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Optimal Control of Constrained Time Delay Systems

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Abstract

In this paper, a new approach based on embedding method for finding an approximate solution for a wide range of nonlinear optimal control problems with delays in state and control variables subject to mixed-control state constraints is introduced. First, the problem is transformed to a new optimal measure problem which is an infinite dimensional linear programming problem and then this new problem is approximated by a finite dimensional one. The approximate values of the optimal control, optimal state and optimal objective function are obtained by solving the corresponding finite dimensional linear programming problem. The effectiveness and applicability of the proposed idea is illustrated by several numerical examples.

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1 Introduction

The dynamics of many control systems may be expressed by time-delay differential equations. Delays often occur in the transmission of material or information between different parts of the systems. Communication systems, power systems, transmission systems, chemical processing systems and economics systems are examples of time delay systems. Time-delay systems are also used to model several different mechanisms in the dynamics of epidemics. In recent decades, optimal control problems with delays and obtaining their approximate solutions are very important issues in control theory and have attracted much attention of many researchers and investigators. Let us briefly review some papers concerning different classes of control problems.

Karatishvili [12] was first to provide a maximum principle for optimal control problems with a constant state delay. In [13], he gave similar results for control problems with pure control delays. Bader [1] used collocation methods to

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solve the boundary value problem for the retarded state variable and advanced adjoint variable. Dadebo and Luus [5] used the differential dynamic programming method with a moderate number of stages. Chen et al. [4] used an iterative dynamic programming method for solving time-delayed optimal control problems. Optimal control problems with constant delays in state and control variables and mixed control-state inequality constraints (based on the use of Pontryagin-type minimum(maximum) principle) were considered by Göllmann et al. [9].
Our aim is to propose a measure theoretical approach to solve optimal control problems which are governed by a nonlinear time delay system with mixed control-state constraints. This method which is on the basis of embedding process is an extension of the work by Rubio [15] to a class of nonlinear time delay systems. During last two decades, many methods based on Rubio's work have been proposed for designing optimal solution for various systems by Kamyad et al. [10, 11], Farahi et al. [8] and Effati et al. [6, 7].
This paper is organized as follows. Section 2 introduces the formulation of the retarded optimal control problem (ROCP). The variational form of the problem (ROCP) will be shown in Section 3. In Section 4 the new form of the problem is transmitted to measure space and in Section 5 the problem in measure space which is an infinite dimensional linear programming problem is approximated by a finite dimensional one. Computing the admissible pair is done in Section 6. In Section 7, three numerical examples are presented, demonstrating the proposed method's simplicity in solving time-delayed optimal control problems. Conclusions are finally made in Section 8.

2 System Description

Consider the following retarded optimal control problem (ROCP) with mixed control-state inequality constraints:

$$\text{Minimize } J(x(.),u(.)) = g(x(t_f)) + \int_{t_0}^{t_f} L(t,x(t),x(t-r),u(t),u(t-s))dt \qquad (1)$$

subject to

$$\begin{array}{lll} \dot{x}(t) = f(t,x(t),x(t-r),u(t),u(t-s)), & & a.e. \quad t \in [t_0,t_f], \\ x(t) = \phi(t), & t \in [t_0-r,t_0] & \\ u(t) = \theta(t), & t \in [t_0-s,t_0] & \\ H(t,x(t),u(t)) \leq 0, & t \in [t_0,t_f] & \end{array} \qquad (2)$$

such that the following conditions hold:

1. The set $I = [t_0,t_f]$ is a time interval, $\Delta t = t_f - t_0$ is a rational number and $\Re$ is the set of all real numbers.

2. The set A is a closed and bounded subset in $\Re^n$, $x(.) : I \longrightarrow A \subseteq \Re^n$ is the state function of the system which is absolutely continuous on I. Also it is assumed that

$$x(t) = [x_1(t) \;\; x_2(t) \ldots \;\; x_n(t)]^T,$$

$$x_j(t) \in [\beta_{1_j}, \beta_{2_j}], \qquad j = 1, 2, \ldots, n,$$

where β_{1_j} and β_{2_j} are appropriate known real numbers.

3. The set U is a closed and bounded subset in $\Re^m$, $u(.) : I \longrightarrow U \subseteq \Re^m$ is the control function of the system which is piecewise continuous and Lebesgue-measurable on I. Also it is assumed that

$$u(t) = [u_1(t) \;\; u_2(t) \ldots \;\; u_m(t)]^T,$$

$$u_i(t) \in [\alpha_{1_i}, \alpha_{2_i}], \qquad i = 1, 2, \ldots, m,$$

where α_{1_i} and α_{2_i} are appropriate known real numbers.

