AMO-Advanced Modeling and Optimization, Volume 14, Number 1, 2012

An Algorithm for Solving an Integer Linear Fractional / Quadratic Bilevel Programming Problem

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Abstract

In this paper, an integer bilevel programming problem is considered in which the upper level objective function is linear fractional and the lower level objective function is quadratic. The variables at both the levels are related by the set of linear constraints. An algorithm is developed to find the integer solution for the bilevel programming problem. Applying the Kuhn-Tucker conditions at the lower level, the bilevel programming problem is converted to a linear fractional programming problem with complementarity constraints. Gomory's cut is applied to find an integer solution of the linear fractional programming problem which satisfies the complementarity constraints and hence determines the optimum integer solution of the given bilevel programming problem. A computational approach is also given to solve the above problem by converting the integer variables to 0–1 variables. The method is illustrated with the help of an example.

Keywords: Fractional programming, Quadratic programming, Integer programming, Bilevel programming, Kuhn-Tucker conditions Gomory's cut, LINDO (6.1)

Classification No.

Primary 90C10; 90C20

Secondary 90C32

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* **AMO – Advanced Modeling and Optimization. ISSN: 1841-4311**

Introduction

The Bilevel programming problem (BLPP) is defined as

(BLPP) $\underset{X}{\text{Max}}\, f(X, Y)$

where Y solves

$\underset{Y}{\text{Max}}\, F(X, Y)$

subject to $(X, Y) \in S$,

where $S = \{(X, Y) : AX + BY \leq b;\ X, Y \geq 0\}$.

In (BLPP), two decision makers are located at two different hierarchical levels, upper level and lower level, each controlling independently a separate set of decision variables. Both the levels are interested in optimizing their own objectives. The objectives at each level are conflicting in nature.

(BLPP) has been used by researchers in several fields ranging from economics to transportation engineering. (BLPP) is used to model problems involving multiple decision makers. These problems include traffic signal optimization [17], structural design [15] and genetic algorithms [13]. (BLPP) has been developed and studied by Bialas and Karwan [7,8] in the year 1982, 1984; Candler and Townsley [10] in 1982; Bard [3,4,5] in the year 1983, 84, 92 developed different techniques for solving (BLPP).

Linear Fractional programming problem is studied by many authors [2,7,8,11]. Charnes.et.al. formulated linear fractional programming problem to a linear programming problem by applying the transformation. A Quadratic Fractional Programming Problem has been worked upon earlier also by many authors. Cabot et.al. [9] in the year 1970 solved the non-convex quadratic minimization problem by ranking the extreme points. R.Gupta and M.C. Puri [12] in 1994 have developed an algorithm for ranking the extreme points of Quadratic Fractional Programming Problem.

Most applications of bilevel programming that have appeared in the literature, deal with central economic planning at the regional level. In this context, the government is considered as the leader who controls a set of policy variables such as tax rates, import quotas and price supports. The particular industry targeted for regulation is viewed as the follower. In most cases, the follower tries to maximize net income subject to the prevailing technological, economic and government constraints. Fractional programs arise in various circumstances in management science as well as other areas. Maximization of productivity, maximization of return on investment, maximization of cost/time give rise to a fractional program.

Based on the Kuhn-Tucker conditions and the duality theory, Wang et al. [19] has derived necessary and sufficient optimality conditions for linear-quadratic bilevel programs. A parametric method for solving bilevel programming problem has been discussed by Faisca, Dua, Rustem, Saraiva and Pistikopoulos [14].

Here, in this paper we have taken linear fractional function at the upper level and quadratic function at the lower level. An algorithm is developed in which the lower level problem using the Kuhn-Tucker conditions is combined with the upper level problem to form a fractional programming problem with complementarity conditions. Then,using Gomory's cut an integer solution of the bilevel programming problem is obtained.

Mathematical Formulation

The linear fractional / quadratic integer bilevel programming problem (LFQIBPP) is given by

(LFQIBPP) $\underset{X_1}{\text{Max}}\, Z_1(X) = \dfrac{c_1X_1 + c_2X_2 + \alpha}{d_1X_1 + d_2X_2 + \beta}$

where for a given X_1, X_2 solves

$$\underset{X_2}{\text{Max}}\, Z_2(X) = e^T X + \frac{1}{2} X^T Q X$$

subject to $X \in S$,

where $S = \{X = (X_1, X_2) \in \mathbb{R}^{n_1+n_2} : A_1X_1 + A_2X_2 \le b,\ X_1, X_2 \ge 0$ and integers$\}$.

