Reduction of discrete interval systems based on pole clustering and improved Padé approximation: a computer-aided approach

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***Abstract*:** This paper presents a computer aided method for model reduction of discrete interval systems in which the denominator of the approximant is obtained by pole-clustering method while the numerator is derived by matching the time moments as well as Markov parameters of high-order interval system (HOIS) to those of its approximant. The time moments and Markov parameters for discrete interval systems are derived and presented in generalized forms. The proposed method is substantiated by a worked example.

***Index Terms*:** Discrete interval system, Kharitonov Polynomials, Model order reduction, Padé approximation, Pole-clustering method.

1. Introduction

Since the simplification of high-order systems plays an important role in many engineering problems, the model reduction has received considerable attention of researchers. Various techniques [1]-[12] have been proposed for model reduction of continuous-time as well as discrete-time systems. Several methods based on power series expansion of the system transfer function for order reduction of high-order systems are available in the literature. One of these methods, namely, Padé approximation [1] has found to be very useful in theoretical physical research [13]-[14] due to being computationally simple. But the approximant obtained using Padé approximation often leads to be unstable even though the original system is stable. To resolve the problem of stability, Routh approximation method [5] was introduced. However, it is noticed that the Routh approximation does not always lead to good approximant since this method tends to approximate only low frequency behaviour of high-order system [15]. Various attempts have been made to improve the Routh approximation method [6]-[9].

Some of the above methods have been extended through interval arithmetic to derive approximant for high-order interval systems [16]-[22]. Routh-Padé approximation [16] has been presented for model reduction of continuous interval systems where the denominator is obtained by direct truncation of the Routh-table and the numerator is obtained by matching the time moments of the high-order interval system to those of its approximant. In this method, the inversion of denominator of transfer function of the high-order interval system (or approximant) is necessary for obtaining the time moments. Further, a method for obtaining the approximant of high-order discrete interval system is presented in [22] where the denominator of approximant is obtained by retaining dominant poles of the high-order system while the numerator is obtained by matching first r time moments of the high-order interval system to those of the its approximant. In this method, only the time moments are considered for deriving the approximant. However, for better overall time response, both the time moments as well as Markov parameters should be considered as time moments and Markov parameters correspond to steady state and transient state matching, respectively.

In this paper, a method is proposed for the model reduction of high-order discrete interval systems where the denominator is obtained using the pole clustering technique [23]-[25] and the numerator is obtained by matching first r time moments and Markov parameters of high-order interval system to those of its approximant. The time moments and Markov parameters are presented in generalized forms and obtained without inverting the system transfer function. The brief outline of this paper is organized as follows: section-2 covers proposed reduction method, a numerical example is included in section-3 to substantiate the method and finally, this paper is concluded in section-4.

2. Proposed Reduction Method

Consider a stable discrete interval system given by the transfer function

$$G(z)=\frac{[b_0^-,b_0^+]+[b_1^-,b_1^+]z+\cdots+[b_{n-1}^-,b_{n-1}^+]z^{n-1}}{[a_0^-,a_0^+]+[a_1^-,a_1^+]z+\cdots+[a_n^-,a_n^+]z^n} \tag{1}$$

where $[b_i^-,b_i^+]$ for $i=0,1,\cdots,n-1$ are numerator coefficients of $G(z)$ with b_i^- and b_i^+ as lower and upper bounds of interval $[b_i^-,b_i^+]$, respectively, and $[a_i^-,a_i^+]$ for $i=0,1,\cdots,n$ are denominator coefficients of $G(z)$ with a_i^- and a_i^+ as lower and upper bounds of interval $[a_i^-,a_i^+]$, respectively.

The transfer function $G(z)$ can also be written in its power series expansion around $z=1$ and $z=\infty$, given as

$$G(z)=[t_0^-,t_0^+]+[t_1^-,t_1^+](z-1)+\cdots+[t_n^-,t_n^+](z-1)^n+\cdots \tag{2}$$

(expansion around $z=1$)

$$G(z)=[M_1^-,M_1^+]z^{-1}+[M_2^-,M_2^+]z^{-2}+\cdots+[M_n^-,M_n^+]z^{-n}+\cdots \tag{3}$$

(expansion around $z=\infty$)

where $[t_i^-,t_i^+]$ for $i=0,1,\cdots$ are proportional to the time moments of HOIS, and $[M_i^-,M_i^+]$ for $i=1,2,\cdots$ are Markov parameters of HOIS.

Assuming $[b_i^-,b_i^+]=\boldsymbol{b}_i$ for $i=0,1,\cdots,n-1$, $[a_i^-,a_i^+]=\boldsymbol{a}_i$ for $i=0,1,\cdots,n$, $[t_i^-,t_i^+]=\boldsymbol{t}_i$ for $i=0,1,\cdots$ and $[M_i^-,M_i^+]=\boldsymbol{M}_i$ for $i=1,2,\cdots$, (1), (2) and (3) become (4), (5) and (6), respectively.

$$G(z)=\frac{\boldsymbol{b}_0+\boldsymbol{b}_1z+\cdots+\boldsymbol{b}_{n-1}z^{n-1}}{\boldsymbol{a}_0+\boldsymbol{a}_1z+\cdots+\boldsymbol{a}_nz^n} \tag{4}$$

$$=\boldsymbol{t}_0+\boldsymbol{t}_1(z-1)+\cdots+\boldsymbol{t}_n(z-1)^n+\cdots \tag{5}$$

$$=\boldsymbol{M}_1z^{-1}+\boldsymbol{M}_2z^{-2}+\cdots+\boldsymbol{M}_nz^{-n}+\cdots \tag{6}$$