4. The boundary conditions $x(t_0) = x_0 \in A \subseteq \Re^n$ and $x(t_f) = x_f \in A \subseteq \Re^n$ which are the initial and final states, respectively, are satisfied where $x(t_f)$ may be known or unknown.

5. The pair $p = (x(.), u(.))$ satisfies the system (2) almost everywhere on I^0, the interior set of I.

6. The numbers r and s are constant, known, positive and rational, $\phi(t) = [\phi_1(t) \;\; \phi_2(t) \ldots \;\; \phi_n(t)]^T$ and $\theta(t) = [\theta_1(t) \;\; \theta_2(t) \ldots \;\; \theta_m(t)]^T$ are known piecewise continuous vector functions.

7. The function $g : \Re^n \longrightarrow \Re$ is assumed to be continuously differentiable and known.

8. The function $L : I \times \Re^n \times \Re^n \times \Re^m \times \Re^m \longrightarrow \Re$ is nonlinear, piecewise continuous, measurable and known.

9. The vector functions $f : I \times \Re^n \times \Re^n \times \Re^m \times \Re^m \longrightarrow \Re^n$ and $H : I \times \Re^n \times \Re^m \longrightarrow \Re^q$ are nonlinear and piecewise continuous.

Definition 2.1 *A pair $p = (x(.), u(.))$ is called an admissible pair for the problem (ROCP), if the state $x(.)$ and the control $u(.)$ satisfy conditions 2-5.*

We assume that the set of all admissible pairs is nonempty and denote it by W.

3 The Variational Formulation

Let

$$\Omega = I \times \prod_{j=1}^{j=n} [\beta_{1_j}, \beta_{2_j}] \times \prod_{i=1}^{i=m} [\alpha_{1_i}, \alpha_{2_i}] = I \times A \times U.$$

Select the auxiliary function

$$\eta(t) = [\eta_1(t) \;\; \eta_2(t) \ldots \;\; \eta_n(t)]^T,$$

where

$$\eta_j(t) \in C^1[t_0, t_f], \qquad j = 1, 2, \dots, n,$$

are arbitrary nonzero continuously differentiable functions.
Multiplying of the left and right sides of the differential equation (2) in $\eta^T(t)$ and adding $\dot{\eta}^T(t)x(t)$ to the both sides of (2) and integrating of the both sides over the time interval $I = [t_0, t_f]$ yields

$$\int_{t_0}^{t_f} \eta^T(t) f(t, x(t), x(t-r), u(t), u(t-s))dt + \int_{t_0}^{t_f} \dot{\eta}^T(t)x(t)dt = \Delta\eta, \tag{3}$$

where

$$\Delta\eta = \eta^T(t_f)x(t_f) - \eta^T(t_0)x(t_0).$$

Let $I^0 = (t_0, t_f)$ and $D(I^0)$ be the space of all infinitely differentiable functions with compact support in $I^0 = (t_0, t_f)$. Select $\eta(t) = \psi(t)$, where $\psi \in D(I^0)$ then ψ has zero value at the initial point t_0 and also at the terminal point t_f, i. e. $\psi(t_0) = \psi(t_f) = 0$. So we have

$$\int_{t_0}^{t_f} \psi^T(t) f(t, x(t), x(t-r), u(t), u(t-s))dt + \int_{t_0}^{t_f} \dot{\psi}^T(t)x(t)dt = \Delta\psi = 0. \tag{4}$$

Also by choosing the functions which are dependent only on time, we have

$$\int_{t_0}^{t_f} \beta(t, x(t), u(t))dt = a_\beta, \qquad \beta \in C_1(\Omega), \tag{5}$$

where $C_1(\Omega)$ is the space of all functions in $C(\Omega)$ that depend only on time and a_β is the integral of β on I.
We need to convert $g(x(t_f))$ in (1) to an integral form. So by differentiation of the function g we have

$$\frac{dg(x(t))}{dt} = \nabla g(x(t))\dot{x}(t), \tag{6}$$

where

$$\nabla g(x(t)) = (\frac{\partial g(x(t))}{\partial x_1}, \frac{\partial g(x(t))}{\partial x_2}, \dots, \frac{\partial g(x(t))}{\partial x_n}),$$

$$\frac{dx}{dt} \equiv \dot{x}(t) = (\dot{x}_1(t), \dot{x}_2(t), \dots, \dot{x}_n(t)).$$