$c_1, d_1 \in \mathbb{R}^{n_1};\ c_2, d_2 \in \mathbb{R}^{n_2};\ \alpha, \beta \in \mathbb{R};\ A_1 \in \mathbb{R}^{m\times n_1};\ A_2 \in \mathbb{R}^{m\times n_2};\ b \in \mathbb{R}^m;$

$e = (e_1, e_2) \in \mathbb{R}^{n_1+n_2}$.

Q is an $((n_1 + n_2) \times (n_1 + n_2))$ symmetric positive semi-definite matrix. Here, $S \subset \mathbb{R}^{n_1+n_2}$ defines the common constraint region and it is assumed that the feasible region S is closed and bounded.

It is also assumed that $(d_1X_1 + d_2X_2 + \beta) > 0$, $\forall\ (X_1, X_2) \in S$. The feasible region of the lower level problem, for a given X_1, is defined as

$$S(X_1) = \{X_2 \in \mathbb{R}^{n_2} : (X_1, X_2) \in S\}.$$

Relaxed Problem

Consider the relaxed problem of (LFQIBPP), in which the integral condition is not considered.

Define the relaxed problem (RLFQBPP) as

(RLFQBPP) $\underset{X_1}{\text{Max}}\, Z_1(X) = \dfrac{c_1X_1 + c_2X_2 + \alpha}{d_1X_1 + d_2X_2 + \beta}$

where for a given X_1, X_2 solves

$$\underset{X_2}{\text{Max}}\, Z_2(X) = e^T X + \frac{1}{2} X^T Q X$$

subject to $X \in S'$,

where $S' = \{X = (X_1, X_2) \in \mathbb{R}^{n_1+n_2} : A_1X_1 + A_2X_2 \le b;\ X_1, X_2 \ge 0\}$.

Consider the lower level problem of (RLFQBPP) for a given X_1,

$$\underset{X_2}{\text{Max}}\, Z_2(X) = e^T X + \frac{1}{2} X^T Q X$$

$$= e_1^T X_1 + e_2^T X_2 + \frac{1}{2}[X_1 \quad X_2]^T \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

subject to $A_2 X_2 \le b - A_1 X_1$

$X_2 \ge 0.$

Here , $f(X) = e_1^T X_1 + e_2^T X_2 + \frac{1}{2} X_1^T Q_{11} X_1 + X_1^T Q_{12} X_2 + \frac{1}{2} X_2^T Q_{22} X_2$

$g(X) = b - A_2 X_1 - A_2 X_2 \ge 0$.

Define the Lagrangian function $L(X, \lambda)$ as

$$L(X, \lambda) = f(X) + \lambda^T g(X)$$

where $\lambda \ge 0$ is a vector of Lagrange's multipliers.

Applying the Kuhn-Tucker conditions, we have

$$\frac{\partial L}{\partial X_2} \le 0 \Rightarrow e_2 + X_1^T Q_{12} + Q_{22} X_2 - A_2^T \lambda \le 0 \qquad (1)$$

$$\frac{\partial L}{\partial \lambda} \ge 0 \Rightarrow g(X) \ge 0 \Rightarrow b - A_1 X - A_2 X_2 \ge 0 \qquad (2)$$

$$X_2^T \frac{\partial L}{\partial X_2} = 0 \Rightarrow X_2^T (e_2 + X_1^T Q_{12} + Q_{22} X_2 - A_2^T \lambda) = 0 \qquad (3)$$

$$\lambda^T \frac{\partial L}{\partial \lambda} = 0 \Rightarrow \lambda^T (b - A_1 X_1 - A_2 X_2) = 0 \qquad (4)$$

In equation (2), introducing the surplus variable, we get

$$b - A_1X_1 - A_2X_2 - Iy = 0$$

or $$A_1X_1 + A_2X_2 + Iy = b \qquad (5)$$

$$y \geq 0$$

From equations (4) and(5)

$$\lambda^T y = 0 \qquad (6)$$

In equation (1), introducing the slack variable, we get

$$e_2 + X_1^T Q_{12} + Q_{22}X_2 - A_2^T\lambda + Iu = 0$$

$$\Rightarrow \quad -X_1^T Q_{12} + Q_{22}X_2 + A_2^T\lambda - Iu = e_2 \qquad (7)$$

$$u \geq 0$$

Using equations (3) and(7)

$$X_2^T u = 0 \qquad (8)$$

Thus, the given (RLFQBPP) problem reduces to a linear fractional programming problem (RLFPP), given by

(RLFPP) $$\underset{X_1,X_2}{\text{Max}}\, Z_1(X) = \frac{c_1X_1 + c_2X_2 + \alpha}{d_1X_1 + d_2X_2 + \beta}$$

subject to $A_1X_1 + A_2X_2 + Iy = b$

$$-X_1^T Q_{12} - Q_{22}X_2 + A_2^T\lambda - Iu = e_2 \qquad (9)$$

$$X_1, X_2, \lambda, y, u \geq 0$$

with the condition $\lambda^T y = 0$ and $X_2^T u = 0$.

