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ONE PARAMETER FAMILY OF $\mathfrak{b}-\mathbf{m}_2$ DEVELOPABLE SURFACES OF BIHARMONIC NEW TYPE $\mathfrak{b}-$SLANT HELICES IN $\boldsymbol{Sol}^3$

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Abstract. In this paper, we study inextensible flows of $\mathfrak{b}-\mathbf{m}_2$ developable surfaces of biharmonic new type $\mathfrak{b}-$slant helix in the $\boldsymbol{Sol}^3$. We characterize one parameter family of the $\mathfrak{b}-\mathbf{m}_2$ developable surfaces in terms of their Bishop curvatures. Finally, we illustrate our results.

1. Introduction

Developable surfaces are especially important to the home boatbuilder because they are often working with sheet materials like plywood, steel or aluminum. Developable surfaces can be formed from flat sheets without stretching, so the forces required to form sheet materials into developable surfaces are much less than for other surfaces. In some cases, particularly with plywood, the forces required to form non-developable surfaces could be so large that the material is damaged internally when it is formed. Another advantage of developable surfaces is that the development, or flattened out shape, of such a surface is exact.

This study is organised as follows: Firstly, we study inextensible flows of $\mathfrak{b}-\mathbf{m}_2$ developable surfaces of biharmonic new type $\mathfrak{b}-$slant helix in the $\boldsymbol{Sol}^3$. Finally, characterize one parameter family of the $\mathfrak{b}-\mathbf{m}_2$ developable surfaces in terms of their Bishop curvatures.

2. Riemannian Structure of Sol Space $\boldsymbol{Sol}^3$

Sol space, one of Thurston's eight 3-dimensional geometries, can be viewed as $\mathbb{R}^3$ provided with Riemannian metric

$$g_{\boldsymbol{Sol}^3} = e^{2z}dx^2 + e^{-2z}dy^2 + dz^2,$$

where (x, y, z) are the standard coordinates in $\mathbb{R}^3$.

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Note that the Sol metric can also be written as:

$$g_{\boldsymbol{Sol}^3} = \sum_{i=1}^{3} \omega^i \otimes \omega^i,$$

where

$$\omega^1 = e^z dx, \quad \omega^2 = e^{-z} dy, \quad \omega^3 = dz,$$

and the orthonormal basis dual to the 1-forms is

$$\mathbf{e}_1 = e^{-z}\frac{\partial}{\partial x}, \quad \mathbf{e}_2 = e^{z}\frac{\partial}{\partial y}, \quad \mathbf{e}_3 = \frac{\partial}{\partial z}. \tag{2.1}$$

Proposition 2.1. *For the covariant derivatives of the Levi-Civita connection of the left-invariant metric $g_{\boldsymbol{Sol}^3}$, defined above the following is true:*

$$\nabla = \begin{pmatrix} -\mathbf{e}_3 & 0 & \mathbf{e}_1 \\ 0 & \mathbf{e}_3 & -\mathbf{e}_2 \\ 0 & 0 & 0 \end{pmatrix}, \tag{2.2}$$

where the (i,j)-element in the table above equals $\nabla_{\mathbf{e}_i}\mathbf{e}_j$ for our basis

$$\{\mathbf{e}_k, k = 1, 2, 3\} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}.$$

Lie brackets can be easily computed as:

$$[\mathbf{e}_1, \mathbf{e}_2] = 0, \quad [\mathbf{e}_2, \mathbf{e}_3] = -\mathbf{e}_2, \quad [\mathbf{e}_1, \mathbf{e}_3] = \mathbf{e}_1.$$

The isometry group of $\boldsymbol{Sol}^3$ has dimension 3. The connected component of the identity is generated by the following three families of isometries:

$$\begin{aligned} (x, y, z) &\to (x + c, y, z), \\ (x, y, z) &\to (x, y + c, z), \\ (x, y, z) &\to \left(e^{-c}x, e^{c}y, z + c\right). \end{aligned}$$