By integrating of (6) over the interval $I = [t_0, t_f]$, we have

$$g(x(t_f)) - g(x(t_0)) = \int_{t_0}^{t_f} \frac{dg(x(t))}{dt} = \int_{t_0}^{t_f} \nabla g(x(t))\dot{x}(t)dt,$$

or

$$g(x(t_f)) = g(x(t_0)) + \int_{t_0}^{t_f} \nabla g(x(t))\dot{x}(t)dt, \tag{7}$$

where $g(x(t_0))$ is constant.
Furthermore we need to convert the inequality constraints $H(t, x(t), u(t))$ in (2)

to an integral form. One can show that the inequality constraints in (2) are equivalent to the following equalities (see [3]):

$$\int_{t_0}^{t_f} (H_i(t, x(t), u(t)) + |H_i(t, x(t), u(t))|)dt = 0, \qquad i = 1, 2, \ldots, q, \tag{8}$$

where H_i is the ith component of H. Now the problem (ROCP) is transformed to the following form:

$$\begin{aligned}\text{Minimize } J(x(.), u(.)) \;=\; & \int_{t_0}^{t_f} \{\nabla g(x(t)) f(t, x(t), x(t-r), u(t), u(t-s)) \\ & + L(t, x(t), x(t-r), u(t), u(t-s))\}dt + g(x(t_0))\end{aligned}$$

subject to

$$\begin{aligned}
&\int_{t_0}^{t_f} \eta^T(t) f(t, x(t), x(t-r), u(t), u(t-s))dt + \int_{t_0}^{t_f} \dot{\eta}^T(t) x(t)dt = \Delta\eta, && \eta \in C^1(I)\\
&\int_{t_0}^{t_f} \psi^T(t) f(t, x(t), x(t-r), u(t), u(t-s))dt + \int_{t_0}^{t_f} \dot{\psi}^T(t) x(t)dt = 0, && \psi \in D(I^0)\\
&\int_{t_0}^{t_f} \beta(t, x(t), u(t))dt = a_\beta, && \beta \in C_1(\Omega),\\
&\int_{t_0}^{t_f} (H_i(t, x(t), u(t)) + |H_i(t, x(t), u(t))|)dt = 0, && i = 1, 2, \ldots, q.
\end{aligned} \tag{9}$$

Now for each pair $p = (x(.), u(.)) \in W$, we define the mapping

$$\Lambda_p : F \in C(\Omega) \longrightarrow \int_{t_0}^{t_f} F(t, x(t), u(t))dt, \tag{10}$$

where $C(\Omega)$ indicates the space of all continuous functions on Ω. The functional Λ_p is well-defined, linear, nonnegative and continuous (for more details see [15]). Now the problem (9) is as the following

$$\text{Minimize } J(p) = \Lambda_p(f_0) + g(x(t_0))$$

subject to

$$\begin{aligned}
&\Lambda_p(f_\eta + x_{\dot{\eta}}) = \Delta\eta, && \eta \in C^1(I)\\
&\Lambda_p(f_\psi + x_{\dot{\psi}}) = 0, && \psi \in D(I^0)\\
&\Lambda_p(\beta) = a_\beta, && \beta \in C_1(\Omega)\\
&\Lambda_p(H_i + |H_i|) = 0, && i = 1, 2, \ldots, q,
\end{aligned} \tag{11}$$

where

$$\begin{aligned}
&f_0 = \nabla g(x(t)) f(t, x(t), x(t-r), u(t), u(t-s)) + L(t, x(t), x(t-r), u(t), u(t-s)),\\
&f_\eta = \eta^T(t) f(t, x(t), x(t-r), u(t), u(t-s)),\\
&x_{\dot{\eta}} = \dot{\eta}^T(t) x(t).
\end{aligned}$$

The optimization problem (11) is an infinite dimensional linear programming problem. The problem (ROCP) is now to find an appropriate Λ_P on the set $C(\Omega)$ such that satisfies the constraints (11) and minimizes $\Lambda_p(f_0)$. The continuous mapping Λ_p is called a Radon measure. The problem (11) is called the variational form of the problem (ROCP).