Here, I is the identity matrix of appropriate dimension and the condition $\lambda^T y = 0$ and $X_2^T u = 0$ represents the complementarity condition.

The above problem (RLFPP) is a relaxed problem in which the integrality condition is not considered. If we impose the integer restriction on the above problem, it becomes an integer linear fractional programming problem (ILFPP), defined as,

(ILFPP) $$\max_{X_1, X_2} Z_1(X) = \frac{c_1X_1 + c_2X_2 + \alpha}{d_1X_1 + d_2X_2 + \beta}$$

subject to $A_1X_1 + A_2X_2 + Iy = b$ (10)

$$-X_1^TQ_{12} - Q_{22}X_2 + A_2^T\lambda - Iu = e_2$$

λ, y, u ≥ 0

X_1, $X_2 \geq 0$ and integers,

with the condition that $\lambda^T y = 0$ and $X_2^T u = 0$.

The problem (RLFPP) without the complementarity condition is a linear fractional programming problem whose optimal solution will be at an extreme point. We are interested in finding an integer solution of (RLFPP) which satisfies the complementarity conditions. It will be the solution of (ILFPP) and hence that of the given bilevel programming problem.

(RLFPP) problem can be rewritten as

$$\max_{X_1, X_2} Z_1(X) = \frac{z^1}{z^2} = \frac{c_1^1x_1^1 + c_1^2x_1^2 + \ldots + c_1^{n_1}x_1^{n_1} + c_2^1x_2^1 + \ldots + c_2^{n_2}x_2^{n_2} + \alpha}{d_1^1x_1^1 + d_1^2x_1^2 + \ldots + d_1^{n_1}x_1^{n_1} + d_2^1x_2^1 + \ldots + d_2^{n_2}x_2^{n_2} + \beta}$$

subject to

$$a_{11}^1x_1^1 + a_{12}^1x_1^2 + \ldots + a_{1,n_1}^1x_1^{n_1} + a_{11}^2x_2^1 + a_{12}^2x_{12}^2 + \ldots + a_{1,n_2}^2x_2^{n_2} + y_1 = b_1$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$a_{m1}^1x_1^1 + a_{m2}^1x_1^2 + \ldots + a_{m,n_1}^1x_1^{n_1} + a_{m1}^2x_2^1 + a_{m2}^2x_2^2 + \ldots + a_{m,n_2}^2x_2^{n_2} + y_m = b_m$$

$$-q_{12}^{11}x_1^1 - q_{12}^{12}x_1^2 - \ldots - q_{12}^{1,n_2}x_1^{n_1} - q_{22}^{11}x_2^1 - q_{22}^{12}x_2^2 - \ldots - q_{22}^{1,n_2}x^{n_2} + a_{11}^2\lambda_1 + \ldots + a_{m_1}^2\lambda_m + u_1 = e_2^1$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$-q_{12}^{n_1,1}x_1^1 - q_{12}^{n_1,2}x_1^2 - \ldots - q_{12}^{n_1,n_2}x_1^{n_1} - q_{22}^{n_2,1}x_2^1 - \ldots - q_{22}^{n_1,n_2}x_2^{n_2} + a_{1,n_2}^2\lambda_1 + \ldots + a_{m_1,n_2}^2\lambda_m + u_{n_2} = e_2^{n_2}$$

X_1, X_2, λ, y, u ≥ 0

with the condition $\lambda^T y = 0$ and $X_2^T u = 0$, where $c_1=(c_1^1,c_1^2,....,c_1^{n_1})\in\mathbb{R}^{n_1}$; $c_2=(c_2^1,c_2^2,....,c_2^{n_2})\in\mathbb{R}^{n_2}$;

$d_1=(d_1^1,d_1^2,....,d_1^{n_1})\in\mathbb{R}^{n_1}$; $d_2=(d_2^1,d_2^2,....,d_2^{n_2})\in\mathbb{R}^{n_2}$;