Suppose, it is desired to obtain a stable rth -order ($r<n$) approximant described by the transfer function

$$G_r(z)=\frac{[\hat{b}_0^-,\hat{b}_0^+]+[\hat{b}_1^-,\hat{b}_1^+]z+\cdots+[\hat{b}_{r-1}^-,\hat{b}_{r-1}^+]z^{r-1}}{[\hat{a}_0^-,\hat{a}_0^+]+[\hat{a}_1^-,\hat{a}_1^+]z+\cdots+[\hat{a}_r^-,\hat{a}_r^+]z^r} \tag{7}$$

where $[\hat{b}_i^-,\hat{b}_i^+]$ for $i=0,1,\cdots,r-1$ and $[\hat{a}_i^-,\hat{a}_i^+]$ for $i=0,1,\cdots,r$ are, respectively, numerator and denominator coefficients of $G_r(z)$.

The transfer function $G_r(z)$ can also be written in its power series expansion around $z=1$ and $z=\infty$, given as

$$G_r(z)=[\hat{t}_0^-,\hat{t}_0^+]+[\hat{t}_1^-,\hat{t}_1^+](z-1)+\cdots+[\hat{t}_r^-,\hat{t}_r^+](z-1)^r+\cdots \tag{8}$$

(expansion around $z=1$)

$$G_r(z)=[\hat{M}_1^-,\hat{M}_1^+]z^{-1}+[\hat{M}_2^-,\hat{M}_2^+]z^{-2}+\cdots+[\hat{M}_r^-,\hat{M}_r^+]z^{-r}+\cdots \tag{9}$$

(expansion around $z=\infty$)

where $[\hat{t}_i^-,\hat{t}_i^+]$ for $i=0,1,\cdots$ are proportional to the time moments of approximant, and $[\hat{M}_i^-,\hat{M}_i^+]$ for $i=1,2,\cdots$ are Markov parameters of approximant.

Assuming $[\hat{b}_i^-,\hat{b}_i^+]=\hat{\boldsymbol{b}}_i$ for $i=0,1,\cdots,r-1$, $[\hat{a}_i^-,\hat{a}_i^+]=\hat{\boldsymbol{a}}_i$ for $i=0,1,\cdots,r$, $[\hat{t}_i^-,\hat{t}_i^+]=\hat{\boldsymbol{t}}_i$ for $i=0,1,\cdots$

and $[\hat{M}_i^-, \hat{M}_i^+] = \hat{\boldsymbol{M}}_i$ for $i = 1, 2, \cdots$, (7), (8) and (9) become (10), (11) and (12), respectively.

$$G_r(z) = \frac{\hat{\boldsymbol{b}}_0 + \hat{\boldsymbol{b}}_1 z + \cdots + \hat{\boldsymbol{b}}_{r-1} z^{r-1}}{\hat{\boldsymbol{a}}_0 + \hat{\boldsymbol{a}}_1 z + \cdots + \hat{\boldsymbol{a}}_r z^r} \tag{10}$$

$$= \hat{\boldsymbol{t}}_0 + \hat{\boldsymbol{t}}_1 (z-1) + \cdots + \hat{\boldsymbol{t}}_r (z-1)^r + \cdots \tag{11}$$

$$= \hat{\boldsymbol{M}}_1 z^{-1} + \hat{\boldsymbol{M}}_2 z^{-2} + \cdots + \hat{\boldsymbol{M}}_r z^{-r} + \cdots \tag{12}$$

2.1 Calculation of poles of HOIS:

Consider interval polynomial $D(z)$ given by

$$\left.\begin{aligned} D(z) &= [d_0^-, d_0^+] + [d_1^-, d_1^+] z + \cdots + [d_{n-1}^-, d_{n-1}^+] z^{n-1} + z^n \\ &\equiv \boldsymbol{d}_0 + \boldsymbol{d}_1 z + \cdots + \boldsymbol{d}_{n-1} z^{n-1} + z^n \end{aligned}\right\} \tag{13}$$

The poles [22] of discrete interval polynomial $D(z)$ are calculated as follows:
Let L be the interval matrix given as

$$L = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\boldsymbol{d}_0 & -\boldsymbol{d}_1 & -\boldsymbol{d}_2 & \cdots & -\boldsymbol{d}_{n-1} \end{bmatrix} \tag{14}$$

The interval matrix L can be written as

$$L = \left[L_c - \Delta L, L_c + \Delta L\right] \tag{15}$$

where

$$\left.\begin{aligned} l_{cij} &= \frac{1}{2}\left(l_{ij}^- + l_{ij}^+\right) \\ \Delta l_{cij} &= \frac{1}{2}\left(l_{ij}^+ - l_{ij}^-\right) \end{aligned}\right\} \quad i, j = 1, 2, \cdots, n \tag{16}$$

and l_{ij}^-, l_{ij}^+ are the lower and upper bounds of ijth element of L.

