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STRONGLY LACUNARY SUMMABLE DOUBLE SEQUENCE SPACES IN $n-$NORMED SPACES DEFINED BY IDEAL CONVERGENCE AND AN ORLICZ FUNCTION

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Abstract. In this paper we introduce some certain new double sequence spaces via ideal convergence, double lacunary sequence and an Orlicz function in n-normed spaces and examine some properties of the resulting these spaces.

1. INTRODUCTION

Let X be a non-empty set, then a family of sets $I \subset 2^X$ (the class of all subsets of X) is called an *ideal* if and only if for each $A, B \in I$, we have $A \cup B \in I$ and for each $A \in I$ and each $B \subset A$, we have $B \in I$. A non-empty family of sets $F \subset 2^X$ is a *filter* on X if and only if $\varnothing \notin F$, for each $A, B \in F$, we have $A \cap B \in F$ and each $A \in F$ and each $A \subset B$, we have $B \in F$. An ideal I is called *non-trivial ideal* if $I \neq \varnothing$ and $X \notin I$. Clearly $I \subset 2^X$ is a non-trivial ideal if and only if $\mathcal{F} = \mathcal{F}(I) = \{X/A : A \in I\}$ is a filter on X. A non-trivial ideal $I \subset 2^X$ is called *admissible* if and only if $\{\{x\} .x \in X\} \subset I$. Further details on ideals of 2^X can be found in Kostyrko, et.al [3]. The notion was further investigated by Salat, et.al [4] and others.

By the convergence of a double sequence we mean the convergence on the Pringsheim sense that is, a double sequence $x = (x_{k,l})$ has Pringsheim limit L (denoted by $P - \lim x = L$) provided that given $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that $|x_{k,l} - L| < \varepsilon$ whenever $k, l > n$, [1]. We shall write more briefly as "*$P-$convergent*".

The double sequence $x = (x_{k,l})$ is bounded if there exists a positive number M such that $|x_{k,l}| < M$ for all k and l. Let l_∞^2 the space of all bounded double such that

$$\|x_{k,l}\|_{(\infty,2)} = \sup_{k,l} |x_{k,l}| < \infty.$$

The double sequence $\theta_{r,s} = \{(k_r, l_s)\}$ is called *double lacunary sequence* [5] if there exist two increasing of integers such that

$$k_o = 0,\ h_r = k_r - k_{r-1} \to \infty \text{ as } r \to \infty \text{ and } l_o = 0,\ \bar{h}_s = l_s - l_{s-1} \to \infty \text{ as } s \to \infty.$$

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Notations: $k_{r,s} = k_r l_s$, $h_{r,s} = h_r \bar{h}_s$, $\theta_{r,s}$ is determined by

$$I_{r,s} = \{(k,l) : k_{r-1} < k \leq k_r \text{ and } l_{s-1} < l \leq l_s\}, q_r = \frac{k_r}{k_{r-1}}, \bar{q}_s = \frac{l_s}{l_{s-1}} \text{ and } q_{r,s} = q_r \bar{q}_s.$$

Recall in [6] that an *Orlicz function* M is continuous, convex, nondecreasing function define for $x > 0$ such that $M(0) = 0$ and $M(x) > 0$. If convexity of Orlicz function is replaced by $M(x+y) \leq M(x) + M(y)$ then this function is called the *modulus function* and characterized by Ruckle [7]. An Orlicz function M is said to satisfy Δ_2-*condition* for all values u, if there exists $K > 0$ such that $M(2u) \leq KM(u)$, $u \geq 0$.

Lemma 1. *Let M be an Orlicz function which satisfies Δ_2-condition and let $0 < \delta < 1$. Then for each $t \geq \delta$, we have $M(t) < Kt\delta^{-1}M(2)$ for some constant $K > 0$.*

A double sequence space X is said to be *solid* or *normal* if $(\alpha_{k,l} x_{k,l}) \in X$, and for all double sequences $\alpha = (\alpha_{k,l})$ of scalars with $|\alpha_{k,l}| \leq 1$ for all $k, l \in \mathbb{N}$.