$X_1=(x_1^1,x_1^2,....,x_1^{n_1})\in\mathbb{R}^{n_1}$; $X_2=(x_2^1,x_2^2,....,x_2^{n_2})\in\mathbb{R}^{n_2}$

$y=(y_1,....,y_m)\in\mathbb{R}^m$ $\lambda=(\lambda_1,....,\lambda_m)\in\mathbb{R}^m$;

$b=(b_1,....,b_m)\in\mathbb{R}^m$; $u=(u_1,....,u_{n_2})\in\mathbb{R}^{n_2}$;

$e_2=(e_2^1,....,e_2^{n_2})\in\mathbb{R}^{n_2}$.

$$A_1=\begin{bmatrix} a_{11}^1 & a_{12}^1 & .... & a_{1,n_1}^1 \\ ... & ... & ... & ... \\ a_{m_1}^1 & a_{m_2}^1 & ... & a_{m,n_1}^1 \end{bmatrix}\in\mathbb{R}^{m\times n_1},\qquad A_2=\begin{bmatrix} a_{11}^2 & a_{12}^2 & .... & a_{1,n_2}^2 \\ ... & ... & ... & ... \\ a_{m_1}^2 & a_{m_2}^2 & ... & a_{m,n_2}^2 \end{bmatrix}\in\mathbb{R}^{m\times n_2}$$

$$Q_{12}=\begin{bmatrix} q_{12}^{11} & .... & q_{12}^{1,n_2} \\ ... & ... & ... \\ q_{12}^{n_1,1} & ... & q_{12}^{n_1,n_2} \end{bmatrix}\in\mathbb{R}^{n_1\times n_2},\ Q_{22}=\begin{bmatrix} q_{22}^{11} & .... & q_{22}^{1,n_2} \\ ... & ... & ... \\ q_{22}^{n_2,1} & ... & q_{22}^{n_2,n_2} \end{bmatrix}\in\mathbb{R}^{n_2\times n_2}$$

The optimal solution of (RLFPP) problem can be expressed in the form of

$$\text{Max } Z_1(X)=\frac{\bar{c}_0+\sum_{j\in N_B}\bar{c}_j x_j}{\bar{d}_0+\sum_{j\in N_B}\bar{d}_j x_j}$$

subject to $x_i = x_i^* - \sum_{j\in N_B}\alpha_{ij}x_j,\ \ j\in N_B,\ i\in B$

x_i, x_j ≥ 0 and integers ,$\forall$ $i\in$ B, $j\in$ N_B,

where B is the set of basic variables and N_B is the set of non-basic variables,

$\bar{c}_j = (z_j^1 - c_j)$ is the reduced cost of the numerator,

$\bar{d}_j = (z_j^2 - d_j)$ is the reduced cost of the denominator,

$\Delta_j = z^1(z_j^2 - d_j) - z^2(z_j^1 - c_j)$ is the total reduced cost.

At the optimality, all $\Delta_j \leq 0, \quad j \in N_B$.

The solution for (RLFPP) is given by

$$x_i = x_i^*, \qquad x_j = 0, \qquad \text{Max } Z_1(X) = \frac{\bar{c}_0}{\bar{d}_0}.$$

Let the current optimal solution of (RLFPP) does not satisfy the integrality condition. Let the basic variable x_k be not an integer in the current optimal solution which is otherwise required to be an integer.

Let the equation of x_k be given by

$$x_k = x_k^* - \sum_{j \in N_B} \alpha_{kj} x_j$$

This implies

$$x_k = [x_k^*] + f_k - \sum_{j \in N_B} \{[\alpha_{kj}] + f_{kj}\} x_j$$

or $$x_k - [x_k^*] + \sum_{j \in N_B} [\alpha_{kj}] x_j = f_k - \sum_{j \in N_B} f_{kj} x_j$$

where $[x_k^*]$ is the integral part and f_k is the fractional part of x_k, where x_k is the basic variable. $[\alpha_{kj}]$ is the integral part and f_{kj} is the fractional part of α_{kj}.

The necessary condition for the variable x_k to be integral is that

$$f_k - \sum_{j \in N_B} f_{kj} x_j \equiv 0 \pmod 1$$

But $$f_k - \sum_{j \in N_B} f_{kj} x_j < f_k < 1$$

Therefore, the necessary condition for the integrability becomes

$$f_k - \sum_{j \in N_B} f_{kj} x_j \leq 0$$

or $$s - \sum_{j \in N_B} f_{kj} x_j = -f_k$$

This is the Gomory's cut, where s is a slack variable. This is the required cut which should be applied to the optimal table of (RLFPP) problem.

