DISCRETE OPTIMIZATION MODEL FOR THE 2-DIMENSIONAL ENERGIZED WAVES

Sunday A. Reju[1]
Mathematics & Statistics Department
Polytechnic of Namibia
Windhoek, Namibia
sunnyareju@gmail.com

Victor O. Waziri
Department of Cyber Security Science
Federal University of Technology
Minna, Nigeria
onomzavictor@gmail.com

Immanuel. K. Ajibola
Mathematics & Statistics Department
Polytechnic of Namibia
Windhoek, Namibia
iajibola@polytechnic.edu.na

Abstract

Energized waves are waves characterized by diffusion effects [Odio et al, 1997, Pain, 2005] and such waves are common in acoustics, ocean waves and many other wave propagation phenomena that involve propagation of energy. [Reju, 1995] was the first to apply the Extended Conjugate Gradient Method (ECGM) to the optimal control of classical wave propagation problem and with a specific extension to energized waves by [Waziri, 2006]. Computational procedures of Reju and Waziri are semi-analytic in nature utilizing in each case a direct methodology that is finally employed in the implementation of the ECGM. This paper however employs a discretization procedure via the 4th order Runge-Kutta algorithm to the search of optimal solutions of the 2-dimensional energised wave propagation problem and so contributing to the unified theory of the foregoing diffusive and kinetic Hamiltonian approach [Reju et al, 1999] preceding the implementation of the various versions of ECGM [Reju, et al, 2001; Waziri and Reju, 2006]. Moreover, the paper compares the discrete model optimal results with the semi-analytic approach of [Waziri and Reju, 2006] with its associated microscopic phenomena.

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[1] Corresponding Author
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1. Introduction

We consider the following optimal control problem [Waziri and Reju, 2006]:

$$\min_{z,u} J(z,u) = \min_{z,u} \int_0^y \int_0^x \int_0^t [z^2(x,y,t) + u^2(x,y,t)]\, dtdxdy \tag{1.1}$$

subject to the energised wave problem:

$$\frac{\partial^2 z(x,y,t)}{\partial t^2} + \frac{\partial z(x,y,t)}{\partial t} = \frac{\partial^2 z(x,y,t)}{\partial x^2} + \frac{\partial^2 z(x,y,t)}{\partial y^2} + u(x,y,t) \tag{1.2}$$

$$z(x,y,0) = z_0(x,y),\ z(x,0,t) = z(0,y,t) = 0 \tag{1.3}$$

$$\frac{\partial z(0,y,t)}{\partial t} = \frac{\partial z(x,0,t)}{\partial t} = 0 \tag{1.4}$$

We define the following Hamiltonian akin to [Singh and Titli, 1978] for the problem:

$$H(x,y,t) = z^2(x,y,t) + u^2(x,y,t) + \lambda^T \left[\int_0^t \left[\frac{\partial^2 z(x,y,s)}{\partial x^2} + \frac{\partial^2 z(x,y,s)}{\partial y^2} + u(x,y,t) \right] ds \right] \tag{1.5}$$

Similar to (Reju, 2001), in obtaining the resulting expressions from the first order necessary optimality conditions, we set

$$f(z,u) = \int_0^t \left[\frac{\partial^2 z(x,y,s)}{\partial x^2} + \frac{\partial^2 z(x,y,s)}{\partial y^2} + u(x,y,t) \right] ds \tag{1.6}$$

$$g(z,u) = z^2(x,y,t) + u^2(x,y,t) \tag{1.7}$$

Thus the optimality equations become (Waziri and Reju, 2006)

$$z(x,y,t) = \frac{\partial u(x,y,t)}{\partial t} \tag{1.8}$$

Now assume that $z(x,y,t)$ has the following Fourier form with its associated control $u(x,y,t)$

$$z(x,y,t) = \sum_{i=1}^{\infty} \alpha_i(t) \sin \pi i x \sin \pi i y \tag{1.9}$$

$$u(x,y,t) = \sum_{i=1}^{\infty} u_i(t) \sin \pi i x \sin \pi i y \tag{1.10}$$

From equation, we have

$$z(x,y,t) = \sum_{i=1}^{\infty} \frac{\partial u_i(t)}{\partial t} \sin \pi i x \sin \pi i y \tag{1.11}$$