4 Optimization in Measure Space

By the Riesz's representation theorem, every Radon measure Λ_p can be corresponding to a regular, finite and unique Borel measure. So there exists a Borel measure μ on Ω such that

$$\Lambda_p(F) = \int_\Omega F(t,x,u)d\mu = \mu(F), \qquad F \in C(\Omega). \tag{12}$$

For more details (12) see [15].
Suppose that $M^+(\Omega)$ denotes the space of all positive Borel measures on Ω. Consider the following functional

$$J : M^+(\Omega) \longrightarrow \Re,$$

$$J(\mu) = \int_\Omega f_0(t,x,u)d\mu + g(x(t_0)) = \mu(f_0) + g(x(t_0)).$$

Now the problem (11) is equivalent to the following problem

$$\text{Minimize } J(\mu) = \int_\Omega f_0(t,x,u)d\mu + g(x(t_0)) = \mu(f_0) + g(x(t_0))$$

subject to

$$\begin{array}{ll} \mu(f_\eta + x_{\dot\eta}) = \Delta\eta, & \eta \in C^1(I) \\ \mu(f_\psi + x_{\dot\psi}) = 0, & \psi \in D(I^0) \\ \mu(\beta) = a_\beta, & \beta \in C_1(\Omega) \\ \mu(H_i + |H_i|) = 0, & i = 1, 2, \ldots, q. \end{array} \tag{13}$$

Assume that $Q \subset M^+(\Omega)$ is the set of all positive Borel measures which satisfy infinite constraints (13). Now if we equip the space $M^+(\Omega)$ with weak*-topology then it can be proved that Q is compact and the functional $J : Q \longrightarrow \Re$ is continuous. Also the space $M^+(\Omega)$ with weak*-topology is a Hausdorff space. These conditions guarantee that there exists an optimal measure μ^* in the set Q in which $\mu^*(f_0) \leq \mu(f_0)$, for any $\mu \in Q$ (for more details and proofs see [15]).

5 Approximation

It is obvious that the problem (13) is an infinite dimensional linear programming problem in the measure space in which all the functions in (13) are linear in the variable μ. Now the linear programming problem (13) can be solved with different approaches. One of these approaches is the use of approximation. This work is done in two phases:

5.1 The First Approximation

Assume that the linear combinations of functions $\{\eta_i : i \in \mathcal{N}\}$, $\{\psi_j : j \in \mathcal{N}\}$ and $\{\beta_s : s \in \mathcal{N}\}$ are uniformly dense in $C^1(I)$, $D(I^0)$ and $C_1(\Omega)$ respectively, where

$\mathcal{N}$ denotes the set of all natural numbers. So for the approximation of the infinite constraints (13) to the finite ones we shall consider only a finite number of these functions. Now for the first set of the constraints (13) we choose the functions $\{\eta_i : i = 1, 2, \dots, M_1\}$ as the following

$$\eta_i(t) = t^{i-1}, \qquad t \in I, \quad i = 1, 2, \dots, M_1.$$

For the second set of the constraints (13) we choose the functions $\{\psi_j : j = 1, 2, \dots, M_2\}$ as the following

$$\psi_j(t) = \sin \frac{2\pi j(t - t_0)}{t_f - t_0}, \qquad t \in I, \qquad j = 1, 2, \dots, M_2,$$

or

$$\psi_j(t) = 1 - \cos \frac{2\pi j(t - t_0)}{t_f - t_0}, \qquad t \in I, \qquad j = 1, 2, \dots, M_2.$$

Also for the third set of the constraints (13) we choose $\{\beta_s : s = 1, 2, \dots, L\}$ as the following

$$\beta_s(t) = \begin{cases} 1, & t \in I_s \\ 0, & t \notin I_s \end{cases}$$

where

$$I_s = [t_0 + \frac{(s-1)(t_f - t_0)}{L}, t_0 + \frac{s(t_f - t_0)}{L}], \qquad s = 1, 2, \dots, L,$$

are the subintervals of $I = [t_0, t_f]$ (for more details see [15]).

5.2 The Second Approximation

Applying the second approximation, we get a finite dimensional linear programming problem which its solution is an approximate solution for the problem (ROCP). By Proposition III. 2 of [15] we obtain an approximation to the optimal measure μ^* by a finite combination of atomic measures as the following form

$$\mu^* = \sum_{k=1}^{M_1+M_2+L} \alpha_k^* \delta(z_k^*), \tag{14}$$

where $z_k^* \in \Omega$, the coefficients $\alpha_k^* \geq 0$, $(k = 1, 2, \dots, M_1 + M_2 + L)$ and $\delta(.)$ is the unitary atomic measure which is defined as $\delta(z)(F) = F(z)$ for each $F \in C(\Omega)$, $z \in \Omega$. Using (14) one can conclude that the optimization problem (13) is equivalent to a nonlinear optimization problem in which the unknown parameters are z_k^*, α_k^*, $(k = 1, 2, \dots, M_1 + M_2 + L)$. We need to transform this nonlinear programming problem to a linear form. For this reason we use a linear approximation. If ω is a dense subset of Ω, $N >> M_1 + M_2 + L$ and $z_k \in \omega$, $(k = 1, 2, \dots, N)$ are known then the corresponding nonlinear optimization problem can be approximated by the following statement (see [15]):