3. Biharmonic New Type $\mathfrak{b}$–Slant Helices in Sol Space $\boldsymbol{Sol}^3$

Assume that $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ be the Frenet frame field along γ. Then, the Frenet frame satisfies the following Frenet–Serret equations:

$$\begin{aligned} \nabla_{\mathbf{t}}\mathbf{t} &= \kappa\mathbf{n}, \\ \nabla_{\mathbf{t}}\mathbf{n} &= -\kappa\mathbf{t} + \tau\mathbf{b}, \\ \nabla_{\mathbf{t}}\mathbf{b} &= -\tau\mathbf{n}, \end{aligned} \tag{3.1}$$

where κ is the curvature of γ and τ its torsion and

$$\begin{aligned} &g_{\boldsymbol{Sol}^3}(\mathbf{t}, \mathbf{t}) = 1, \ g_{\boldsymbol{Sol}^3}(\mathbf{n}, \mathbf{n}) = 1, \ g_{\boldsymbol{Sol}^3}(\mathbf{b}, \mathbf{b}) = 1, \\ &g_{\boldsymbol{Sol}^3}(\mathbf{t}, \mathbf{n}) = g_{\boldsymbol{Sol}^3}(\mathbf{t}, \mathbf{b}) = g_{\boldsymbol{Sol}^3}(\mathbf{n}, \mathbf{b}) = 0. \end{aligned} \tag{3.2}$$

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative. The Bishop frame is expressed as

$$
\begin{aligned}
\nabla_{\mathbf{t}}\mathbf{t} &= k_1\mathbf{m}_1 + k_2\mathbf{m}_2, \\
\nabla_{\mathbf{t}}\mathbf{m}_1 &= -k_1\mathbf{t}, \\
\nabla_{\mathbf{t}}\mathbf{m}_2 &= -k_2\mathbf{t},
\end{aligned}
\tag{3.3}
$$

where

$$
\begin{aligned}
&g_{\boldsymbol{Sol}^3}(\mathbf{t},\mathbf{t}) = 1,\ g_{\boldsymbol{Sol}^3}(\mathbf{m}_1,\mathbf{m}_1) = 1,\ g_{\boldsymbol{Sol}^3}(\mathbf{m}_2,\mathbf{m}_2) = 1, \\
&g_{\boldsymbol{Sol}^3}(\mathbf{t},\mathbf{m}_1) = g_{\boldsymbol{Sol}^3}(\mathbf{t},\mathbf{m}_2) = g_{\boldsymbol{Sol}^3}(\mathbf{m}_1,\mathbf{m}_2) = 0.
\end{aligned}
\tag{3.4}
$$

Here, we shall call the set $\{\mathbf{t},\mathbf{m}_1,\mathbf{m}_2\}$ as Bishop trihedra, k_1 and k_2 as Bishop curvatures and $\delta(s) = \arctan\frac{k_2}{k_1}$, $\tau(s) = \delta'(s)$ and $\kappa(s) = \sqrt{k_1^2 + k_2^2}$.

Bishop curvatures are defined by

$$
\begin{aligned}
k_1 &= \kappa(s)\cos\delta(s), \\
k_2 &= \kappa(s)\sin\delta(s).
\end{aligned}
$$

With respect to the orthonormal basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ we can write

$$
\begin{aligned}
\mathbf{t} &= t^1e_1 + t^2e_2 + t^3e_3, \\
\mathbf{m}_1 &= m_1^1\mathbf{e}_1 + m_1^2\mathbf{e}_2 + m_1^3\mathbf{e}_3, \\
\mathbf{m}_2 &= m_2^1\mathbf{e}_1 + m_2^2\mathbf{e}_2 + m_2^3\mathbf{e}_3.
\end{aligned}
\tag{3.5}
$$

Theorem 3.1. *$\gamma : I \longrightarrow \boldsymbol{Sol}^3$ is a biharmonic curve according to Bishop frame if and only if*

$$
\begin{aligned}
k_1^2 + k_2^2 &= \text{constant} \neq 0, \\
k_1'' - \left[k_1^2 + k_2^2\right]k_1 &= -k_1\left[2m_2^3 - 1\right] - 2k_2m_1^3m_2^3, \\
k_2'' - \left[k_1^2 + k_2^2\right]k_2 &= 2k_1m_1^3m_2^3 - k_2\left[2m_1^3 - 1\right].
\end{aligned}
\tag{3.6}
$$

Definition 3.2. *A regular curve $\gamma : I \longrightarrow \boldsymbol{Sol}^3$ is called a new type slant helix provided the unit vector $\mathbf{m}_2$ of the curve γ has constant angle $\mathcal{M}$ with some fixed unit vector u, that is*

$$
g_{\boldsymbol{Sol}^3}(\mathbf{m}_2(s), u) = \cos\mathcal{M} \text{ for all } s \in I.
\tag{3.7}
$$

The condition is not altered by reparametrization, so without loss of generality we may assume that new type slant helices have unit speed. The second slant helices can be identified by a simple condition on natural curvatures.