Hence the energised wave equation becomes

$$\sum_{i=1}^{\infty} \frac{\partial^3 u_i(t)}{\partial t^3} \sin \pi i x \sin \pi i y + \sum_{i=1}^{\infty} \frac{\partial^2 u_i(t)}{\partial t^2} \sin \pi i x \sin \pi i y$$

$$= -2i^2\pi^2 \sum_{i=1}^{\infty} \frac{\partial u_i(t)}{\partial t} \sin \pi i x \sin \pi i y + \sum_{i=1}^{\infty} u_i(t) \sin \pi i x \sin \pi i y \tag{1.12}$$

or

$$\sum_{i=1}^{\infty} \frac{\partial^3 u_i(t)}{\partial t^3} + \sum_{i=1}^{\infty} \frac{\partial^2 u_i(t)}{\partial t^2} = -2i^2\pi^2 \sum_{i=1}^{\infty} \frac{\partial u_i(t)}{\partial t} + \sum_{i=1}^{\infty} u_i(t) \tag{1.13}$$

Our optimal control problem therefore becomes

$$\min_u \int_0^1 \left[\left(u_1^2(t) + u_2^2(t) + \cdots + u_n^2(t) \right) + \left(\frac{\partial u_1(t)}{\partial t} + \frac{\partial u_2(t)}{\partial t} + \cdots + \frac{\partial u_n(t)}{\partial t} \right) \right] dt \tag{1.14}$$

subject to the following constraint system

$$\left.\begin{aligned}
\frac{\partial^3 u_1(t)}{\partial t^3} &= -\frac{\partial^2 u_1(t)}{\partial t^2} - 2(1)^2\pi^2 \frac{\partial u_1(t)}{\partial t} + u_1(t) \\
\frac{\partial^3 u_2(t)}{\partial t^3} &= -\frac{\partial^2 u_2(t)}{\partial t^2} - 2(2)^2\pi^2 \frac{\partial u_2(t)}{\partial t} + u_2(t) \\
&\cdots \\
\frac{\partial^3 u_n(t)}{\partial t^3} &= -\frac{\partial^2 u_n(t)}{\partial t^2} - 2(n)^2\pi^2 \frac{\partial u_n(t)}{\partial t} + u_n(t)
\end{aligned}\right\} \tag{1.15}$$

In the next section we construct the discrete model.

2. Discrete Model Formulation

We start our discretization model by reducing the general 3rd order differential equation

$$\frac{\partial^3 u_n(t)}{\partial t^3} = -\frac{\partial^2 u_n(t)}{\partial t^2} - 2(n)^2\pi^2 \frac{\partial u_n(t)}{\partial t} + u_n(t) \tag{2.1}$$

to a system of three 1st order equations as follows:

Set

$$y_{n1} = u_n,\ y_{n2} = u'_{n1},\ y_{n3} = u''_{n1} \tag{2.2}$$

Hence we have

$$y'_{n1} = u'_{n1} \ = 0y_{n1} + y_{n2} + 0y_{n3} \tag{2.3}$$

$$y'_{n2} = u''_{n1} \ = 0y_{n1} + 0y_{n2} + y_{n3} \tag{2.4}$$

$$y'_{n3} = u'''_{n1} = y_{n1} - 2(n)^2\pi^2 y_{n2} - y_{n3} \tag{2.5}$$

The equation therefore becomes the following system

$$\begin{bmatrix} y'_{n1} \\ y'_{n2} \\ y'_{n3} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -2(n)^2\pi^2 & -1 \end{bmatrix} \begin{bmatrix} y_{n1} \\ y_{n2} \\ y_{n3} \end{bmatrix} \tag{2.6}$$

Solving the above system by using an appropriate numerical method would finally yield the following solutions:

$$u(x,y,t) = \sum_{n=1}^{\infty} u_n(t) \sin \pi nx \sin \pi ny = \sum_{i=1}^{\infty} y_{n1}(t) \sin \pi nx \sin \pi ny \tag{2.7}$$

$$z(x,y,t) = \sum_{n=1}^{\infty} \frac{\partial u_n(t)}{\partial t} \sin \pi nx \sin \pi ny = \sum_{n=1}^{\infty} y_{n2}(t) \sin \pi nx \sin \pi ny \tag{2.8}$$