Let $n \in \mathbb{N}$ and X be a real vector space of dimension d, where $n \leq d$. A real-valued function $\|.,...,.\|$ on X satisfying the following four conditions:

(i) $\|x_1, x_2, ..., x_n\| = 0$ if and only if $x_1, x_2, ..., x_n$ are linearly dependent,
(ii) $\|x_1, x_2, ..., x_n\|$ is invariant under permutation,
(iii) $\|\alpha x_1, x_2, ..., x_n\| = |\alpha| \|x_1, x_2, ..., x_n\|, \alpha \in \mathbb{R}$,
(iv) $\|x_1 + x_1', x_2, ..., x_n\| \leq \|x_1, x_2, ..., x_n\| + \|x_1', x_2, ..., x_n\|$

is called an $n-$*norm* on X, and the pair $(X, \|.,...,.\|)$ is called an $n-$*normed space* [2].

A trivial example of $n-$normed space is $X = \mathbb{R}$ equipped with the following Euclidean $n-$norm:

$$\|x_1, x_2, ..., x_n\|_E = abs \left(\begin{vmatrix} x_{11}...x_{1n} \\ ... \\ x_{n1}...x_{nn} \end{vmatrix} \right)$$

where $x_i = (x_{i1}, ..., x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \ldots, n$.

2. MAIN RESULTS

Let I_2 be an ideal of $2^{\mathbb{N}\times\mathbb{N}}$, $\theta_{r,s}$ be a double lacunary sequence, M be an Orlicz function, $p = (p_{k,l})$ be a bounded double sequence of strictly positive real numbers and $(X, \|.,...,.\|)$ be an $n-$normed space. Further $w(n-X)$ denotes $X-$valued sequence space. Now, we define the following double sequence spaces:

$$w_{\theta_{r,s}}^{I_2}[M, p, \|.,...,.\|]_o = \Big\{ x = (x_{k,l}) \in w(n-X) : \forall\varepsilon > 0,$$

$$\left\{ (r,s) \in I_{r,s} : \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} \left[M\left(\left\| \frac{x_{k,l}}{\rho}, z_1, z_2, ..., z_{n-1} \right\| \right) \right]^{p_{k,l}} \geq \varepsilon \right\} \in I_2$$

$$\text{for some } \rho > 0 \text{ and for every } z_1, z_2, ..., z_{n-1} \in X \Big\},$$

$$w_{\theta_{r,s}}^{I_2}[M,p,\|.,...,.\|]=\Big\{x=(x_{k,l})\in w(n-X):\forall\varepsilon>0,$$
$$\left\{(r,s)\in I_{r,s}:\ \frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s}}\left[M\left(\left\|\frac{x_{k,l}-L}{\rho},z_1,z_2,...,z_{n-1}\right\|\right)\right]^{p_{k,l}}\geq\varepsilon\right\}\in I_2$$
$$\text{for some } \rho>0, L\in X \text{ and for every } z_1,z_2,...,z_{n-1}\in X\Big\},$$

$$w_{\theta_{r,s}}^{I_2}[M,p,\|.,...,.\|]_\infty=\Big\{x=(x_{k,l})\in w(n-X):\exists K>0,$$
$$\left\{(r,s)\in I_{r,s}:\ \frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s}}\left[M\left(\left\|\frac{x_{k,l}}{\rho},z_1,z_2,...,z_{n-1}\right\|\right)\right]^{p_{k,l}}\geq K\right\}\in I_2$$
$$\text{for some } \rho>0 \text{ and for every } z_1,z_2,...,z_{n-1}\in X\Big\},$$

and

$$w_{\theta_{r,s}}[M,p,\|.,...,.\|]_\infty=\Big\{x=(x_{k,l})\in w(n-X):\exists K>0,$$
$$\frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s}}\left[M\left(\left\|\frac{x_{k,l}}{\rho},z_1,z_2,...,z_{n-1}\right\|\right)\right]^{p_{k,l}}\leq K$$
$$\text{for some } \rho>0 \text{ and for every } z_1,z_2,...,z_{n-1}\in X\Big\}.$$

