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HALF DOMINATION ARRANGEMENTS IN REGULAR AND SEMI-REGULAR TESSELLATION TYPE GRAPHS

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Abstract. We study the problem of half-domination sets of vertices in vertex-transitive infinite graphs generated by regular or semi-regular tessellations of the plane. In some cases, the results obtained are sharp and in the rest, we show upper bounds for the average densities of vertices in half-domination sets.

1. Introduction

By a *tiling* of the plane one understands a countable union of closed sets (called tiles) whose union is the whole plane and with the property that every two of these sets have disjoint interiors. The term tessellation is a more modern one that is used mostly for special tilings. We are going to be interested in the tilings in which the closed sets are either all copies of one single regular convex polygon (regular tessellations) or several ones (semi-regular tessellations) and in which each vertex has the same *vertex arrangement* (the number and order of regular polygons meeting at a vertex). Also we are considering the *edge-to-edge* restriction, meaning that every two tiles either do not intersect or intersect along a common edge, or at a common vertex. According to [4], there are three regular edge-to-edge tessellations and eight semi-regular tessellations (see [4], pages 58-59). The generic tiles in a regular tessellation, or in a semi-regular one, are usually called *prototiles*. For instance, in a regular tessellation the prototiles are squares, equilateral triangles or regular hexagons. We will refer to these tessellations by an abbreviation that stands for the ordered tuple of positive integers that give the so called vertex arrangement (i.e. the number of sides of the regular polygons around a vertex starting with the smallest size and taking into account counterclockwise order). The abbreviation is usually using the convention with the powers similar to that used in the standard prime factorization of natural numbers. So, the regular tessellation with squares is referred to as (4^4). We refer to Figure 1 for the rest of the notation.

Each such tessellation is *periodic* in the sense that there exists a cluster of tiles formed by regular polygons, which by translations generated by only two vectors, say v_1 and v_2, covers the whole plane, and the resulting tiling is the given tessellation (Figure 1).

For each such tessellation $\mathcal{T}$, we associate an infinite graph, $G_{\mathcal{T},\infty}$ in the following way. For each regular polygon in the tessellation we have a vertex in $G_{\mathcal{T},\infty}$, and the edges in this graph are determined by every two polygons that share a common edge within the tessellation. This way, for instance, for the regular

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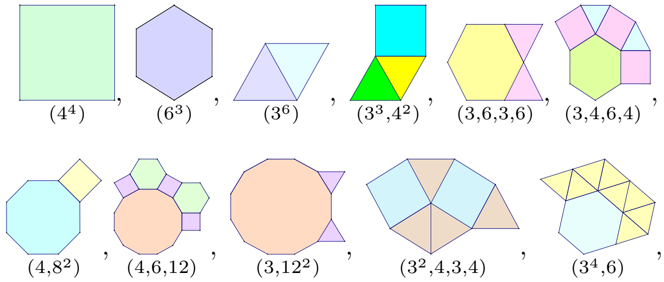


FIGURE 1. Minimal clusters

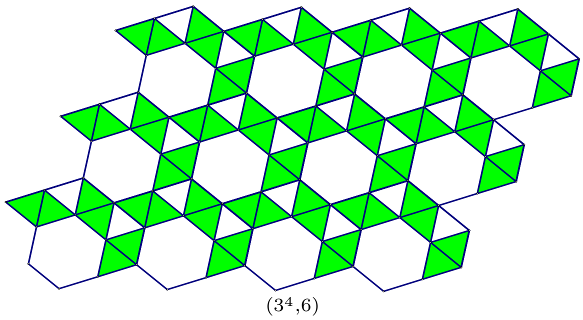


FIGURE 2. $G_{(3^4,6),4,3}$

tessellation with squares (4^4), we obtain what is usually called the infinite *grid* graph. Also, if $m, n \in \mathbf{N}$ we can construct a graph $G_{\mathcal{T},m,n}$ in the same way as before using only m copies of the cluster of tiles generating $\mathcal{T}$ shown in Figure 1 in the direction of v_1 and n copies in the direction of v_2. For $\mathcal{T} = (3^4, 6)$, $m = 4$ and $n = 3$ we obtain the graph generated by the polygons in the portion of the tessellation $\mathcal{T}$ shown in Figure 2.