Now we employ the following 4th order Runge-Kutta method to discretize the system

$$\begin{bmatrix} y'_{n1} \\ y'_{n2} \\ y'_{n3} \end{bmatrix} = \begin{bmatrix} f(t, y_{n1}) \\ f(t, y_{n2}) \\ f(t, y_{n3}) \end{bmatrix} \tag{2.9}$$

as follows, using the equivalence expressions $y_{n1} = y^{(n1)},\ y_{n2} = y^{(n2)},\ y_{n3} = y^{(n3)}$

$$y_{i+1}^{(n1)} = y_i^{(n1)} + \frac{1}{6}\left[k_1^{(n1)} + 2k_2^{(n1)} + 3k_3^{(n1)} + k_4^{(n1)}\right] \tag{2.10}$$

$$y_{i+1}^{(n2)} = y_i^{(n2)} + \frac{1}{6}\left[k_1^{(n2)} + 2k_2^{(n2)} + 3k_3^{(n2)} + k_4^{(n2)}\right] \tag{2.11}$$

$$y_{i+1}^{(n3)} = y_i^{(n3)} + \frac{1}{6}\left[k_1^{(n3)} + 2k_2^{(n3)} + 3k_3^{(n3)} + k_4^{(n3)}\right] \tag{2.12}$$

where

$$k_1^{(n1)} = hf\left(t_i, y_i^{(n1)}\right), \;\; k_2^{(n1)} = hf\left(t_i + \frac{h}{2}, y_i^{(n1)} + \frac{1}{2}k_1^{(n1)}\right) \tag{2.13}$$

$$k_3^{(n1)} = hf\left(t_i + \frac{h}{2}, y_i^{(n1)} + \frac{1}{2}k_2^{(n1)}\right), \qquad k_4^{(n1)} = hf\left(t_i + h, y_i^{(n1)} + k_3^{(n1)}\right) \tag{2.14}$$

$$k_1^{(n2)} = hf\left(t_i, y_i^{(n2)}\right), \qquad k_2^{(n2)} = hf\left(t_i + \frac{h}{2}, y_i^{(n2)} + \frac{1}{2}k_1^{(n2)}\right) \tag{2.15}$$

$$k_3^{(n2)} = hf\left(t_i + \frac{h}{2}, y_i^{(n2)} + \frac{1}{2}k_2^{(n2)}\right), \qquad k_4^{(n2)} = hf\left(t_i + h, y_i^{(n2)} + k_3^{(n2)}\right) \tag{2.16}$$

$$k_1^{(n3)} = hf\left(t_i, y_i^{(n3)}\right), \qquad k_2^{(n3)} = hf\left(t_i + \frac{h}{2}, y_i^{(n3)} + \frac{1}{2}k_1^{(n3)}\right) \tag{2.17}$$

$$k_3^{(n3)} = hf\left(t_i + \frac{h}{2}, y_i^{(n3)} + \frac{1}{2}k_2^{(n3)}\right), \qquad k_4^{(n3)} = hf\left(t_i + h, y_i^{(n3)} + k_3^{(n3)}\right) \tag{2.18}$$

Following the Runge-Kutta implementation, then we have the optimal state and control given by

$$u(x, y, t) = \sum_{i=1}^{\infty} y_{i+1}^{(n1)}(t) \sin \pi i x \sin \pi i y \tag{2.19}$$

$$z(x, y, t) = \sum_{i=1}^{\infty} y_{i+1}^{(n2)}(t) \sin \pi i x \sin \pi i y \tag{2.20}$$

3. A Model Example

Using the following input data

$$u_n(0) = \frac{\partial u_n(0)}{\partial t} = 0.5;\; t \in [0,1]; = \frac{(1-0)}{n}, \mathrm{n} = 15 \tag{3.1}$$

we have the following iterated solutions from the Runge-Kutta algorithm implementation.