The real part λ_i^σ and imaginary part λ_i^ω of the ith eigenvalue λ_i^h of L are given as

$$\left.\begin{aligned} \lambda_i^\sigma(L) &= \left[\lambda_i^\sigma\left(L_c - \Delta L \circ M^i\right), \lambda_i^\sigma\left(L_c + \Delta L \circ M^i\right)\right] \\ \lambda_i^\omega(L) &= \left[\lambda_i^\omega\left(L_c - \Delta L \circ N^i\right), \; \lambda_i^\omega\left(L_c + \Delta L \circ N^i\right)\right] \\ & \qquad\qquad\qquad\qquad i = 1, 2, \cdots, n \end{aligned}\right\} \tag{17}$$

where $\circ$ denotes component wise multiplication; elements of matrices M^i and N^i are given by

$$\left.\begin{aligned} M_{k,j}^i &= \operatorname{sgn}\left(n_{\sigma k}^i m_{\sigma j}^i + n_{\omega k}^i m_{\omega j}^i\right) \\ N_{k,j}^i &= \operatorname{sgn}\left(n_{\sigma k}^i m_{\omega j}^i - n_{\omega k}^i m_{\sigma j}^i\right) \end{aligned}\right\} \tag{18}$$

where *m* and *n* are, respectively, the ith eigenvector and reciprocal eigenvector of L_c, with σ and ω denoting the real and imaginary parts, respectively.

2.2 Procedure to obtain time moments of HOIS and approximant:

Putting $z = \xi + 1$ in (4), $G(z)$ becomes

$$G(\xi) = \frac{\boldsymbol{b}_0 + \boldsymbol{b}_1(\xi+1) + \cdots + \boldsymbol{b}_{n-1}(\xi+1)^{n-1}}{\boldsymbol{a}_0 + \boldsymbol{a}_1(\xi+1) + \cdots + \boldsymbol{a}_n(\xi+1)^n} \tag{19}$$

$$= \frac{\boldsymbol{B}_0 + \boldsymbol{B}_1\xi + \cdots + \boldsymbol{B}_{n-1}\xi^{n-1}}{\boldsymbol{A}_0 + \boldsymbol{A}_1\xi + \cdots + \boldsymbol{A}_n\xi^n} \tag{20}$$

$G(\xi)$ in (20) can be expanded through interval arithmetic (Appendix I) around $\xi = 0$ as given in (21)-(23):

$$G(\xi) = \boldsymbol{t}_0\left\{1 + \boldsymbol{t}_1\xi\left\{1 + \boldsymbol{t}_2\xi\left\{1 + \boldsymbol{t}_3\xi\left\{1 + \left(\begin{aligned} &\left(\frac{(\boldsymbol{p}_4 - \boldsymbol{q}_4)}{\boldsymbol{t}_1\boldsymbol{t}_2\boldsymbol{t}_3} - \frac{\boldsymbol{q}_3}{\boldsymbol{t}_2\boldsymbol{t}_3} - \frac{\boldsymbol{q}_2}{\boldsymbol{t}_3} - \boldsymbol{q}_1\right)\xi + \cdots + \\ &\left(\frac{(\boldsymbol{p}_{n-1} - \boldsymbol{q}_{n-1})}{\boldsymbol{t}_1\boldsymbol{t}_2\boldsymbol{t}_3} - \frac{\boldsymbol{q}_{n-2}}{\boldsymbol{t}_2\boldsymbol{t}_3} - \frac{\boldsymbol{q}_{n-3}}{\boldsymbol{t}_3} - \boldsymbol{q}_{n-4}\right)\xi^{n-4} \\ &+ \left(\frac{-\boldsymbol{q}_n}{\boldsymbol{t}_1\boldsymbol{t}_2\boldsymbol{t}_3} - \frac{\boldsymbol{q}_{n-1}}{\boldsymbol{t}_2\boldsymbol{t}_3} - \frac{\boldsymbol{q}_{n-2}}{\boldsymbol{t}_3} - \boldsymbol{q}_{n-3}\right)\xi^{n-3} + \\ &\left(\frac{-\boldsymbol{q}_n}{\boldsymbol{t}_2\boldsymbol{t}_3} - \frac{\boldsymbol{q}_{n-1}}{\boldsymbol{t}_3} - \boldsymbol{q}_{n-2}\right)\xi^{n-2} + \\ &\left(\frac{-\boldsymbol{q}_n}{\boldsymbol{t}_3} - \boldsymbol{q}_{n-1}\right)\xi^{n-1} - \boldsymbol{q}_n\xi^n \end{aligned}\right) \div \left(\begin{aligned} &1 + \\ &\boldsymbol{q}_1\xi + \\ &\boldsymbol{q}_2\xi^2 + \\ &\cdots + \\ &\boldsymbol{q}_{n-1}\xi^{n-1} + \\ &\boldsymbol{q}_n\xi^n \end{aligned}\right)\right\}\right\}\right\}\right\} \tag{21}$$

$$= \boldsymbol{t}_0 + \boldsymbol{t}_0\boldsymbol{t}_1\xi + \boldsymbol{t}_0\boldsymbol{t}_1\boldsymbol{t}_2\xi^2 + \cdots + \left(\prod_{i=0}^{n}\boldsymbol{t}_i\right)\xi^n + \cdots \tag{22}$$

$$= \boldsymbol{\zeta}_0 + \boldsymbol{\zeta}_1\xi + \boldsymbol{\zeta}_2\xi^2 + \cdots + \boldsymbol{\zeta}_n\xi^n + \cdots \tag{23}$$