The following well-known inequality will be used in this study:
If $0\leq\inf_{k,l}p_{k,l}=H_o\leq p_{k,l}\leq\sup_{k,l}=H<\infty$, $D=\max\left(1,2^{H-1}\right)$, then

$$|x_{k,l}-y_{k,l}|^{p_{k,l}}\leq D\left\{|x_{k,l}|^{p_{k,l}}+|y_{k,l}|^{p_{k,l}}\right\}$$

for all $k,l\in\mathbb{N}$ and $x_{k,l},y_{k,l}\in\mathbb{C}$. Also $|x_{k,l}|^{p_{k,l}}\leq\max\left(1,|x_{k,l}|^H\right)$ for all $x_{k,l}\in\mathbb{C}$.

Theorem 1. *The sets* $w_{\theta_{r,s}}^{I_2}[M,p,\|,.,\|]_o$, $w_{\theta_{r,s}}^{I_2}[M,p,\|,.,\|]$ *and* $w_{\theta_{r,s}}^{I_2}[M,p,\|,.,\|]_\infty$ *are linear spaces over the complex field* $\mathbb{C}$.

Proof. We will prove only for $w_{\theta_{r,s}}^{I_2}[M,p,\|,.,\|]_o$ and the others can be proved similarly. Let $x,y\in w_{\theta_{r,s}}^{I_2}[M,p,\|,.,\|]_o$ and $\alpha,\beta\in\mathbb{C}$. Then

$$\left\{(r,s)\in I_{r,s}:\frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s}}\left[M\left(\left\|\frac{x_{k,l}}{\rho_1},z_1,z_2,...,z_{n-1}\right\|\right)\right]^{p_{k,l}}\geq\frac{\varepsilon}{2}\right\}\in I_2,\text{ for some }\rho_1>0$$

and

$$\left\{(r,s)\in I_{r,s}:\frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s}}\left[M\left(\left\|\frac{y_{k,l}}{\rho_2},z_1,z_2,...,z_{n-1}\right\|\right)\right]^{p_{k,l}}\geq\frac{\varepsilon}{2}\right\}\in I_2,\text{ for some }\rho_2>0.$$

Since $\|.,...,.\|$ is a $n-$norm and M is an Orlicz function, the following inequality holds:

$$\frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s}}\left[M\left(\left\|\frac{\alpha x_{k,l}+\beta y_{k,l}}{|\alpha|\rho_1+|\beta|\rho_2},z_1,z_2,...,z_{n-1}\right\|\right)\right]^{p_{k,l}}$$
$$\leq\frac{D}{h_{r,s}}\sum_{(k,l)\in I_{r,s}}\left[\frac{|\alpha|}{|\alpha|\rho_1+|\beta|\rho_2}M\left(\left\|\frac{x_{k,l}}{\rho_1},z_1,z_2,...,z_{n-1}\right\|\right)\right]^{p_{k,l}}$$
$$+\frac{D}{h_{r,s}}\sum_{(k,l)\in I_{r,s}}\left[\frac{|\beta|}{|\alpha|\rho_1+|\beta|\rho_2}M\left(\left\|\frac{y_{k,l}}{\rho_2},z_1,z_2,...,z_{n-1}\right\|\right)\right]^{p_{k,l}}$$
$$\leq\frac{D}{h_{r,s}}\sum_{(k,l)\in I_{r,s}}\left[M\left(\left\|\frac{x_{k,l}}{\rho_1},z_1,z_2,...,z_{n-1}\right\|\right)\right]^{p_{k,l}}$$
$$+\frac{D}{h_{r,s}}\sum_{(k,l)\in I_{r,s}}\left[M\left(\left\|\frac{y_{k,l}}{\rho_2},z_1,z_2,...,z_{n-1}\right\|\right)\right]^{p_{k,l}}$$