Table 3.1: Runge-Kutta Solutions

t	y_{n1}	y_{n1}	y_{n1}
0	0.5	0.5	0.5
0.07	0.5340	0.5116	-0.1474
0.13	0.5673	0.4816	-0.7385
0.2	0.5974	0.4154	-1.2272
...	...	...	...
0.8	0.5781	-0.3001	0.2451
0.87	0.5589	-0.2706	0.6317

0.93	0.5426	-0.2177	0.9391
1	0.5303	-0.1476	1.1465

Therefore we have from (2.19) and (2.20)

$$u(x,y,.) = \sum_{i=1}^{15} y_{i+1}^{(n1)}(.) \sin \pi ix \sin \pi iy$$

$$= (0.5) \sin \pi x \sin \pi y + (0.5340) \sin 2\pi x \sin 2\pi y$$

$$+(0.5673) \sin 3\pi x \sin 3\pi y + \cdots + (0.5303) \sin 15\pi x \sin 15\pi y \qquad (3.2)$$

$$z(x,y,.) = \sum_{i=1}^{15} y_{i+1}^{(n2)}(.) \sin \pi ix \sin \pi iy$$

$$= (0.5) \sin \pi x \sin \pi y + (0.5116) \sin 2\pi x \sin 2\pi y$$

$$+(0.4816) \sin 3\pi x \sin 3\pi y + \cdots + (-0.1476) \sin 15\pi x \sin 15\pi y \qquad (3.3)$$

Now, for a complete kinetic and diffusive solution, using the relations

$$y_{n1} = u_n,\ y_{n2} = u'_{n1},\ y_{n3} = u''_{n1} \qquad (3.4)$$

we have

$$y_{n1} = u_n = y_{n2}t;\ y_{n2} = u'_{n1} = y_{n3}t \qquad (3.5)$$

Thus the optimal solutions become

$$u(x,y,t) = \sum_{i=1}^{15} y_{i+1}^{(n2)} t \sin \pi ix \sin \pi iy \qquad (3.6)$$

$$z(x,y,t) = \sum_{i=1}^{15} y_{i+1}^{(n3)} t \sin \pi ix \sin \pi iy \qquad (3.7)$$

Using the above input data with step size 0.01, we have the following simulated solutions in surface plots for our optimal state and control (noting that for a 3-dimensional plot, we need to fix one of the spatial variables).

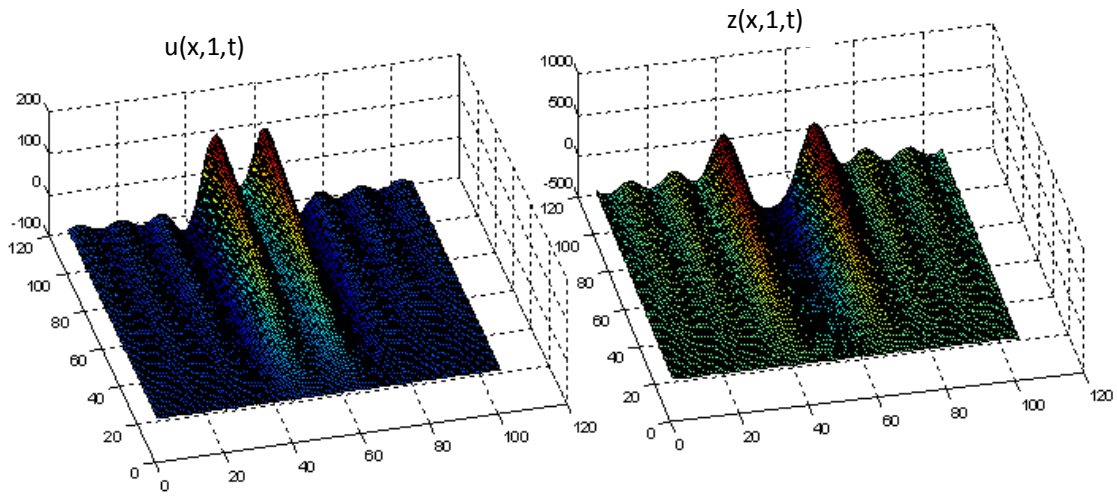

Figure 3.1 Surface Plot for state z(x,1,t) and control u(x,1,t) for n = 15, y = 0.5