where

$$\boldsymbol{\zeta}_k = \prod_{i=0}^{k}\boldsymbol{t}_i \qquad k = 0, 1, 2, \cdots \tag{24}$$

with

$$\boldsymbol{t}_k = \begin{cases} \boldsymbol{B}_0 / \boldsymbol{A}_0 & k = 0 \\ \boldsymbol{p}_1 - \boldsymbol{q}_1 & k = 1 \\ \left((\boldsymbol{p}_k - \boldsymbol{q}_k) - \sum_{i=1}^{k-1}\boldsymbol{q}_i\left(\prod_{j=1}^{k-i}\boldsymbol{t}_j\right)\right) \div \left(\prod_{l=1}^{k-1}\boldsymbol{t}_l\right) & k \geq 2 \end{cases} \tag{25}$$

where

$$\boldsymbol{p}_i = \begin{cases} \dfrac{\boldsymbol{B}_i}{\boldsymbol{B}_0} & \forall\, i \in [0, n-1] \\ 0 & \forall\, i \geq n \end{cases} \tag{26}$$

and

$$\boldsymbol{q}_i = \begin{cases} \dfrac{\boldsymbol{A}_i}{\boldsymbol{A}_0} & \forall\, i \in [0, n] \\ 0 & \forall\, i \geq n+1 \end{cases} \tag{27}$$

Putting $\xi = z - 1$ in (23), $G(z)$ then can be obtained as follows

$$\begin{aligned} G(z) = {} & \boldsymbol{\zeta}_0 + \boldsymbol{\zeta}_1(z-1) + \boldsymbol{\zeta}_2(z-1)^2 + \cdots \\ & + \boldsymbol{\zeta}_n(z-1)^n + \cdots \end{aligned} \tag{28}$$

Therefore, the time moments of HOIS [26]-[27], in terms of $\boldsymbol{\zeta}_i$ parameters, are

$$\boldsymbol{T}_i = \begin{cases} \boldsymbol{\zeta}_i & i=0 \\ (-1)^i \sum_{j=1}^{i} \frac{1}{j!}(\partial)^j w_{ij}\left(\boldsymbol{\zeta}_j\right) & i=1,2,\cdots \end{cases} \tag{29}$$

where ∂ is the sampling frequency and w_{ij} is defined as

$$w_{ij} = \begin{cases} 1 & j=1 \\ 0 & i<j \\ 1 & i=j \\ w_{i-1,j-1}+j\,w_{i-1,j} & i>j \end{cases} \tag{30}$$

Similarly, time moments of the approximant are given by

$$\hat{\boldsymbol{T}}_i = \begin{cases} \hat{\boldsymbol{\zeta}}_i & i=0 \\ (-1)^i \sum_{j=1}^{i} \frac{1}{j!}(\rho)^j w_{ij}\left(\hat{\boldsymbol{\zeta}}_j\right) & i=1,2,\cdots \end{cases} \tag{31}$$

where ∂ is sampling frequency, w_{ij} is given by (30) and $\hat{\boldsymbol{\zeta}}_i$ for $i=0,1,2,\cdots$ have the same meaning as $\boldsymbol{\zeta}_i$ for $i=0,1,2,\cdots$ in (24) for HOIS.

2.3 Procedure to obtain Markov parameters of HOIS and approximant:

Expanding (4) through interval arithmetic (Appendix I) around $z=\infty$, $G(z)$ becomes

$$G(z)=\boldsymbol{\mu}_1 z^{-1}\left\{1+\boldsymbol{\mu}_2 z^{-1}\left\{1+\boldsymbol{\mu}_3 z^{-1}\left\{1+\boldsymbol{\mu}_4 z^{-1}\left\{\begin{pmatrix} 1+\left(\frac{(\boldsymbol{\hbar}_{n-5}-\boldsymbol{\lambdabar}_{n-4})}{\boldsymbol{\mu}_2\boldsymbol{\mu}_3\boldsymbol{\mu}_4}-\frac{\boldsymbol{\lambdabar}_{n-3}}{\boldsymbol{\mu}_3\boldsymbol{\mu}_4}-\frac{\boldsymbol{\lambdabar}_{n-2}}{\boldsymbol{\mu}_4}\right)z^{-1} \\ +\cdots+ \\ \left(\frac{(\boldsymbol{\hbar}_0-\boldsymbol{\lambdabar}_1)}{\boldsymbol{\mu}_2\boldsymbol{\mu}_3\boldsymbol{\mu}_4}-\frac{\boldsymbol{\lambdabar}_2}{\boldsymbol{\mu}_3\boldsymbol{\mu}_4}-\frac{\boldsymbol{\lambdabar}_3}{\boldsymbol{\mu}_4}\right)z^{-(n-4)}+ \\ \left(\frac{-\boldsymbol{\lambdabar}_0}{\boldsymbol{\mu}_2\boldsymbol{\mu}_3\boldsymbol{\mu}_4}-\frac{\boldsymbol{\lambdabar}_1}{\boldsymbol{\mu}_3\boldsymbol{\mu}_4}-\frac{\boldsymbol{\lambdabar}_2}{\boldsymbol{\mu}_4}\right)z^{-(n-3)}+ \\ \left(\frac{-\boldsymbol{\lambdabar}_0}{\boldsymbol{\mu}_3\boldsymbol{\mu}_4}-\frac{\boldsymbol{\lambdabar}_1}{\boldsymbol{\mu}_4}\right)z^{-(n-2)}+\left(\frac{-\boldsymbol{\lambdabar}_0}{\boldsymbol{\mu}_4}\right)z^{-(n-1)} \end{pmatrix} \div \begin{pmatrix} 1+ \\ \boldsymbol{\lambdabar}_{n-1}z^{-1}+ \\ \boldsymbol{\lambdabar}_{n-2}z^{-2}+ \\ \cdots+ \\ \boldsymbol{\lambdabar}_1 z^{-(n-1)}+ \\ \boldsymbol{\lambdabar}_0 z^{-n} \end{pmatrix}\right\}\right\}\right\}\right\} \tag{32}$$