$$\text{Minimize} \ \sum_{k=1}^{N} \alpha_k f_0(z_k) + g(x(t_0))$$

subject to

$$
\begin{aligned}
&\sum_{k=1}^{N} \alpha_k (f_{\eta_i} + x_{\dot{\eta}_i})(z_k) = \Delta\eta_i, && i = 1, 2, \ldots, M_1 \\
&\sum_{k=1}^{N} \alpha_k (f_{\psi_j} + x_{\dot{\psi}_j})(z_k) = 0, && j = 1, 2, \ldots, M_2 \\
&\sum_{k=1}^{N} \alpha_k (\beta_s)(z_k) = a_{\beta_s}, && s = 1, 2, \ldots, L \\
&\sum_{k=1}^{N} \alpha_k (H_i + |H_i|)(z_k) = 0, && i = 1, 2, \ldots, q. \\
&\alpha_k \geq 0, && k = 1, 2, \ldots, N, \qquad (15)
\end{aligned}
$$

where

$$
\begin{aligned}
&f_0(z_k) = \nabla g(x_k) f(t_k, x_k, x(t_k - r), u_k, u(t_k - s)) \\
&+L(t_k, x_k, x(t_k - r), u_k, u(t_k - s)), \\
&f_{\eta_i}(z_k) = \eta_i^T(t_k) f(t_k, x_k, x(t_k - r), u_k, u(t_k - s)), \\
&x_{\dot{\eta}_i}(z_k) = \dot{\eta}_i{}^T(t_k) x_k,
\end{aligned}
$$

and $z_k = (t_k, x_k, u_k) \in \omega$, $(k = 1, 2, \ldots, N)$ are constructed by dividing the sets I, A, U into the number of equal subsets and a_{β_s} is the integral of β_s on I.
Note that the states and controls which have delays are still unknown. So we have the following Lemma.

Lemma 5.1 *Let $L = D_I$ and $n = D_x \times D_u$ where D_I, D_x and D_u are the number of divisions of I, A and U respectively. For $(p = 1, 2, \ldots, L)$ select $t_{(p-1)n+1} = t_{(p-1)n+2} = \ldots = t_{pn}$, then the problem (15) is converted to the following linear programming problem in which L is selected such that $\frac{Lr}{\Delta t}$ and $\frac{Ls}{\Delta t}$ be natural numbers.*

$$
\text{Minimize } \sum_{i=0}^{n-1} \sum_{p=1}^{L} \alpha_{np-i} f_0(z_{np-i}) + g(x(t_0))
$$

subject to

$$
\begin{aligned}
&\sum_{i=0}^{n-1} \sum_{p=1}^{L} \alpha_{np-i} (f_{\eta_i} + x_{\dot{\eta}_i})(z_{np-i}) = \Delta\eta_i, && && i = 1, 2, \ldots, M_1 \\
&\sum_{i=0}^{n-1} \sum_{p=1}^{L} \alpha_{np-i} (f_{\psi_j} + x_{\dot{\psi}_j})(z_{np-i}) = 0, && && j = 1, 2, \ldots, M_2 \\
&\sum_{i=0}^{n-1} \sum_{p=1}^{L} \alpha_{np-i} (\beta_s)(z_{np-i}) = a_{\beta_s}, && && s = 1, 2, \ldots, L \\
&\sum_{i=0}^{n-1} \sum_{p=1}^{L} \alpha_{np-i} (H_i + |H_i|)(z_{np-i}) = 0, && && i = 1, 2, \ldots, q. \\
&\alpha_{np-i} \geq 0, && p = 1, 2, \ldots, L, && i = 0, 1, \ldots, n-1, \qquad (16)
\end{aligned}
$$