Clearly, the choice of the tiles on Figure 1 is not unique but whatever one takes for these tiles it is not going to be relevant in our calculation of densities. For each such graph as before, and one of its vertices v, let us denote by $d(v)$ the number of adjacent vertices to v, known as the degree of the vertex v. We define a set of vertices S to be *half-dependent* if for each vertex $v \in S$ the number of adjacent vertices to v that are in S is less than or equal to $\lfloor \frac{d(v)}{2} \rfloor$.

Let $m, n \in \mathbf{N}$ be arbitrary, and for each graph $G_{\mathcal{T},m,n}$ we denote by $\rho_{\mathcal{T},m,n}$ the maximum cardinality of a half-dependent set in $G_{\mathcal{T},m,n}$, divided by the number of vertices of $G_{\mathcal{T},m,n}$. Hence one may consider the number

$$\rho_{\mathcal{T}} = \limsup_{m,n\to\infty} \rho_{\mathcal{T},m,n}, \tag{1}$$

which represents, heuristically speaking, the highest proportion in which one can distribute the vertices in a half-domination set. For instance we will show that in the case of a regular tessellation with regular hexagons (see Figure 3)(b) the number defined above is $\frac{2}{3}$.

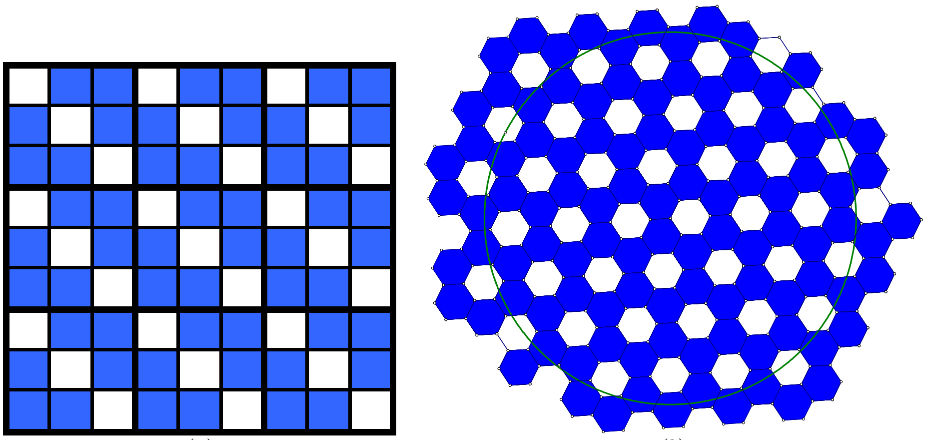


FIGURE 3. Regular tessellations with squares and regular hexagons

In the regular tessellation with squares we have shown in [7] that the number defined in (1) is also $\frac{2}{3}$ (Figure 3(a)). In this paper we are interested in the values of $\rho_{\mathcal{T}}$ for these types of tessellations.

2. Various techniques

First we are going to use one of main ideas in [6] and [7], and some classical linear optimization techniques.

2.1. **Integer Linear Programming Systems.** The regular tessellation (6^3) can be treated simply in the following way. We are going to work with the graph $G_{6^3,n,n}$ obtained from translating n-times a prototile so that the each two neighboring tiles share an edge and then translate the whole row of n-tiles in such a way that two neighboring rows fit together to give the tessellation of a rhombic n-by-n region. We assign to each tile a variable $x_{i,j}$ which can be 0 or 1: if a tile is part of the half-dominating set (colored blue in Figure 3b) of maximum cardinality then its variable $x_{i,j}$ is equal to 1, and if it is not, $x_{i,j}=0$. For most of the vertices of $G_{6^3,n,n}$ the degree is 6 so each vertex in the half-dominating set, denoted by V, must have at most 3 other vertices adjacent to it which are in V. Let us denote by $N(i,j)$ the adjacent indices to the vertex indexed with (i,j). We can write the half-domination condition as

$$x^{\star}_{i,j} := \sum_{(k,l)\in N(i,j)} x_{k,l} \le 3 \ \text{ for all (i,j) with } 1<i,j<n, (i,j)\in V.$$