From the above inequality we get

$$\left\{(r,s)\in I_{r,s}:\ \frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s}}\left[M\left(\left\|\frac{\alpha x_{k,l}+\beta y_{k,l}}{|\alpha|\rho_1+|\beta|\rho_2},z_1,z_2,...,z_{n-1}\right\|\right)\right]^{p_{k,l}}\geq\varepsilon\right\}$$

$$\subset\left\{(r,s)\in I_{r,s}:\ \frac{D}{h_{r,s}}\sum_{(k,l)\in I_{r,s}}\left[M\left(\left\|\frac{x_{k,l}}{\rho_1},z_1,z_2,...,z_{n-1}\right\|\right)\right]^{p_{k,l}}\geq\frac{\varepsilon}{2}\right\}$$
$$\cup\left\{(r,s)\in I_{r,s}:\ \frac{D}{h_{r,s}}\sum_{(k,l)\in I_{r,s}}\left[M\left(\left\|\frac{y_{k,l}}{\rho_2},z_1,z_2,...,z_{n-1}\right\|\right)\right]^{p_{k,l}}\geq\frac{\varepsilon}{2}\right\}.$$

Two sets on the right hand side belong to I_2 and this completes the proof.

It is also easy verify that the space $w_{\theta_{r,s}}[M,p,\|.,...,.\|]_\infty$ is also a linear space. □

Theorem 2. *For fixed $(n,m)\in\mathbb{N}\times\mathbb{N}$, $w_{\theta_{r,s}}[M,p,\|.,...,.\|]_\infty$ paranormed space with respect to the paranorm defined by*

$$h_{(n,m)}(x)=\inf\left\{\rho^{\frac{p_{n,m}}{H}}>0:\left(\sup_{r,s}\frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s}}\left[M\left(\left\|\frac{x_{k,l}}{\rho},z_1,z_2,...,z_{n-1}\right\|\right)\right]^{p_{k,l}}\right)^{\frac{1}{H}}\leq 1,\right.$$
$$\left.\textit{for some } \rho>0 \textit{ and for every } z_1,z_2,...,z_{n-1}\in X\right\}.$$

Proof. $h_{(n,m)}(\theta)=0$ and $h_{(n,m)}(-x)=h_{(n,m)}(x)$ are easy to prove, so we omit them. Let us take $x,y\in w_{\theta_{r,s}}[M,p,\|.,...,.\|]_\infty$. Let

$$A(x)=\left\{\rho>0:\sup_{r,s}\frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s}}\left[M\left(\left\|\frac{x_{k,l}}{\rho},z_1,z_2,...,z_{n-1}\right\|\right)\right]^{p_{k,l}}\leq 1,\forall z\in X\right\}$$

and

$$A(y)=\left\{\rho>0:\sup_{r,s}\frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s}}\left[M\left(\left\|\frac{y_{k,l}}{\rho},z_1,z_2,...,z_{n-1}\right\|\right)\right]^{p_{k,l}}\leq 1,\forall z\in X\right\}.$$

Let $\rho_1\in A(x)$ and $\rho_2\in A(y)$. If $\rho=\rho_1+\rho_2$, then we have

$$\begin{aligned}
&\sup_{r,s}\frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s}}\left[M\left(\left\|\frac{x_{k,l}+y_{k,l}}{\rho},z_1,z_2,...,z_{n-1}\right\|\right)\right]\\
\leq&\frac{\rho_1}{\rho_1+\rho_2}\sup_{r,s}\frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s}}\left[M\left(\left\|\frac{x_{k,l}}{\rho_1},z_1,z_2,...,z_{n-1}\right\|\right)\right]\\
&+\frac{\rho_2}{\rho_1+\rho_2}\sup_{r,s}\frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s}}\left[M\left(\left\|\frac{y_{k,l}}{\rho_1},z_1,z_2,...,z_{n-1}\right\|\right)\right].
\end{aligned}$$