$$=\boldsymbol{\mu}_1 z^{-1}+\boldsymbol{\mu}_1\boldsymbol{\mu}_2 z^{-2}+\boldsymbol{\mu}_1\boldsymbol{\mu}_2\boldsymbol{\mu}_3 z^{-3}+\cdots+\left(\prod_{i=1}^{n}\boldsymbol{\mu}_i\right)z^{-n}+\cdots \tag{33}$$

$$=\boldsymbol{M}_1 z^{-1}+\boldsymbol{M}_2 z^{-2}+\boldsymbol{M}_3 z^{-3}+\cdots+\boldsymbol{M}_n z^{-n}+\cdots \tag{34}$$

where

$$\boldsymbol{\mu}_k = \begin{cases} \dfrac{\boldsymbol{b}_{n-1}}{\boldsymbol{a}_n} & k=1 \\ \boldsymbol{\hbar}_{n-2}-\boldsymbol{\lambdabar}_{n-1} & k=2 \\ \left((\boldsymbol{\hbar}_{n-k}-\boldsymbol{\lambdabar}_{n-k+1})-\sum_{i=1}^{k-2}\boldsymbol{\lambdabar}_{n-i}\left(\prod_{j=2}^{k-i}\boldsymbol{\mu}_j\right)\right)\div\left(\prod_{l=2}^{k-1}\boldsymbol{\mu}_l\right) & k\geq 3 \end{cases} \tag{35}$$

with

$$\hbar_i = \begin{cases} \dfrac{\boldsymbol{b}_i}{\boldsymbol{b}_{n-1}} & \forall\, i \in [0, n-1] \\ 0 & \forall\, i \geq n \end{cases} \tag{36}$$

and

$$\lambda_i = \begin{cases} \dfrac{\boldsymbol{a}_i}{\boldsymbol{a}_n} & \forall\, i \in [0, n] \\ 0 & \forall\, i \geq n+1 \end{cases} \tag{37}$$

Hence, the Markov parameters of HOIS are

$$\boldsymbol{M}_k = \prod_{i=1}^{k} \boldsymbol{\mu}_i \qquad k = 1, 2, \cdots \tag{38}$$

Similarly, Markov parameters of the approximant are given by

$$\hat{\boldsymbol{M}}_k = \prod_{i=1}^{k} \hat{\boldsymbol{\mu}}_i \qquad k = 1, 2, \cdots \tag{39}$$

where $\hat{\boldsymbol{\mu}}_i$ for $i = 1, 2, \cdots$ have the same meaning as $\boldsymbol{\mu}_i$ for $i = 1, 2, \cdots$ in (35) for HOIS.

2.4 Procedure to obtain denominator polynomial of approximant:

In pole clustering technique [23]-[25], cluster center is obtained by grouping the poles of HOIS which is based on relative distance between the poles and desired order. In the process of modeling, separate clusters should be made for real poles and complex conjugate poles and then each cluster center or pair of cluster centers is replaced by single pole or pair of complex conjugate poles of approximant, respectively.

For obtaining rth order approximant, r cluster centers should be obtained. The inverse distance measure (IDM) criterion [23]-[25] is used for pole clustering. The steps for IDM criteion follow as:

Step 1: Form r cluster partitions from the poles of HOIS by collecting the real and complex conjugate poles of HOIS in separate cluster partitions.

Step 2: Calculate the cluster centers:

The cluster center for real poles is obtained as

$$\delta^C = \left\{ \left(\sum_{i=1}^{k} \left(\frac{1}{\alpha_i} \right) \right) \div k \right\}^{-1} \tag{40}$$

where δ^C is cluster center of cluster partition in which k real poles $(\alpha_1, \alpha_2, \cdots, \alpha_k)$ are grouped.

The pair of cluster centers for complex conjugate poles in the form of $\delta^R \pm j\delta^I$ is obtained as

$$\left. \begin{aligned} \delta^R &= \left\{ \left(\sum_{i=1}^{l} \left(\frac{1}{\alpha_i^R} \right) \right) \div l \right\}^{-1} \\ \delta^I &= \left\{ \left(\sum_{i=1}^{l} \left(\frac{1}{\alpha_i^I} \right) \right) \div l \right\}^{-1} \end{aligned} \right\} \tag{41}$$

where δ^R and δ^I are, respectively, real and imaginary part of cluster pair $\delta^R \pm j\delta^I$ in which l pairs of complex conjugate poles $\left(\left(\alpha_1^R \pm j\alpha_1^I\right), \left(\alpha_2^R \pm j\alpha_2^I\right), \cdots, \left(\alpha_l^R \pm j\alpha_l^I\right) \right)$ are grouped.