For a vertex not in V we simply have no restriction on the above sum (maximum is 6). The trick is to write an inequality that encompasses both situations. In this case, we see that the following inequality accomplishes exactly that:

$$3x_{i,j} + \sum_{(k,l)\in N(i,j)} x_{k,l} \le 6 \ \text{ for all (i,j) with } 1<i,j<n.$$

For the vertices on the boundary we have similar inequalities. Summing all these inequalities up, we obtain, $3|V'|+6|V'|+\Sigma\le 6n^2$, where V' are the vertices in V in the interior and Σ are the number of those on the boundary. Since clearly $\lim\frac{\Sigma}{n^2}=0$ we see that $\lim\frac{|V|}{n^2}\le\frac{2}{3}$. In Figure 3b, we exemplify an arrangement which accomplishes this density. Hence, $\rho_{6^3}=\frac{2}{3}$.

In the case of the tessellation with equilateral triangles, the inequality above changes into

$$2x_{\triangle}+\sum_{\triangle'\in N(\triangle)}x_{\triangle'}\le 3 \text{ for all } \triangle s \text{ not at the boundary.}$$

The argument above gives the inequality $\rho_{3^6}\le\frac{3}{5}$. The arrangement in Figure 4 has a density which is only $\frac{9}{16}$. In this case it is not possible to achieve the density $\frac{3}{5}$.

2.2. **Toroidal type graphs.** One way to avoid to deal with the boundary tiles, is to form toroidal type graphs obtained from tessellations. We have shown in [6] that this does not affect the maximum density, in the sense that both situations tend at the limit to the same density value. Let us begin with the case $\mathcal{T}=3^6$. For $n\in\mathbb{N}$, we take the parallelogram which gives the graph $G_{3^6,n,n}$ and identify the opposite edges (without changing the direction). This gives rise to a similar graph which we will denote by $T_{3^6,n}$. This graph is regular and the equations we get can be easily described and implemented in LPSolve. Let us introduce the variables $x_{i,j,k}$, $i,j\in\{0,1,2,...,n-1\}$ and $k\in\{1,2\}$, in the following way. The index (i,j) refers to the translation of the minimal parallelogram (Figure 1) i places in the horizontal direction and j places in the 60° direction. The index k corresponds to the lower $(k=1)$ and upper equilateral triangle within this parallelogram $(k=0)$. We get a number of $2n^2$ vertices of this regular graph. The optimization equations are simply

$$(2)\qquad \begin{cases} 2x_{i,j,1}+x_{i,j,2}+x_{u,j,2}+x_{i,v,2}\le 3,\ where\ u\equiv i-1\ (mod\ n),\\ \quad v\equiv j-1\ (mod\ n), u,v\in\{0,1,2,...,n-1\}\\ 2x_{i,j,2}+x_{i,j,1}+x_{u,j,1}+x_{i,v,1}\le 3,\ where\ w\equiv i+1\ (mod\ n),\\ \quad t\equiv j+1\ (mod\ n), w,t\in\{0,1,2,...,n-1\}, i,j\in\{0,1,2,...,n-1\}.\end{cases}$$

We denote by $\rho_{3^6,n}$ the best densities for this graph. Observe that $\rho_{3^6,n}\le\rho_{3^6,m}$ if n divides m. Also, as proved in [6], we have $\rho_{3^6}=\sup_n\rho_{3^6,n}$. We have the following densities:

n	1	2	3	4	5	6	7	8	9
$\rho_{3^6,n}$	0	$\frac{1}{2}$	$\frac{5}{9}$	$\frac{9}{16}$	$\frac{14}{25}$	$\frac{5}{9}$	$\ge\frac{27}{49}$	$\ge\frac{9}{16}$	$\ge\frac{5}{9}$