Thus

$$\sup_{r,s}\frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s}}\left[M\left(\left\|\frac{x_{k,l}+y_{k,l}}{\rho_1+\rho_2},z_1,z_2,...,z_{n-1}\right\|\right)\right]^{p_{k,l}}\leq 1$$

and

$$\begin{aligned}
h_{(n,m)}(x+y)&=\inf\left\{(\rho_1+\rho_2)^{\frac{p_{n,m}}{H}}:\rho_1\in A(x)\text{ and }\rho_2\in A(y)\right\}\\
&\leq\inf\left\{(\rho_1)^{\frac{p_{n,m}}{H}}:\rho_1\in A(x)\right\}+\inf\left\{(\rho_2)^{\frac{p_{n,m}}{H}}:\rho_2\in A(y)\right\}\\
&=h_{(n,m)}(x)+h_{(n,m)}(y).
\end{aligned}$$

Now, let $\lambda_{k,l}^u\to\lambda$, where $\lambda_{k,l}^u,\lambda\in\mathbb{C}$ and $h_{(n,m)}\left(x_{k,l}^u-x_{k,l}\right)\to 0$ as $u\to\infty$. We have to show that $h_{(n,m)}\left(\lambda_{k,l}^u x_{k,l}^u-\lambda x_{k,l}\right)\to 0$ as $u\to\infty$. Let $\lambda_{k,l}\to\alpha$, where $\lambda_{k,l},\lambda\in\mathbb{C}$ and $h_{(n,m)}\left(x_{k,l}^u-x_{k,l}\right)\to 0$ as $u\to\infty$. Let

$$\begin{aligned}
A(x^u)=\Bigg\{&\rho_u>0:\sup_{r,s}\frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s}}\left[M\left(\left\|\frac{x_{k,l}^u}{\rho_u},z_1,z_2,...,z_{n-1}\right\|\right)\right]^{p_{k,l}}\leq 1,\\
&\forall z_1,z_2,...,z_{n-1}\in X\Bigg\}.
\end{aligned}$$

and

$$\begin{aligned}
A(x^u-x)=\Bigg\{&\rho_u^{\imath}>0:\sup_{r,s}\frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s}}\left[M\left(\left\|\frac{x_{k,l}^u-x_{k,l}}{\rho_u^{\imath}},z_1,z_2,...,z_{n-1}\right\|\right)\right]^{p_{k,l}}\leq 1,\\
&\forall z_1,z_2,...,z_{n-1}\in X\Bigg\}.
\end{aligned}$$

If $\rho_u \in A(x^u)$ and $\rho_u^\imath \in A(x^u - x)$ then we observe that

$$M\left(\left\|\frac{\lambda_{k,l}^u x_{k,l}^u - \lambda x_{k,l}}{\rho_u\left|\lambda_{k,l}^u - \lambda\right| + \rho_u^\imath|\lambda|}, z_1, z_2, ..., z_{n-1}\right\|\right)$$

$$\leq M\left(\left\|\frac{\lambda_{k,l}^u x_{k,l}^u - \lambda x_{k,l}^u}{\rho_u\left|\lambda_{k,l}^u - \lambda\right| + \rho_u^\imath|\lambda|}, z_1, z_2, ..., z_{n-1}\right\| + \left\|\frac{\lambda x_{k,l}^u - \lambda x_{k,l}}{\rho_u\left|\lambda_{k,l}^u - \lambda\right| + \rho_u^\imath|\lambda|}, z_1, z_2, ..., z_{n-1}\right\|\right)$$

$$\leq \frac{\rho_u\left|\lambda_{k,l}^u - \lambda\right|}{\rho_u\left|\lambda_{k,l}^u - \lambda\right| + \rho_u^\imath|\lambda|} M\left(\left\|\frac{x_{k,l}^u}{\rho_u}, z_1, z_2, ..., z_{n-1}\right\|\right)$$

$$+ \frac{\rho_u^\imath|\lambda|}{\rho_u\left|\lambda_{k,l}^u - \lambda\right| + \rho_u^\imath|\lambda|} M\left(\left\|\frac{x_{k,l}^u - x_{k,l}}{\rho_u^\imath}, z_1, z_2, ..., z_{n-1}\right\|\right).$$