Since all $x_j = 0$, $j \in N_B$, it follows that $s = -f_k$ which is infeasible. This means that after applying the cut, the solution becomes optimal but is not feasible.

To solve the above table, find the departing variable according as

$$\bar{b}_r = \underset{i}{\mathrm{Min}}\{\bar{b}_i, \bar{b}_i < 0\}.$$

Choose the entering variable according as

$$\mathrm{Max}\left\{\left|\frac{\Delta_j}{y_{rj}}\right| : y_{rj} < 0\right\}.$$

Proceed with this process till all $\bar{b}_i \geq 0$.

If the resulting optimum solution is an integer, the process ends. Otherwise, the process is repeated. After getting the integer solution, check whether the complementarity conditions $X_2^T u = 0$ and $\lambda^T y = 0$ are also satisfied. If the complementarity conditions are not satisfied, enter that variable into the basis such that $X_2^T u = 0$ and $\lambda^T y = 0$ is satisfied. The optimal integer solution so obtained will be the solution of the bilevel programming problem.

Algorithm for solving linear fractional / quadratic integer bilevel programming problem

Step 1 Consider the problem (LFQIBPP) defined as

$$\underset{X_1}{\text{Max}}\, Z_1(X) = \frac{c_1X_1 + c_2X_2 + \alpha}{d_1X_1 + d_2X_2 + \beta}$$

where for a given X_1, X_2 solves

$$\underset{X_2}{\text{Max}}\, Z_2(X) = e^T X + \frac{1}{2} X^T Q X$$

subject to $A_1X_1 + A_2X_2 \leq b$

$X_1, X_2 \geq 0$ and integers.

Step 2 Define the relaxed problem (RLFQBPP) of (LFQIBPP) problem, without considering the integer condition.

Step 3 Consider the follower's problem for a given value of X_1.

Let $f(X) = e_1^T X_1 + e_2^T X_2 + \frac{1}{2} X_1^T Q_{11} X_1 + X_1^T Q_{12} X_2 + \frac{1}{2} X_2^T Q_{22} X_2$

and $g(X) = b - A_1X_1 - A_2X_2 \geq 0.$

Define the Lagrangian function $L(X, \lambda) = f(X) + \lambda^T g(X)$.

Apply the Kuhn Tucker conditions and convert (RLFQBPP) problem to (RLFPP) problem with the condition that $\lambda^T y = 0$ and $X_2^T u = 0$.

Step 4 Let (X_1^*, X_2^*) be an optimal solution for the problem (RLFPP).

If (X_1^*, X_2^*) is an integer solution, go to step 6, else, go to step 5.

Step 5 Apply Gomory's cut to the optimal table of (RLFPP) problem to find an integer solution. If it is an integer solution go to step 6, otherwise repeat step 5.

Step 6 Check the complementarity conditions $\lambda^T y = 0$ and $X_2^T u = 0$ for the integer solution. If the integer solution satisfies the complementarity conditions, then this is the optimum integer solution of (LFQIBPP) problem. Otherwise, find the next integer solution which satisfies $\lambda^T y = 0$ and $X_2^T u = 0$.

Example 1: Consider a linear fractional/ quadratic integer bilevel programming problem.

(LFQIBPP) $$\max_{x_1} Z_1(x_1, x_2, x_3) = \frac{2 + x_1 - 2x_2 - 2x_3}{3 + x_1 + x_3}$$

where (x_2, x_3) solves

$$\max_{x_2, x_3} Z_2(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 + 2x_1x_2 + 2x_2x_3 - 2x_1 - 7x_2 - 6x_3$$

subject to
$$x_1 + 2x_2 + x_3 \le 10$$
$$x_1 + x_3 \le 2$$
$$3x_1 + x_2 \le 4$$
$$x_1, x_2, x_3 \ge 0 \text{ and integers.}$$

Solution: Consider the relaxed problem for (LFQIBPP), defined as

(RLFQBPP) $\underset{x_1}{\text{Max}}\, Z_1(x_1, x_2, x_3) = \dfrac{2 + x_1 - 2x_2 - 2x_3}{3 + x_1 + x_3}$

where (x_2, x_3) solves

$$\underset{x_2, x_3}{\text{Max}}\, Z_2(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 + 2x_1x_2 + 2x_2x_3 - 2x_1 - 7x_2 - 6x_3$$

subject to $x_1 + 2x_2 + x_3 \le 10$

$x_1 + x_3 \le 2$

$3x_1 + x_2 \le 4$

$x_1, x_2, x_3 \ge 0$.