where

$$
\begin{aligned}
&f_0(z_{np-i}) = \nabla g(x_{np-i}) f(t_{np-i}, x_{np-i}, x_{np-i-\frac{nLr}{\Delta t}}, u_{np-i}, u_{np-i-\frac{nLs}{\Delta t}}) \\
&+L(t_{np-i}, x_{np-i}, x_{np-i-\frac{nLr}{\Delta t}}, u_{np-i}, u_{np-i-\frac{nLs}{\Delta t}}), \\
&f_{\eta_i}(z_{np-i}) = \eta_i^T(t_{np-i}) f(t_{np-i}, x_{np-i}, x_{np-i-\frac{nLr}{\Delta t}}, u_{np-i}, u_{np-i-\frac{nLs}{\Delta t}}), \\
&x_{\dot{\eta}_i}(z_{np-i}) = \dot{\eta}_i^T(t_{np-i}) x_{np-i},
\end{aligned}
$$

and

$$
x_{np-i-\frac{nLr}{\Delta t}} = \phi(t_{np-i} - r), \qquad p = 1, 2, \dots, \frac{Lr}{\Delta t}, \qquad i = 0, 1, \dots, n-1,
$$

$$
u_{np-i-\frac{nLs}{\Delta t}} = \theta(t_{np-i} - s), \qquad p = 1, 2, \dots, \frac{Ls}{\Delta t}, \qquad i = 0, 1, \dots, n-1.
$$

Proof. See [2].

6 Computing the Admissible Pair

Solving the problem (16) the optimal coefficients $\alpha_k^*, (k = 1, 2, \dots, N)$ are attained. We obtain the piecewise continuous control function from analysis Rubio (see [15]) and finally from the differential equation (2), using the 4-step Runge Kutta method, the state function $x(.)$ is obtained.

7 Numerical Examples

Here, we use our approach to obtain approximate optimal solutions of the following three nonlinear time-delayed optimal control problems by solving linear programming (LP) problem (16), via simplex method [14]. All the problems are programmed in MATLAB 9.0 and run on a PC with Processor 2.40 GHz and 4 GB RAM.

Example 7.1 *(Optimal control of a constrained problem). We consider the following optimal control problem with the delay $r = 1$ in the state and $s = 2$ in the control*

$$
\text{Minimize} \int_0^6 (x^2(t) + u^2(t)) dt
$$

subject to

$$
\begin{aligned}
&\dot{x}(t) = x(t-1)u(t-2), && t \in [0, 6] \\
&x(t) = 1, \quad t \in [-1, 0], && x(6) = 0.21 \\
&u(t) = 0, \quad t \in [-2, 0] &&
\end{aligned}
$$

we impose the mixed control-state constraint:

$$
u(t) + x(t) \geq 0.3, \qquad t \in [0, 6].
$$

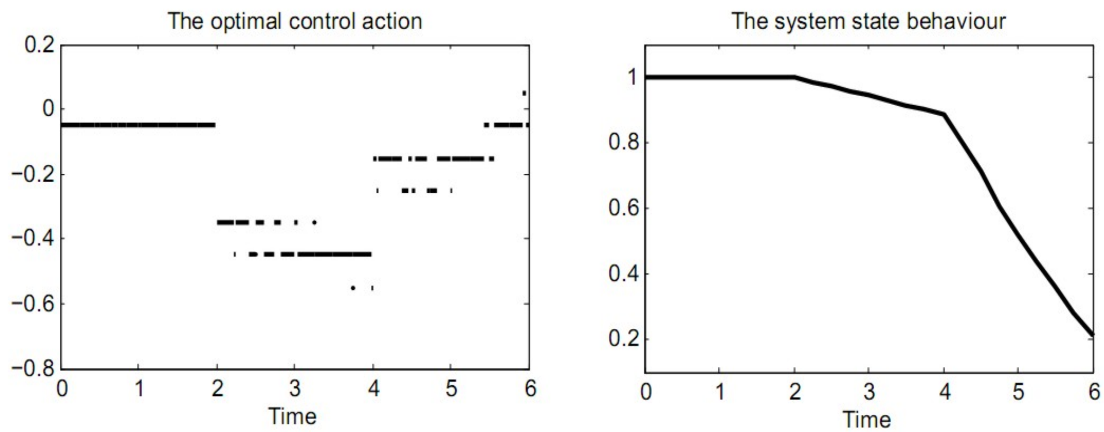

Figure 1: The almost optimal control action and the behaviour of the system state of Example 7.1 using the proposed method.