Step 3: Obtain the denominator of approximant:

Case 1: If all obtained cluster centers are real, the denominator polynomial of rth-order approximant becomes

$$D_r(z) = \prod\nolimits_{i=1}^{r} \left(z - \delta^C \right) \tag{42}$$

Case 2: If one pair of cluster center is complex conjugate and $(r-2)$ cluster centers are real, the denominator polynomial of rth-order approximant becomes

$$D_r(z) = \left(z-(\delta_1^R + j\delta_1^I)\right)\left(z-(\delta_1^R - j\delta_1^I)\right)\prod_{i=1}^{r-2}(z-\delta_i^C) \tag{43}$$

Case 3: If all obtained cluster centers are complex conjugate, the denominator polynomial of rth-order approximant becomes

$$D_r(z) = \prod_{i=1}^{r/2}\left(z-(\delta_i^R + j\delta_i^I)\right)\left(z-(\delta_i^R - j\delta_i^I)\right) \tag{44}$$

The denominator polynomial of the approximant is obtained by (42), (43) or (44) depending upon whether obtained cluster centers are all real, mixture of real and imaginary or all imaginary, respectively.

2.5 Procedure to obtain numerator polynomial of approximant:

Once the denominator polynomial of approximant is determined, the numerator polynomial is obtained by matching first r time moments and Markov parameters of HOIS to those of approximant. The time moments of the HOIS and those of the approximant are obtained by (29) and (31), respectively while the Markov parameters of the HOIS and those of the approximant are obtained by (38) and (39), respectively. The numerator parameters of approximant are obtained by matching first time moments and Markov parameters of HOIS to those of its approximant as given below

Case 1: r even

$$\left.\begin{aligned} &\boldsymbol{T}_i - \hat{\boldsymbol{T}}_i = 0 \qquad i = 0,1,\cdots,(r/2-1) \\ &\boldsymbol{M}_i - \hat{\boldsymbol{M}}_i = 0 \qquad i = 1,2,\cdots,r/2 \end{aligned}\right\} \tag{45}$$

Case 2: r odd

$$\left.\begin{aligned} &\boldsymbol{T}_i - \hat{\boldsymbol{T}}_i = 0 \qquad i = 0,1,\cdots,(r-1)/2 \\ &\boldsymbol{M}_i - \hat{\boldsymbol{M}}_i = 0 \qquad i = 1,2,\cdots,(r-1)/2 \end{aligned}\right\} \tag{46}$$

3. Numerical Section

Let the transfer function [22] of a third-order interval system be given as

$$G(z) = \frac{[8,10]+[3,4]z+[1,2]z^2}{[0.8,0.85]+[4.9,5]z+[9,9.5]z^2+[6,6]z^3} = \frac{N(z)}{D(z)} \tag{47}$$

Suppose, it is desired to obtain a second-order approximant (r=2) described by the transfer function

$$G_2^p(z) = \frac{[\hat{b}_0^-,\hat{b}_0^+]+[\hat{b}_1^-,\hat{b}_1^+]z}{[\hat{a}_0^-,\hat{a}_0^+]+[\hat{a}_1^-,\hat{a}_1^+]z+[\hat{a}_2^-,\hat{a}_2^+]z^2} \equiv \frac{\hat{\boldsymbol{b}}_0+\boldsymbol{b}_1 z}{\hat{\boldsymbol{a}}_0+\hat{\boldsymbol{a}}_{21}z+\hat{\boldsymbol{a}}_2 z^2} = \frac{\hat{N}_2(z)}{\hat{D}_2(z)} \tag{48}$$

The poles calculated, using (17), of the HOIS (47) are

$$\left.\begin{aligned} \lambda_1^h &= [-0.5340,-0.2680] \\ \lambda_2^h &= [-0.7125,-0.5361] \\ \lambda_3^h &= [-0.8534,-0.7203] \end{aligned}\right\} \tag{49}$$

Since poles (49) are real, thus, using (40), the cluster centers obtained by grouping λ_2 and λ_3 in one cluster partition and λ_1 in another cluster partition are

$$\delta_1^C = [-0.7766,-0.6147]\,,\ \delta_2^C = [-0.5340,-0.2680] \tag{50}$$

and the denominator obtained using (42) is

$$\left.\begin{aligned} \hat{D}_2(z) &= (z-\delta_1^C)(z-\delta_2^C) \\ &= (z-[-0.7766,-0.6147])(z-[-0.5340,-0.2680]) \\ &= [0.1647,0.4147]+[0.8827,1.3106]z+[1,1]z^2 \\ &= [\hat{a}_0^-,\hat{a}_0^+]+[\hat{a}_1^-,\hat{a}_1^+]z+[\hat{a}_2^-,\hat{a}_2^+]z^2 \end{aligned}\right\} \tag{51}$$

First time moment and Markov parameter, as given by (29) and (38), respectively, of the HOIS are

$$\left.\begin{aligned}\boldsymbol{T}_0 &= [0.5621, 0.7729] \\ \boldsymbol{M}_1 &= [0.1667, 0.3333]\end{aligned}\right\} \tag{52}$$

and the first time moment and Markov parameter, as given by (31) and (39), respectively, of the approximant are

$$\left.\begin{aligned}\hat{\boldsymbol{T}}_0 &= \frac{\hat{\boldsymbol{b}}_0 + \hat{\boldsymbol{b}}_1}{\hat{\boldsymbol{a}}_0 + \hat{\boldsymbol{a}}_1 + \hat{\boldsymbol{a}}_2} \\ \hat{\boldsymbol{M}}_1 &= \frac{\hat{\boldsymbol{b}}_1}{\hat{\boldsymbol{a}}_2}\end{aligned}\right\} \tag{53}$$