Define the Lagrangian function as

$L(x_1, x_2, x_3, \lambda) = f(x_1, x_2, x_3) + \lambda^T g(x_1, x_2, x_3)$

$$= (x_1^2 + x_2^2 + x_3^2 + 2x_1x_2 + 2x_2x_3 - 2x_1 - 7x_2 - 6x_3) \\ + \lambda_1(10 - x_1 - 2x_2 - x_3) + \lambda_3(2 - x_1 - x_3) + \lambda_3(4 - 3x_1 - x_2)$$

Applying the Kuhn-Tucker conditions, we have

$$\frac{\partial L}{\partial x_2} \le 0 \Rightarrow \quad 2x_2 + 2x_1 + 2x_3 - 7 - 2\lambda_1 - \lambda_3 \le 0$$

$$\frac{\partial L}{\partial x_3} \le 0 \Rightarrow \quad 2x_3 + 2x_2 - 6 - \lambda_1 - \lambda_2 \le 0$$

$$\frac{\partial L}{\partial \lambda_1} \ge 0 \Rightarrow \quad 10 - x_1 - 2x_2 - x_3 \ge 0 \tag{11}$$

$$\frac{\partial L}{\partial \lambda_2} \ge 0 \Rightarrow \quad 2 - x_1 - x_3 \ge 0$$

$$\frac{\partial L}{\partial \lambda_3} \ge 0 \Rightarrow \quad 4 - 3x_1 - x_2 \ge 0$$

$$x_2 \frac{\partial L}{\partial x_2} = 0 \Rightarrow x_2(2x_1 + 2x_2 + 2x_3 - 2\lambda_1 - \lambda_3 - 7) = 0$$

$$x_3 \frac{\partial L}{\partial x_3} = 0 \Rightarrow x_3(2x_2 + 2x_3 - \lambda_1 - \lambda_2 - 6) = 0$$

$$\lambda_1^T \frac{\partial L}{\partial \lambda_1} = 0 \Rightarrow \lambda_1(10 - x_1 - 2x_2 - x_3) = 0 \qquad (12)$$

$$\lambda_2^T \frac{\partial L}{\partial \lambda_2} = 0 \Rightarrow \lambda_2(2 - x_1 - x_3) = 0$$

$$\lambda_3^T \frac{\partial L}{\partial \lambda_3} = 0 \Rightarrow \lambda_3(4 - 3x_1 - x_2) = 0$$

Using (11) and (12), formulate the problem (RLFPP),

(RLFPP) $\underset{x_1,x_2,x_3}{\text{Max}}\ Z_1(X) = \dfrac{2 + x_1 - 2x_2 - 2x_3}{3 + x_1 + x_3}$

subject to $x_1 + 2x_2 + x_3 + y_1 = 10$

$x_1 + x_3 + y_2 = 2$

$3x_1 + x_2 + y_3 = 4$

$2x_1 + 2x_2 + 2x_3 - 2\lambda_1 - \lambda_3 + u_1 = 7$

$2x_2 + 2x_3 - \lambda_1 - \lambda_2 + u_2 = 6$

$x_2u_1 = 0,\ x_3u_2 = 0,\ \lambda_1y_1 = 0,\ \lambda_2y_2 = 0,\ \lambda_3y_3 = 0$

$x_1, x_2, x_3, \lambda_1, \lambda_2, \lambda_3, y_1, y_2, y_3, u_1, u_2 \geq 0$

Solving the above problem, the optimal table obtained as

			$c_j \to$	1	–2	–2	0	0	0	0	0	0	0	0
			$d_j \to$	1	0	1	0	0	0	0	0	0	0	0
C_B	D_B	V_B	X_B	x_1	x_2	x_3	y_1	y_2	y_3	λ_1	λ_2	λ_3	u_1	u_2
0	0	y_1	26/3	0	5/3	1	1	0	–1/3	0	0	0	0	0
0	0	y_2	2/3	0	–1/3	1	0	1	–1/3	0	0	0	0	0
1	1	x_1	4/3	1	1/3	0	0	0	1/3	0	0	0	0	0
0	0	u_1	13/3	0	4/3	2	0	0	–2/3	–2	0	–1	1	0
0	0	u_2	6	0	2	2	0	0	0	–1	–1	0	0	1
z_1=10/3			$z_j^1 - c_j \to$	0	7/3	2	0	0	1/3	0	0	0	0	0
z_2=13/3			$z_j^2 - d_j \to$	0	1/3	–1	0	0	1/3	0	0	0	0	0
			$\Delta_j \to$	0	–9	–12	0	0	–1/3	0	0	0	0	0

Since $\Delta_j \le 0$, therefore, it is an optimal solution but not an integer solution.