Unfortunately, LPSolve takes too long to finish the analysis for $n\ge 7$ (about 100 variables). In order to reduce the number of variables, and of course taking advantage of the rotational symmetry, one can try to do a different matching of the boundaries as in Figure 5. This corresponds to rotations of 180° around the

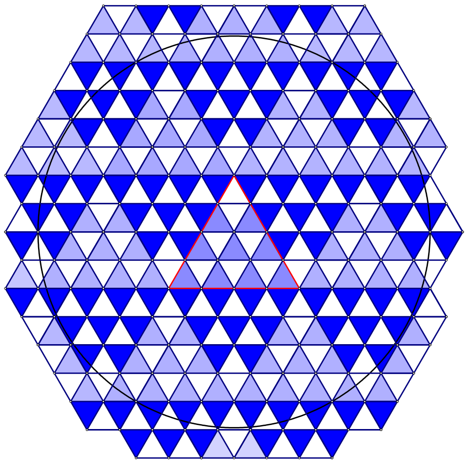


FIGURE 4. Regular tessellation with equilateral triangles

midpoints of the sides for the basic triangle of tiles ABC. For instance, the corner tiles are all adjacent to each other.

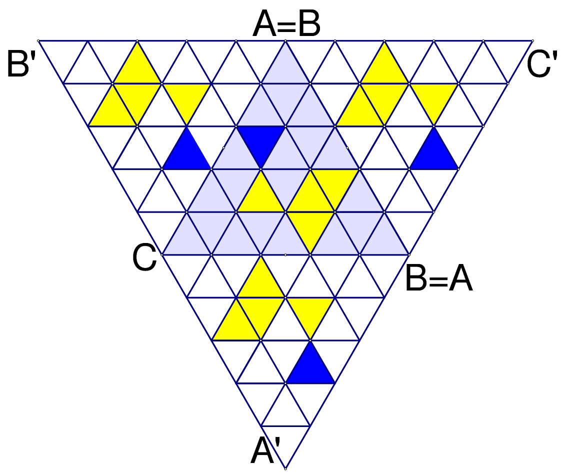


FIGURE 5. Klein type toroidal identification

The equations that we need to use, are those given by (2) and in addition

$$x_{i,j,1} = x_{u,v,2} \text{ where } u \equiv n-1-i \ (mod\ n), \text{ and } v \equiv n-1-j \ (mod\ n),$$

(3)

$$u, v \in \{0,1,2,...,n-1\}, \text{for all } i,j \in \{0,1,2,...,n-1\}.$$

We will refer to the graph obtained by these identifications as $K_{3^6,n}$. This graph is a regular graph with n^2 vertices and degree 3. All the densities are the same as in the table above but LPSolve finishes in a reasonable time and the arrangement in Figure 4 satisfies this new restriction. In the case $n = 4$ we can see that the arrangement in Figure 4 is also the best because if we could add another triangle, that would increase the density to $\frac{5}{8} > \frac{3}{5}$, which we know is not possible by the upper bound established earlier. It is important that the best density in the case $K_{3^6,8}$ is the same as in the case $K_{3^6,4}$. We conjecture that the best density is given by such a matching of two toroidal graphs one having the dimension double the dimension of the other.

2.3. **Upper bounds.** For the problem above we can adopt the method of weighted objective function as described in [7]. We used the weights all equal to 1 in the interior and zero on the boundary for $K_{3^6,13}$. The upper bound obtained is $\rho_{3^6} \leq \frac{70}{121} \approx 0.59375$. This is within $\frac{31}{1936} \approx 0.0160124$ to $\frac{9}{16}$ which makes it plausible that $\rho_{3^6} = \frac{9}{16}$.