To separate a new type slant helix according to Bishop frame from that of Frenet-Serret frame, in the rest of the paper, we shall use notation for the curve defined above as new type $\mathfrak{b}-$slant helix.

We shall also use the following lemma.

Lemma 3.3. *Let $\gamma : I \longrightarrow \boldsymbol{Sol}^3$ be a unit speed curve. Then γ is a new type $\mathfrak{b}-$slant helix if and only if*

$$k_1 = -k_2 \cot \mathcal{M}. \tag{3.8}$$

In the light of above lemma, we express the following result without proof:

Theorem 3.4. *Let $\gamma : I \longrightarrow \boldsymbol{Sol}^3$ be a unit speed non-geodesic biharmonic new type $\mathfrak{b}-$slant helix. Then, the position vector of γ is*

$$\begin{aligned}\gamma(s) = {} & [\frac{\cos \mathcal{M}}{\mathcal{S}_1^2 + \sin^2 \mathcal{M}}[-\mathcal{S}_1 \cos[\mathcal{S}_1 s + \mathcal{S}_2] + \sin \mathcal{M} \sin[\mathcal{S}_1 s + \mathcal{S}_2]] + \mathcal{S}_4 e^{-\sin \mathcal{M} s + \mathcal{S}_3}]\mathbf{e}_1 \\ & + [\frac{\cos \mathcal{M}}{\mathcal{S}_1^2 + \sin^2 \mathcal{M}}[-\sin \mathcal{M} \cos[\mathcal{S}_1 s + \mathcal{S}_2] + \mathcal{S}_1 \sin[\mathcal{S}_1 s + \mathcal{S}_2]] + \mathcal{S}_5 e^{\sin \mathcal{M} s - \mathcal{S}_3}]\mathbf{e}_2 \\ & + [-\sin \mathcal{M} s + \mathcal{S}_3]\mathbf{e}_3,\end{aligned}$$

where $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{S}_4, \mathcal{S}_5$ are constants of integration, [8].

4. Inextensible Flows of $\mathfrak{b} - \mathbf{m}_2$ Developable Surfaces of Biharmonic New Type $\mathfrak{b}-$Slant Helices in Sol Space $\boldsymbol{Sol}^3$

To separate a $\mathbf{m}_2$ developable according to Bishop frame from that of Frenet-Serret frame, in the rest of the paper, we shall use notation for this surface as $\mathfrak{b} - \mathbf{m}_2$ developable.

The purpose of this section is to study $\mathfrak{b}-\mathbf{m}_2$ developable of biharmonic new type $\mathfrak{b}-$slant helix in $\boldsymbol{Sol}^3$.

The $\mathfrak{b}-\mathbf{m}_2$ developable of γ is a ruled surface

$$\mathcal{C}_{new}(s, u) = \gamma(s) + u\mathbf{m}_2. \tag{4.1}$$

Definition 4.1. *A surface evolution $\mathcal{C}_{new}(s, u, t)$ and its flow $\frac{\partial \mathcal{C}_{new}}{\partial t}$ are said to be inextensible if its first fundamental form $\{\mathbf{E}, \mathbf{F}, \mathbf{G}\}$ satisfies*

$$\frac{\partial \mathbf{E}}{\partial t} = \frac{\partial \mathbf{F}}{\partial t} = \frac{\partial \mathbf{G}}{\partial t} = 0. \tag{4.2}$$

Definition 4.2. *We can define the following one-parameter family of developable ruled surface*

$$\mathcal{C}_{new}(s, u, t) = \gamma(s, t) + u\mathbf{m}_2(s, t). \tag{4.3}$$

Hence, we have the following theorem.

Theorem 4.3. *Let $\mathcal{C}_{new}$ be one-parameter family of the $\mathfrak{b}-\mathbf{m}_2$ developable of a unit speed non-geodesic biharmonic new type $\mathfrak{b}-$slant helix. Then $\frac{\partial \mathcal{C}_{new}}{\partial t}$ is inextensible if and only if*

$$\begin{aligned}
&\frac{\partial}{\partial t}((1-uk_1(t))\cos\mathcal{M}(t)\sin[\mathcal{S}_1(t)s+\mathcal{S}_2(t)])^2\\
&+\frac{\partial}{\partial t}((1-uk_1(t))\cos\mathcal{M}(t)\cos[\mathcal{S}_1(t)s+\mathcal{S}_2(t)])^2 \qquad (4.4)\\
&=-\frac{\partial}{\partial t}((1-uk_1(t))\sin\mathcal{M}(t))^2,
\end{aligned}$$

where $\mathcal{S}_1, \mathcal{S}_2$ are smooth functions of time.