Applying the cut on equation x_1, we have

$$\frac{4}{3} = x_1 + \frac{1}{3}x_2 + \frac{1}{3}y_3$$

$$\therefore \quad \frac{1}{3}x_2 + \frac{1}{3}y_3 \ge \frac{1}{3} \quad \text{or} \quad \frac{1}{3}x_2 + \frac{1}{3}y_3 - s_1 = \frac{1}{3}$$

Applying the cut in the above optimal table and solving it, the final table so obtained is

			$c_j \to$	1	–2	–2	0	0	0	0	0	0	0	0	0
			$d_j \to$	1	0	1	0	0	0	0	0	0	0	0	0
C_B	D_B	V_B	X_B	x_1	x_2	x_3	y_1	y_2	y_3	λ_1	λ_2	λ_3	u_1	u_2	s_1
0	0	y_1	9	0	2	1	1	0	0	0	0	0	0	0	–1
0	0	y_2	1	0	0	0	0	1	0	0	0	0	0	0	–1
1	1	x_1	1	1	0	0	0	0	0	0	0	0	0	0	1
0	0	u_1	5	0	2	2	0	0	0	–2	0	–1	1	0	–2
0	0	u_2	6	0	2	2	0	0	0	–1	–1	0	0	1	0
0	0	y_3	1	0	1	0	0	0	1	0	0	0	0	0	–3
z_1=3		$z_j^1 - c_j \to$		0	2	2	0	0	0	0	0	0	0	0	1
z_2=4		$z_j^2 - d_j \to$		0	0	–1	0	0	0	0	0	0	0	0	1
		$\Delta_j \to$		0	–8	–11	0	0	0	0	0	0	0	0	–1

Here, $\Delta_j \le 0$. $x_1 = 1$, $x_2 = 0$, $x_3 = 0$.

Also, $x_2u_1 = 0$, $x_3u_2 = 0$, $\lambda_1 y_1 = 0$, $y_2\lambda_2 = 0$, $\lambda_3 y_3 = 0$.

This is an optimal integer solution for the bilevel programming problem (LFQIBPP), with $Z_1 = 3/4$ and $Z_2 = 1$.

An Alternative Approach for Solving Linear Fractional /Quadratic Integer Bilevel Programming Problem

Consider the problem (LFQIBPP) as

(LFQIBPP) $$\max_{X_1} Z_1(X) = \frac{c_1X_1 + c_2X_2 + \alpha}{d_1X_1 + d_2X_2 + \beta}$$

where for a given X_1, X_2 solves

$$\max_{X_2} Z_2(X) = e^T X + \frac{1}{2} X^T Q X$$

$$= e_1^T X_1 + e_2^T X_2 + \frac{1}{2} X_1^T Q_{11} X_1 + X_1^T Q_{12} X_2 + \frac{1}{2} X_2^T Q_{22} X_2$$

subject to $X \in S$,

$$S = \{X = (X_1, X_2) \in \mathbb{R}^{n_1+n_2} : A_1X_1 + A_2X_2 \le b;\ X_1, X_2 \ge 0 \text{ and integers}\}.$$

The variables in this problem are integers only, they can be converted in the form of 0 or 1 by applying the following transformations.

Replace the integer variables x_k according to the relation

$$x_k = \sum_{n=1}^{N_k} 2^{n-1} y_n \tag{13}$$

where y_n = 0 or 1, and N_k is determined according to the upper limit on the variable x_k.

The 0 – 1 bilevel programming problem so obtained is

(0 – 1 LFQBPP) $\quad \text{Max } Z_1(Y) = \dfrac{g_1 Y_1 + g_2 Y_2 + \gamma}{h_1 Y_1 + h_2 Y_2 + \delta}$

$$\text{Max } Z_2(Y) = p^T Y + \frac{1}{2} Y^T R Y \tag{14}$$

subject to $B_1 Y_1 + B_2 Y_2 \leq b$

$Y_1, Y_2 \in \{0, 1\}$.