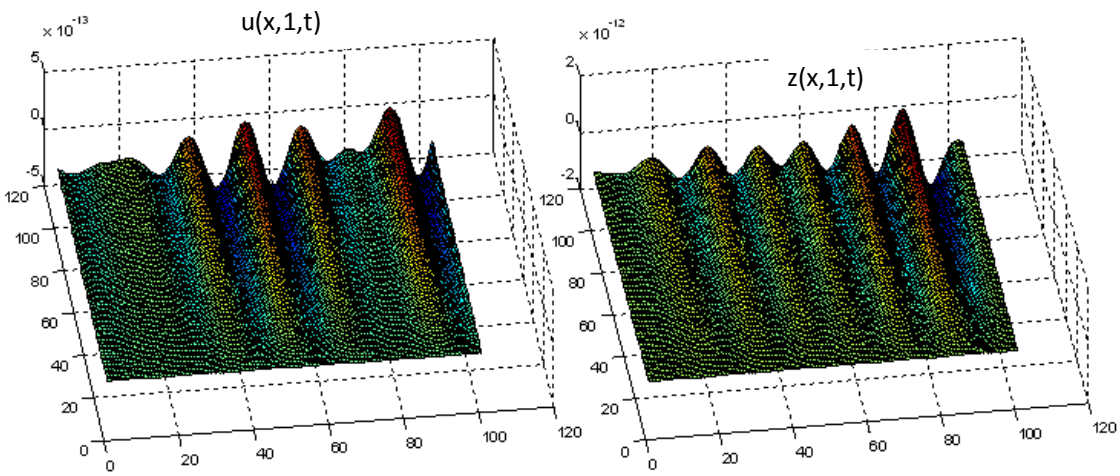

Figure 3.2: Surface Plot for state z(x,1,t) and control u(x,1,t) for n = 15, y = 1

The numerical solutions for a higher dimension (n=25) at y = 1 are shown below.

Table 3.2 Numerical Results for n = 25, y = 1

x_i, t_i	u(x, 1, t)		z(x, 1, t)	
i	n=1	n=25	n=1	n=25
1	0	0	0	0
2	0	0	0	0
…	0	0	0	0
12	0	0	0	0
13	0.0189×10^{-16}	0.0354×10^{-16}	0.0189×10^{-16}	-0.5799×10^{-16}
14	0.0360×10^{-16}	0.0674×10^{-16}	0.0036×10^{-15}	-0.1103×10^{-15}
…	…	…	…	
23	0.0750×10^{-32}	0.1403×10^{-32}	0.0075×10^{-31}	-0.2298×10^{-31}
24	0	0	0	0
25	0.0378×10^{-16}	0.0708×10^{-16}	0.0038×10^{-15}	-0.1160×10^{-15}

…	…	…	…	
98	0.0360×10^{-15}	0.0674×10^{-15}	0.0036×10^{-14}	-0.1103×10^{-14}
99	0.0189×10^{-15}	0.0354×10^{-15}	0.0189×10^{-15}	-0.5799×10^{-15}
100	0.0750×10^{-31}	0.1403×10^{-31}	0.0075×10^{-30}	-0.2298×10^{-30}

A comparison of the above results with the semi-analytic model of [Waziri and Reju, 2006] is presented in the next section.

4. Conclusion

The discrete model of the 2-dimensional energised waves extends the results of [Waziri and Reju, 2006] on analysis of 2-dimensional energised waves with a source. Comparing the results with our new discrete model, we have the following:

n	Semi-Analytic Model	Semi-Analytic Model	Discrete Model	Discrete Model
	u(x, y, t)	z(x, y, t)	u(x, y, t)	z(x, y, t)
5	6×10^{-28}	1×10^{-27}	1.6×10^{-32}	1×10^{-32}
10	6×10^{-28}	1×10^{-27}	3×10^{-32}	2×10^{-32}
30	1×10^{-27}	4×10^{-27}	8×10^{-32}	5×10^{-32}
40	6×10^{-28}	2×10^{-27}	1×10^{-31}	2×10^{-31}
60	1×10^{-27}	5×10^{-27}	1.5×10^{-31}	2×10^{-31}

Thus the discrete model yet depicts propagation fields characterized by creeping motions and microscopic events, commonly occurring in some tectonic dynamics as noted in the earlier semi-analytic results of (Reju et al, 2001) and obviously reflected again in the energised wave model of [Waziri and Reju, 2006]. The discretization procedure provides expressions (3.6) and (3.7) for the complete construction of a discretized version of the ECGM control operator for the 2-Dimensional energized wave equation earlier provided by [Waziri and Reju, 2006].

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