Proof. Assume that $\mathcal{C}_{new}(s,u,t)$ be a one-parameter family of the $\mathfrak{b}-\mathbf{m}_2$ developable of a unit speed non-geodesic biharmonic new type $\mathfrak{b}-$slant helix.

From our assumption, we get the following equation

$$\begin{aligned}
\mathbf{m}_2 &= \sin\mathcal{M}(t)\sin[\mathcal{S}_1(t)s+\mathcal{S}_2(t)]\mathbf{e}_1+\sin\mathcal{M}(t)\cos[\mathcal{S}_1(t)s+\mathcal{S}_2(t)]\mathbf{e}_2\\
&\quad+\cos\mathcal{M}(t)\mathbf{e}_3, \qquad (4.5)
\end{aligned}$$

where $\mathcal{S}_1, \mathcal{S}_2$ are smooth functions of time.

On the other hand, using Bishop formulas Eq.(3.3) and Eq.(2.1), we have

$$\mathbf{m}_1=\cos[\mathcal{S}_1(t)s+\mathcal{S}_2(t)]\mathbf{e}_1-\sin[\mathcal{S}_1(t)s+\mathcal{S}_2(t)]\mathbf{e}_2. \qquad (4.6)$$

Using above equation and Eq.(4.5), we get

$$\mathbf{t}=\cos\mathcal{M}(t)\sin[\mathcal{S}_1(t)s+\mathcal{S}_2(t)]\mathbf{e}_1+\cos\mathcal{M}(t)\cos[\mathcal{S}_1(t)s+\mathcal{S}_2(t)]\mathbf{e}_2-\sin\mathcal{M}(t)\mathbf{e}_3. \qquad (4.7)$$

Furthermore, we have the natural frame $\{(\mathcal{C}_{new})_s, (\mathcal{C}_{new})_u\}$ given by

$$\begin{aligned}
(\mathcal{C}_{new})_s &= (1-uk_1(t))\cos\mathcal{M}(t)\sin[\mathcal{S}_1(t)s+\mathcal{S}_2(t)]\mathbf{e}_1\\
&\quad+(1-uk_1(t))\cos\mathcal{M}(t)\cos[\mathcal{S}_1(t)s+\mathcal{S}_2(t)]\mathbf{e}_2-(1-uk_1(t))\sin\mathcal{M}(t)
\end{aligned}$$

and

$$\begin{aligned}
(\mathcal{C}_{new})_u &= \sin\mathcal{M}(t)\sin[\mathcal{S}_1(t)s+\mathcal{S}_2(t)]\mathbf{e}_1+\sin\mathcal{M}(t)\cos[\mathcal{S}_1(t)s+\mathcal{S}_2(t)]\mathbf{e}_2\\
&\quad+\cos\mathcal{M}(t)\mathbf{e}_3. \qquad (4.8)
\end{aligned}$$

The components of the first fundamental form are

$$\begin{aligned}
\frac{\partial \mathbf{E}}{\partial t} &= \frac{\partial}{\partial t}((1-uk_1(t))\cos\mathcal{M}(t)\sin[\mathcal{S}_1(t)s+\mathcal{S}_2(t)])^2\\
&\quad+\frac{\partial}{\partial t}((1-uk_1(t))\cos\mathcal{M}(t)\cos[\mathcal{S}_1(t)s+\mathcal{S}_2(t)])^2\\
&\quad+\frac{\partial}{\partial t}((1-uk_1(t))\sin\mathcal{M}(t))^2,\\
\frac{\partial \mathbf{F}}{\partial t} &= 0, \qquad (4.9)\\
\frac{\partial \mathbf{G}}{\partial t} &= 0.
\end{aligned}$$

Hence, $\frac{\partial \mathcal{C}_{new}}{\partial t}$ is inextensible if and only if Eq.(4.4) is satisfied. This concludes the proof of theorem.