Let the sets $I = [0, 6]$, $A = [0.2, 1]$ and $U = [-0.8, 0.2]$ are divided into 24, 8 and 10 subintervals, respectively. So $\Omega = I \times A \times U$ is divided into $N = 1920$ subintervals. Now if $M_1 = 2, \eta_i(t) = t^{i-1}$ for $i = 1, 2,\ M_2 = 22, \psi_j(t) = \sin \frac{2\pi j(t)}{6}$ for $j = 1, 2, \ldots, 22$ and $L = 24$, then by solving a linear programming problem corresponding to (16), the optimal control and the state function are obtained. The minimum value of the cost functional using the proposed method is $J^* = 3.0969$ with a CPU time of 7.694775 seconds while the minimum value of the cost functional using the Euler discretization method described in [9] is $J^* = 3.108259352$ with a CPU time of 65.8 seconds. The graphs of the piecewise continuous control function and the state function are shown in Figure 1.

Example 7.2 *(Optimal control of a renewable resource). We discuss the optimal control of a logistic growth process. A well-known example is optimal fishing, where the fact that overfishing reduces the profit for the fishing industry in the long run indicates the importance of developing of a long-time fishing strategy.*
Let $x(t)$ denote the biomass population and $u(t)$ the harvesting effort. In the following control model with fixed final time $t_f > 0$, only the state variable $x(t)$ has a delay $r \geq 0$:

$$\text{Maximize } \int_0^{t_f} e^{-dt}(pu(t) - c_E x(t)^{-1} u(t)^3)dt$$

subject to

$$\begin{aligned} &\dot{x}(t) = ax(t)(1 - \frac{x(t-r)}{b}) - u(t) \\ &x(t) \equiv x_0, \qquad t \in [-r, 0] \\ &x(t) \geq x_0, \qquad t \in [0, t_f] \\ &u(t) \geq 0, \qquad t \in [0, t_f]. \end{aligned}$$

The data are chosen as follows: market price $p = 2$, discount rate $d = 0.05$, harvesting cost $c_E = 0.2$, growth rates $a = 3$ and $b = 5$, initial value $x_0 = 2$, final time $t_f = 20$ and time delay $r = 0.5$.

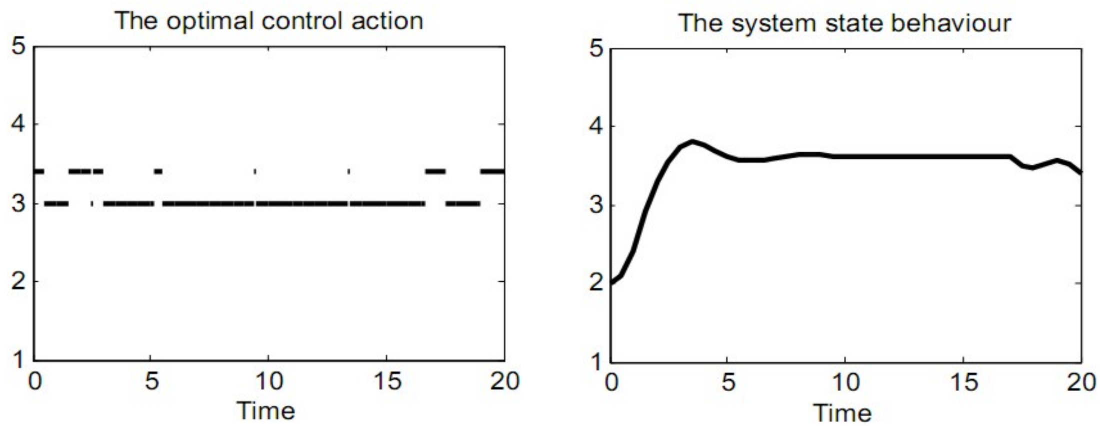

Figure 2: The almost optimal control action and the behaviour of the system state of Example 7.2 using the proposed method.

Let the sets $I = [0, 20]$, $A = [2, 4]$ and $U = [0, 4]$ are divided into 40, 10 and 10 subintervals, respectively. So $\Omega = I \times A \times U$ is divided into $N = 4000$ subintervals. Now if $M_1 = 2, \eta_i(t) = t^{i-1}$ for $i = 1, 2$, $M_2 = 5, \psi_j(t) = \sin \frac{2\pi j(t)}{20}$ for $j = 1, 2, \ldots, 5$ and $L = 40$, then by solving a linear programming problem corresponding to (16), the optimal control and the state function are obtained. The minimum value of the cost functional using the proposed method is $J^* = 57.2245$ with a CPU time of 32.166890 seconds while the minimum value of the cost functional using the discretization method described in [9] is $J^* = 56.876896$ with a CPU time of 2688.78 seconds. The graphs of the piecewise continuous control function and the continuous state function are shown in Figure 2.