Using (45), numerator parameters are obtained as

$$\left.\begin{aligned}\boldsymbol{T}_0 &= \hat{\boldsymbol{T}}_0 \\ \boldsymbol{M}_1 &= \hat{\boldsymbol{M}}_1\end{aligned}\right\} \Rightarrow \left\{\begin{aligned}\hat{\boldsymbol{b}}_0 &= [0.9841, 1.7731] \\ \hat{\boldsymbol{b}}_1 &= [0.1667, 0.3333]\end{aligned}\right. \tag{54}$$

Thus, from (51) and (54), the second-order approximant obtained is

$$G_2^p(z) = \frac{[0.9841, 1.7731] + [0.1667, 0.3333]z}{[0.1647, 0.4147] + [0.8827, 1.3106]z + [1,1]z^2} \tag{55}$$

and the second-order approximant proposed in [22] is

$$G_2^o(z) = \frac{[0.8845, 0.9] + [0.5921, 0.6055]z}{[0.1437, 0.3805] + [0.8041, 1.2465]z + [1,1]z^2} \tag{56}$$

The step and impulse responses of some of the systems constructed with the help of Kharitonov polynomials (Appendix II) of numerator and denominator of HOIS, approximant in [22] and proposed approximant are shown in Fig. 1-2 and Fig. 3-4, respectively.

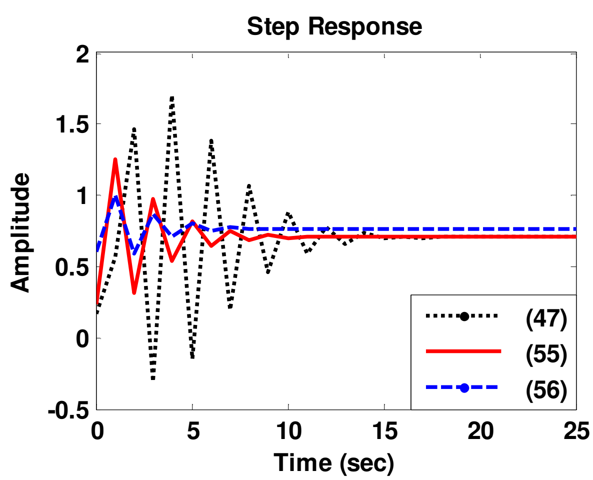

Fig. 1. Step Responses of HOIS and approximants.

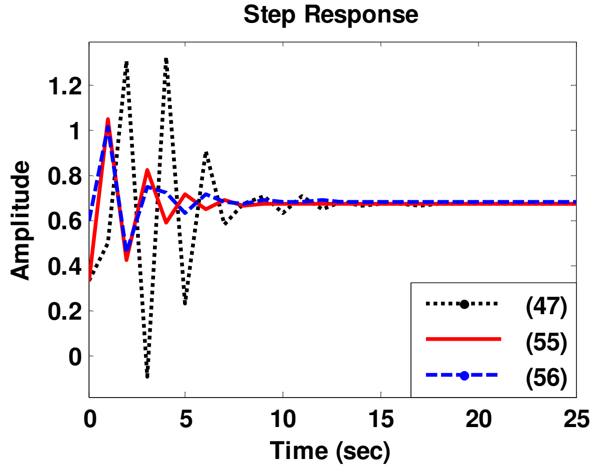

Fig. 2. Step Responses of HOIS and approximants.

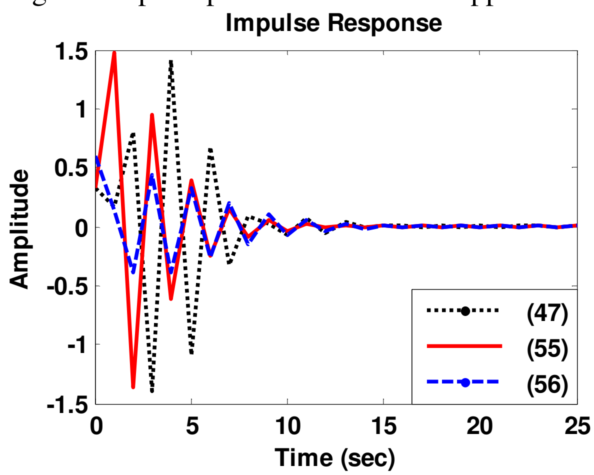

Fig. 3. Impulse Responses of HOIS and approximants.

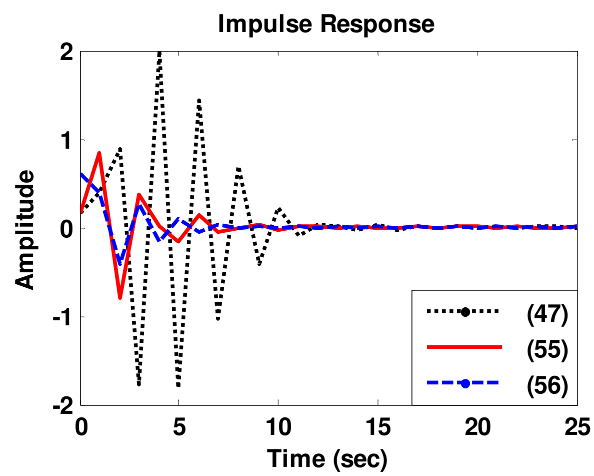

Fig. 4. Impulse Responses of HOIS and approximants.