Theorem 4.4. *Let $\mathcal{C}_{new}$ be one-parameter family of the $\mathfrak{b}-\mathbf{m}_2$ developable surface of a unit speed non-geodesic biharmonic new type $\mathfrak{b}-$slant helix. Then, the parametric equations of this family are given by*

$$\begin{aligned}
\boldsymbol{x}_{\mathcal{C}_{new}}(s,u,t) &= -\frac{e^{\sin\mathcal{M}(t)s-\mathcal{S}_3(t)}\cos\mathcal{M}(t)}{\mathcal{S}_1^2(t)+\sin^2\mathcal{M}(t)}\mathcal{S}_1(t)\cos\left[\mathcal{S}_1(t)s+\mathcal{S}_2(t)\right] \\
&+\frac{e^{\sin\mathcal{M}(t)s-\mathcal{S}_3(t)}\cos\mathcal{M}(t)}{\mathcal{S}_1^2(t)+\sin^2\mathcal{M}(t)}\sin\mathcal{M}(t)\sin\left[\mathcal{S}_1(t)s+\mathcal{S}_2(t)\right]] \\
&+\mathcal{S}_4(t)e^{-\sin\mathcal{M}(t)s+\mathcal{S}_3(t)}+u\sin\mathcal{M}(t)\sin\left[\mathcal{S}_1(t)s+\mathcal{S}_2(t)\right],
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{y}_{\mathcal{C}_{new}}(s,u,t) &= -\sin\mathcal{M}(t)\frac{e^{-\sin\mathcal{M}(t)s+\mathcal{S}_3(t)}\cos\mathcal{M}(t)}{\mathcal{S}_1^2(t)+\sin^2\mathcal{M}(t)}\cos\left[\mathcal{S}_1(t)s+\mathcal{S}_2(t)\right] \\
&+\mathcal{S}_1(t)\frac{e^{-\sin\mathcal{M}(t)s+\mathcal{S}_3(t)}\cos\mathcal{M}(t)}{\mathcal{S}_1^2(t)+\sin^2\mathcal{M}(t)}\sin\left[\mathcal{S}_1(t)s+\mathcal{S}_2(t)\right]] \\
&+\mathcal{S}_5(t)e^{\sin\mathcal{M}(t)s-\mathcal{S}_3(t)}+u\sin\mathcal{M}(t)\cos\left[\mathcal{S}_1(t)s+\mathcal{S}_2(t)\right],
\end{aligned} \tag{4.10}$$

$$\boldsymbol{z}_{\mathcal{C}_{new}}(s,u,t) = -\sin\mathcal{M}(t)s+u\cos\mathcal{M}(t)+\mathcal{S}_3(t),$$

where $\mathcal{S}_1,\mathcal{S}_2,\mathcal{S}_3,\mathcal{S}_4,\mathcal{S}_5$ are smooth functions of time.

We can use Mathematica in above theorem, yields

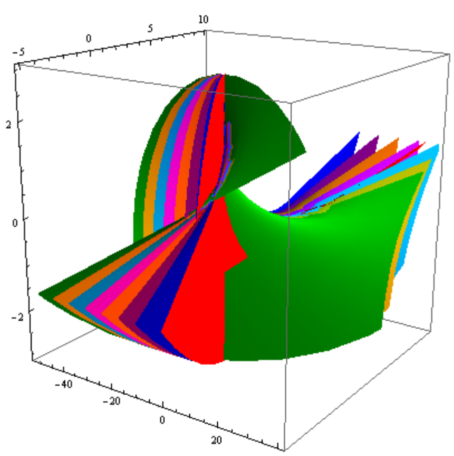

Fig.1.

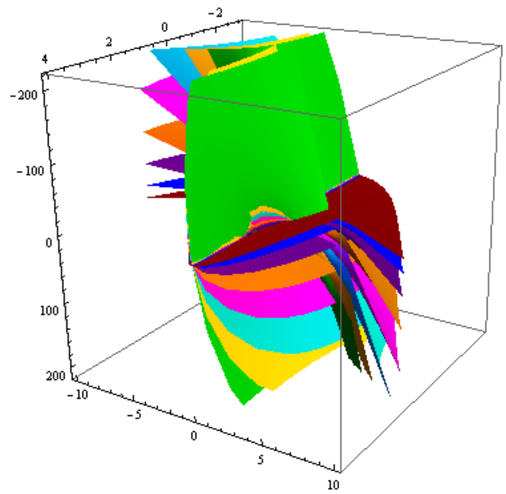

Fig.2.

Fig. 1,2: *The equation (4.10) is illustrated colour Red, Blue, Purple, Orange, Magenta, Cyan, Yellow, Green at the time* $t = 1$, $t = 1.2$, $t = 1.4$, $t = 1.6$, $t = 1.8$, $t = 2$, $t = 2.2$, $t = 2.4$, *respectively.*

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