From this inequality, it follows that

$$\left[M\left(\left\|\frac{\lambda_{k,l}^u x_{k,l}^u - \lambda x_{k,l}}{\rho_u\left|\lambda_{k,l}^u - \lambda\right| + \rho_u^\imath|\lambda|}, z_1, z_2, ..., z_{n-1}\right\|\right)\right]^{p_{k,l}} \leq 1$$

and consequently

$$\begin{aligned} h_{(n,m)}\left(\lambda_{k,l}^u x_{k,l}^u - \lambda x_{k,l}\right) &= \inf\left\{\left(\rho_u\left|\lambda_{k,l}^u - \lambda\right| + \rho_u^\imath|\lambda|\right)^{\frac{p_{n,m}}{H}} : \rho_u \in A(x^u) \right. \\ &\qquad \left. \text{and } \rho_u^\imath \in A(x^u - x)\right\} \\ &\leq \left(\left|\lambda_{k,l}^u - \lambda\right|\right)^{\frac{p_{n,m}}{H}} \inf\left\{(\rho_u)^{\frac{p_{n,m}}{H}} : \rho_u \in A(x^u)\right\} \\ &\quad + (|\lambda|)^{\frac{p_{n,m}}{H}} \inf\left\{(\rho_u^\imath)^{\frac{p_{n,m}}{H}} : \rho_u^\imath \in A(x^u - x)\right\} \\ &\leq \max\left\{|\lambda|, (|\lambda|)^{\frac{p_{n,m}}{H}}\right\} h_{(n,m)}\left(x_{k,l}^u - x_{k,l}\right). \end{aligned}$$

Hence by our assumption the right hand side tends to zero as $u \to \infty$. This completes the proof. □

Corollary 1. *It can be noted that $h = \inf_{n,m\in\mathbb{N}} h_{(n,m)}$ also gives a paranorm on the above sequence spaces. However if one consider the sequence space $w_{\theta_{r,s}}[M,p,\|.,...,.\|]_\infty$ which is larger space than the space $w_{\theta_{r,s}}^{I_2}[M,p,\|.,\ldots,.\|]_\infty$ the construction of the paranorm is not clear and we leave it as an open problem. However it should be noted that for a fixed $F \in I_2$, the space*

$$\begin{aligned} w_{\theta_{r,s}}^F[M,p,\|.,...,.\|]_\infty = \Bigg\{ & x = (x_{k,l}) \in w(n-X) : \exists K > 0, \ \Bigg\{(n,m) \in \mathbb{N}\times\mathbb{N} : \\ & \sup_{(r,s)\in\mathbb{N}\times\mathbb{N}/F} \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} \left[M\left(\left\|\frac{x_{k,l}}{\rho}, z_1, z_2, ..., z_{n-1}\right\|\right)\right]^{p_{k,l}} \geq K\Bigg\} \in I_2, \\ & \textit{for some } \rho > 0 \textit{ and for every } z_1, z_2, ..., z_{n-1} \in X\Bigg\} \end{aligned}$$

which is subspace of the space $w_{\theta_{r,s}}^{I_2}[M,p,\|.,...,.\|]_\infty$ is a paranormed space with the paranorms $h_{(n,m)}$ for $(n,m) \notin F$ and $h_F = \inf_{(n,m)\in\mathbb{N}\times\mathbb{N}/F} h_{(n,m)}$.

