C}_2)\cos\mathfrak{Q}\cosh[\mathcal{C}_0 s+\mathcal{C}_1]+\mathcal{C}_4, \end{aligned} \tag{16}$$

where $\mathcal{C}_0$,$\mathcal{C}_1,\mathcal{C}_2,\mathcal{C}_3$ are constants of integration and

$$\mathcal{C}_0 = \frac{\sqrt{k_2^2-k_1^2}}{\cos\mathfrak{Q}} - \sin\mathfrak{Q}.$$

We may use Mathematica in Theorem 4.3, yields

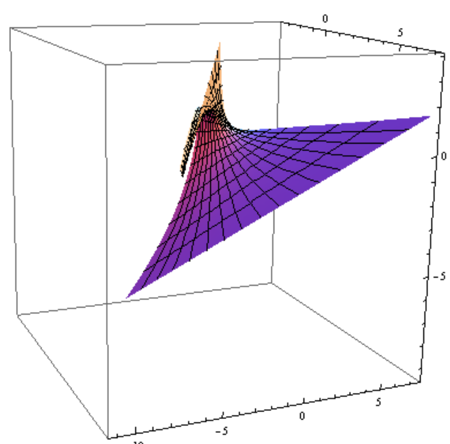

Fig.1.

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