Example 7.3 *(Harmonic oscillator with retarded damping). Consider the following time delayed system:*

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= -x_1(t) - x_2(t-1) + u(t) \end{aligned}$$

with initial conditions

$$\begin{aligned} x_1(0) &= 10, \\ x_2(t) &= 0, \qquad -1 \leq t \leq 0. \end{aligned}$$

The control action is bounded by

$$0 \leq u \leq 6.$$

The performance index to be minimized is given by

$$J = 5x_1^2(t_f) + \frac{1}{2}\int_0^{t_f} u^2(t)dt$$

where $t_f = 2$. In addition, the path of $x_2(t)$ is confined by

$$x_2(t) \geq -6.$$

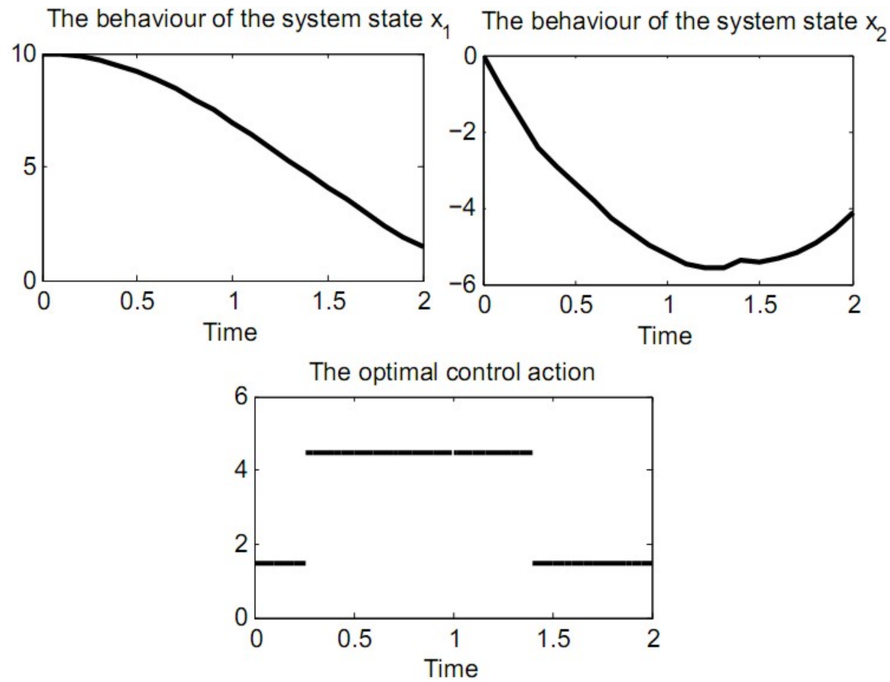

Figure 3: The almost optimal control action and the behaviours of the system states of Example 7.3 using the proposed method.

Let the sets $I = [0, 2]$, $A_1 = [0.6, 10]$, $A_2 = [-6, 0]$ and $U = [0, 6]$ are divided into 20, 10, 3 and 2 subintervals, respectively. So $\Omega = I \times A_1 \times A_2 \times U$ is divided into $N = 1200$ subintervals. Now if $M_1 = 3, \eta_i(t) = t^{i-1}$ for $i = 1, 2, 3$, $M_2 = 3, \psi_j(t) = \sin \frac{2\pi j(t)}{2}$ for $j = 1, 2, 3$ and $L = 20$, then by solving a linear programming problem corresponding to (16), the optimal control and the state functions are obtained. The minimum value of the cost functional using the proposed method is $J^* = 9.1470$ with a CPU time of 5.894848 seconds while the minimum value of the cost functional using the iterative dynamic programming method described in [4] is $J^* = 9.043998$. The graphs of the piecewise continuous control function and the continuous state functions are shown in Figure 3.

8 Conclusions

In this paper, a numerical method based on embedding approach for solving a class of retarded optimal control problems with mixed control-state constraints has successfully been used. The presented approach in this paper is based on some principles of measure theory, functional analysis and linear programming. Our proposed method has some benefits. For example, this method is not iterative and it is self-starting. Furthermore in this approach, the nonlinearity of the constraints and objective functional has not serious effects on the solution. Also this approach yields extremely significantly superior results in comparison with the existing methods. Several issues for optimal control of constrained time delay systems, which could not adequately be addressed in this paper, require further work. For example, optimal control for systems with multiple time lags, time-delayed optimal control problems with time-varying delays, optimal control design for discrete-time nonlinear time-delay systems, time optimal control problems for time delay systems, infinite horizon time delay optimal control problems and other related issues should be studied in more details.These issues are subject to current researches.

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