Theorem 3. *Let M, M_1 and M_2 be Orlicz functions. Then we have*

(i) $w^{I_2}_{\theta_{r,s}}[M_1, p, \|., ..., .\|]_o \subset w^{I_2}_{\theta_{r,s}}[MoM_1, p, \|., ..., .\|]_o$ *provided that $p = (p_{k,l})$ is such that $H_o > 0$.*

(ii) $w^{I_2}_{\theta_{r,s}}[M_1, p, \|., ..., .\|]_o \cap w^{I_2}_{\theta_{r,s}}[M_2, p, \|., ..., .\|]_o \subset w^{I_2}_{\theta_{r,s}}[M_1 + M_2, p, \|., ..., .\|]_o$.

Proof. **(i).** For given $\varepsilon > 0$, we first choose $\varepsilon_o > 0$ such that $\max\left\{\varepsilon_o^H, \varepsilon_o^{H_o}\right\} < \varepsilon$. Now using the continuity of M, choose $0 < \delta < 1$ such that $0 < t < \delta$ implies $M(t) < \varepsilon_o$. Let $x \in w^{I_2}_{\theta_{r,s}}[M_1, p, \|., ..., .\|]_o$. Now from the definition of the space $w^{I_2}[M_1, p, \|., ..., .\|]_o$, for some $\rho > 0$

$$A(\delta) = \left\{(r,s) \in I_{r,s} : \frac{1}{h_{r,s}} \sum_{(k,l) \in I_{r,s}} \left[M_1\left(\left\|\frac{x_{k,l}}{\rho}, z_1, z_2, ..., z_{n-1}\right\|\right)\right]^{p_{k,l}} \geq \delta^H\right\} \in I_2.$$

Thus if $(n,m) \notin A(\delta)$ then

$$\frac{1}{h_{r,s}} \sum_{(k,l) \in I_{r,s}} \left[M_1\left(\left\|\frac{x_{k,l}}{\rho}, z_1, z_2, ..., z_{n-1}\right\|\right)\right]^{p_{k,l}} < \delta^H$$

$$\Rightarrow \sum_{(k,l) \in I_{r,s}} \left[M_1\left(\left\|\frac{x_{k,l}}{\rho}, z_1, z_2, ..., z_{n-1}\right\|\right)\right]^{p_{k,l}} < h_{r,s}\delta^H,$$

$$\Rightarrow \left[M_1\left(\left\|\frac{x_{k,l}}{\rho}, z_1, z_2, ..., z_{n-1}\right\|\right)\right]^{p_{k,l}} < \delta^H \text{ for all } (k,l) \in I_{r,s},$$

$$\Rightarrow M_1\left(\left\|\frac{x_{k,l}}{\rho}, z_1, z_2, ..., z_{n-1}\right\|\right) < \delta \text{ for all } (k,l) \in I_{r,s}.$$

Hence from above inequality and using continuity of M, we must have

$$M\left(M_1\left(\left\|\frac{x_{k,l}}{\rho}, z_1, z_2, ..., z_{n-1}\right\|\right)\right) < \varepsilon_o \text{ for all } (k,l) \in I_{r,s}$$

which consequently implies that

$$\sum_{(k,l) \in I_{r,s}} \left[M\left(M_1\left(\left\|\frac{x_{k,l}}{\rho}, z_1, z_2, ..., z_{n-1}\right\|\right)\right)\right]^{p_{k,l}} < h_{r,s} \max\left\{\varepsilon_o^H, \varepsilon_o^{H_o}\right\} < h_{r,s}\varepsilon,$$

$$\Rightarrow \frac{1}{h_{r,s}} \sum_{(k,l) \in I_{r,s}} \left[M\left(M_1\left(\left\|\frac{x_{k,l}}{\rho}, z_1, z_2, ..., z_{n-1}\right\|\right)\right)\right]^{p_{k,l}} < \varepsilon.$$

This shows that

$$\left\{(r,s) \in I_{r,s} : \frac{1}{h_{r,s}} \sum_{(k,l) \in I_{r,s}} \left[M\left(M_1\left(\left\|\frac{x_{k,l}}{\rho}, z_1, z_2, ..., z_{n-1}\right\|\right)\right)\right]^{p_{k,l}} \geq \varepsilon\right\} \subset A(\delta)$$

and so belongs to I_2. This completes the proof.