Solution Procedure for solving (0 – 1 LFQBPP) Problem

In (0–1 LFQBPP) problem, the follower's problem is a quadratic problem which can be converted to a linear program [18], by making the following conversions,

(1) Any zero-one variable that has a positive exponent is replaced by that variable to the power one.

(2) Any product of 0–1 variables may be changed to a linear 0 – 1 function. Replace the product $y_k y_\ell$ by introducing a 0 – 1 variable $u_{k\ell} = y_k y_\ell$, and adding the following constraints in the constraint set (14)

$y_k + y_\ell - u_{k\ell} \le 1$

$-y_k - y_\ell + 2u_{k\ell} \le 0$ (15)

where $u_{k\ell} \in \{0,1\}$

Based on the above results (0–1) linear fractional quadratic bilevel programming problem can be converted to (0–1) linear bilevel programming problem which can be solved using LINDO (6.1).

Example 2 : Consider the linear fractional quadratic integer bilevel programming problem (LFQIBPP) as in example 1.

The variables considered were integer variables. They are converted to 0–1 variables by the following conversions,

$$x_1 = y_1$$

$$x_2 = y_2 + 2y_3 + 4y_4 \quad (16)$$

$$x_3 = y_5 + 2y_6$$

(LFQIBPP) problem reduces to (0–1 LFQBPP) problem, defined as

(0–1 LFQBPP) $$\max_{y_1} Z_1(Y) = \frac{2 + y_1 - 2y_2 - 8y_3 - 8y_4 - 2y_5 - 4y_6}{3 + y_1 + y_5 + 2y_6}$$

where (y_2, y_3, y_4, y_5,y_6) solves

$$\max_{y_2,\ldots,y_6} Z_2 = y_1^2 + y_2^2 + 4y_3^2 + 16y_4^2 + y_5^2 + 4y_6^2 + 2y_1y_2 + 4y_1y_3$$
$$+8y_1y_4 + 4y_2y_3 + 8y_2y_4 + 2y_2y_5 + 4y_2y_6 + 16y_3y_4$$
$$+4y_3y_5 + 8y_3y_6 + 8y_4y_5 + 16y_4y_6 + 4y_5y_6 - 2y_1$$
$$-7y_2 - 14y_3 - 28y_4 - 6y_5 - 12y_6$$

subject to

$$\begin{aligned} & y_1 + 2y_2 + 4y_3 + 8y_4 + y_5 + 2y_6 \le 10 \\ & y_1 + y_5 + 2y_6 \le 2 \\ & 3y_1 + y_2 + 2y_3 + 4y_4 \le 4 \\ & y_1, y_2, y_3, y_4, y_5, y_6 \in \{0, 1\}. \end{aligned} \tag{17}$$

Solving the leader's problem subject to the constraints (17) by LINDO (6.1), we get $y_1 = 1$, $y_2 = 0$, $y_3 = 0$, $y_4 = 0$, $y_5 = 0$, $y_6 = 0$. Put $y_1 = 1$ in the follower's problem.

Use conversions $y_i^2 = y_i, i = 2,3,4,5,6$ and $y_k y_\ell = u_{k\ell}$ ($k \ne \ell$, $k, \ell = 2, 3, 4, 5, 6$).

Introduce the constraints

$$\begin{aligned} & y_k + y_\ell - u_{k\ell} \le 1, \quad k, \ell = 2, 3, 4, 5, 6 \\ & -y_k - y_\ell + 2u_{k\ell} \le 0 \end{aligned} \tag{18}$$

Solving the follower's problem for $y_1 = 1$, subject to the constraints (17) and (18), using LINDO (6.1), we get $y_2 = 0$, $y_3 = 0$, $y_4 = 0$, $y_5 = 0$, $y_6 = 0$

Put $y_1 = 1$, $y_2 = 0$, $y_3 = 0$, $y_4 = 0$, $y_5 = 0$, $y_6 = 0$, in (16), we get

$x_1 = 1$, $x_2 = 0$ and $x_3 = 0$.

Thus, this is an alternate approach for finding the integer solution of Linear Fractional /Quadratic Integer Bilevel Programming Problem.

Conclusions

In this paper, the linear Fractional/ Quadratic Bilevel Programming Problem is converted to a linear fractional programming problem with complementarity constraints using the Kuhn- Tucker conditions. An integer solution of this problem is obtained by applying the Gomory's cut. An alternate approach for solving this problem using LINDO(6.1) is also given by converting the integer variables to 0-1 variables.

Acknowledgements

The authors are thankful to the referees' for their valuable suggestions and guidance in improving this paper.

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