The worst-case ISEs (Appendix III) of impulse responses for (55) and (56) are given in Table I.

Table I

Worst-case ISE for impulse response

Model	Worst-case ISE
(55)	73.5781
(56)	132.9109

It is noticed from Fig. 1-4 that the overall time response obtained by proposed approximant (55) is better than that of (56) and also worst-case ISE of (55) is lower than that of (56). This confirms the applicability of proposed method to derive approximant for high-order interval systems.

4. Conclusion

A computer aided method is proposed for obtaining approximant of given high-order interval system in which the denominator is obtained by pole-clustering method and the numerator is obtained by matching Markov parameters in addition to time moments. The time moments and Markov parameters are presented in generalized forms and obtained, in contrast to [16], without inverting the system transfer function. The proposed method is validated by a numerical example. The method based on combining neural network with pole clustering [28] is worth mentioning. This problem, in this context, is open to investigation and it would be interesting to compare the present approach with neural network based pole clustering [28].

Appendix I
Interval Arithmetic

The rules of interval arithmetic [19] are defined as follows:

Suppose, $a \equiv [a^-, a^+]$ and $\boldsymbol{b} \equiv [b^-, b^+]$ are two intervals.

Addition:

$$\boldsymbol{a}+\boldsymbol{b} \equiv [a^-,a^+]+[b^-,b^+]=[a^-+b^-,a^++b^+]$$

Subtraction:

$$\boldsymbol{a}-\boldsymbol{b} \equiv [a^-,a^+]-[b^-,b^+]=[a^--b^+,a^+-b^-]$$

Multiplication:

$$\boldsymbol{a}.\boldsymbol{b} \equiv [a^-,a^+][b^-,b^+]=[\min(a^-b^-,a^-b^+,a^+b^-,a^+b^+),\max(a^-b^-,a^-b^+,a^+b^-,a^+b^+)]$$

Division:

$$\boldsymbol{a}/\boldsymbol{b} \equiv [a^-,a^+]/[b^-,b^+]=[a^-,a^+]/[1/b^+,1/b^-]; \quad \boldsymbol{a}/\boldsymbol{a} \equiv [a^-,a^+]/[a^-,a^+]=1$$

Appendix II
Kharitonov Polynomials

Consider a family f of real interval polynomials [29]:

$$\left.\begin{aligned} D(z) &= \boldsymbol{\alpha}_0+\boldsymbol{\alpha}_1 z+\boldsymbol{\alpha}_2 z^2+\cdots+\boldsymbol{\alpha}_{n-1}z^{n-1}+\boldsymbol{\alpha}_n z^n \\ &= [\alpha_0^-,\alpha_0^+]+[\alpha_1^-,\alpha_1^+]z+[\alpha_2^-,\alpha_2^+]z^2+\cdots+[\alpha_n^-,\alpha_n^+]z^n \end{aligned}\right\}$$

The four Kharitonov polynomials associated with f are given below:

$$D_f^{\,1}(z)=\alpha_0^-+\alpha_1^-z+\alpha_2^+z^2+\alpha_3^+z^3+\alpha_4^-z^4+\alpha_5^-z^5+\alpha_6^+z^6+\alpha_7^+z^7+\cdots$$
$$D_f^{\,2}(z)=\alpha_0^++\alpha_1^-z+\alpha_2^-z^2+\alpha_3^+z^3+\alpha_4^+z^4+\alpha_5^-z^5+\alpha_6^-z^6+\alpha_7^+z^7+\cdots$$
$$D_f^{\,3}(z)=\alpha_0^++\alpha_1^+z+\alpha_2^-z^2+\alpha_3^-z^3+\alpha_4^+z^4+\alpha_5^+z^5+\alpha_6^-z^6+\alpha_7^-z^7+\cdots$$
$$D_f^{\,4}(z)=\alpha_0^-+\alpha_1^+z+\alpha_2^+z^2+\alpha_3^-z^3+\alpha_4^-z^4+\alpha_5^+z^5+\alpha_6^+z^6+\alpha_7^-z^7+\cdots$$

APPENDIX III
ISE of Impulse Response

The integral-squared-error (ISE) for impulse responses between $G(z)$ and $G_r(z)$ is obtained as

$$J=\frac{1}{2\pi i}\oint E(z)E(z^{-1})\frac{1}{z}dz$$

where

$$E(z)=G(z)-G_r(z)$$

$$= \frac{B_m(z)}{A_n(z)} \quad m, n = 1, 2, \cdots, 4.$$

Each of $B_m(z)$ and $A_n(z)$ represents four Kharitonov polynomials (Appendix II).

The ISE can recursively [30],[15] be obtained as

$$J = \frac{1}{A_0^n} \sum_{i=0}^{n} \frac{(B_i^i)^2}{A_0^i}$$

where n is the order of error signal $E(z)$ and A_0^n, B_i^i, and A_0^i are defined in [30].

The worst-case ISE [31] is obtained as

$$ISE_{worst-case} = \max_{m,n=1,\cdots,4} J(B_m, A_n)$$

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