(ii) Let $x \in w^{I_2}_{\theta_{r,s}}[M_1, p, \|., ..., .\|]_o \cap w^{I_2}_{\theta_{r,s}}[M_2, p, \|., ..., .\|]_o$. Then the fact that

$$\frac{1}{h_{r,s}} \left[(M_1 + M_2)\left(\left\|\frac{x_{k,l}}{\rho}, z_1, z_2, ..., z_{n-1}\right\|\right)\right]^{p_{k,l}}$$

$$\leq \frac{D}{h_{r,s}} \left[M_1\left(\left\|\frac{x_{k,l}}{\rho}, z_1, z_2, ..., z_{n-1}\right\|\right)\right]^{p_{k,l}} + \frac{D}{nm} \left[M_2\left(\left\|\frac{x_{k,l}}{\rho}, z_1, z_2, ..., z_{n-1}\right\|\right)\right]^{p_{k,l}}$$

gives us the result. □

Theorem 4. ***(i)*** *If* $0 < H_o \leq p_{k,l} < 1$, *then* $w^{I_2}_{\theta_{r,s}}[M,p,\|.,...,.\|]_o \subset w^{I_2}_{\theta_{r,s}}[M,\|.,...,.\|]_o$.

(ii) *If* $1 \leq p_{k,l} \leq H < \infty$, *then* $w^{I_2}_{\theta_{r,s}}[M,\|.,...,.\|]_o \subset w^{I_2}_{\theta_{r,s}}[M,p,\|.,...,.\|]_o$.

(iii) *If* $0 < p_{k,l} < q_{k,l} < \infty$ *and* $\frac{q_{k,l}}{p_{k,l}}$ *is bounded, then* $w^{I_2}_{\theta_{r,s}}[M,p,\|.,...,.\|]_o \subset w^{I_2}_{\theta_{r,s}}[M,q,\|.,...,.\|]_o$.

Proof. The proof is standard, so we omit it. □

Theorem 5. *The sequence spaces* $w^{I_2}_{\theta_{r,s}}[M,p,\|.,...,.\|]_o$, $w^{I_2}_{\theta_{r,s}}[M,p,\|.,...,.\|]$, $w^{I_2}_{\theta_{r,s}}[M,p,\|.,...,.\|]_\infty$ *and* $w_{\theta_{r,s}}[M,p,\|.,...,.\|]_\infty$ *are solid.*

Proof. We give the proof for only $w^{I_2}_{\theta_{r,s}}[M,p,\|.,...,.\|]_o$. The others can be proved similarly. Let $x \in w^{I_2}_{\theta_{r,s}}[M_1,p,\|.,...,.\|]_o$ and $\alpha = (\alpha_{k,l})$ be a double sequence of scalars such that $|\alpha_{k,l}| \leq 1$ for all $k,l \in \mathbb{N}$. Then we have

$$\left\{(r,s) \in I_{r,s} : \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} \left[M\left(\left\|\frac{\alpha_{k,l}x_{k,l}}{\rho}, z_1, z_2, ..., z_{n-1}\right\|\right)\right]^{p_{k,l}} \leq \varepsilon \right\}$$

$$\subset \left\{(r,s) \in I_{r,s} : \frac{T}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} \left[M\left(\left\|\frac{x_{k,l}}{\rho}, z_1, z_2, ..., z_{n-1}\right\|\right)\right]^{p_{k,l}} \leq \varepsilon \right\} \in I_2,$$

where $T = \max_{k,l}\left\{1, |\alpha_{k,l}|^H\right\}$. Hence $\alpha x \in w^{I_2}_{\theta_{r,s}}[M_1,p,\|.,...,.\|]_o$ for all double sequences $\alpha = (\alpha_{k,l})$ with $|\alpha_{k,l}| \leq 1$ for all $k,l \in \mathbb{N}$ whenever $x \in w^{I_2}_{\theta_{r,s}}[M_1,p,\|.,...,.\|]_o$. □

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