For the graph $K_{3^6,n}$, we may add to the system of inequalities (2) and (3) the conditions $0 \leq x_{i,j,1}, x_{i,j,2} \leq 1$ and $\sum_{i,j}(x_{i,j,1}+x_{i,j,2}) = k$. These inequalities describe a polytope in n^2 dimensions. Finding the maximum cardinality of a half dependent set is equivalent to finding the smallest k for which there is no lattice point in the corresponding polytope. There exists a theory which counts the number of lattice points in polytopes which was started by Eugene Ehrhart in 1960's (see [1] and [2]). The theory simplifies significantly if the polytope has integer vertices. Unfortunately our polytope has rational vertices. Theoretically, there exists a quasi-polynomial $P(t,k)$ of degree n^2 which counts the lattice points contained in the dilation of the polytope by t. We want the smallest k_n such that $P(1, k_n + 1) = 0$. Let us make the observation that $P(1, k_n)$ should by relatively a big since the system is invariant under translations (mod n) and under rotations. So, one may expect $P(1, k_n - 1) \approx 3n^2$. There exists several programs which calculate this polynomial from the coefficients of the system of inequalities which define the polytope. One of these programs is called LattE and it written by Matthias Köppe and Jesús A. De Loera. This method remains to be implemented and investigated in a different project. Also, it seems like the number of variables that can be used in this program is also by about 100. However, we think that the method fits very well with the toroidal formulation. If the Ehrhart polynomial can be computed in dimensions k and $2k$, it may give more information of how to find the best density. We will be using other methods which are discussed in the next sections.

3. Semi-regular tessellations

Working with the graphs generated by semi-regular tessellations is more challenging when it comes to implementing the problem into LPSolve. There are also advantages here since the systems have less symmetry and somehow this is a plus for the optimization programs to arrive faster at a maximum.

3.1. **The case $\mathcal{T} = (3^3, 4^2)$.** In Figure 6 (a) we see an arrangement of a half-domination set with a density of $\frac{7}{12}$, which is only $\frac{1}{28}$ off of the upper bound in the next theorem.

Theorem 3.1. *The half-domination density for the tessellation $\mathcal{T} = (3^3, 4^2)$ satisfies*

$$\rho_{(3^3,4^2)} \leq \frac{13}{21}.$$

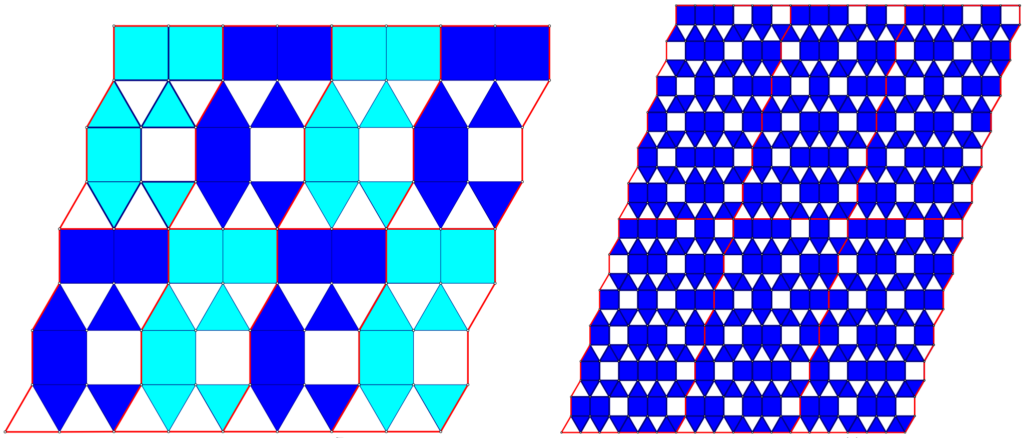


FIGURE 6. Some arrangements for $\mathcal{T} = (3^3, 4^2)$

PROOF. Let us consider a toroidal graph induced by $G_{\mathcal{T},m,n}$ with m and n big and a half domination set corresponding to it. We observe that there are $\frac{3mn}{3} = mn$ squares and $\frac{2}{3}(3mn) = 2mn$ equilateral triangles in $G_{\mathcal{T},m,n}$. As usual let us consider the variables x_v for each vertex v in this graph defined to be 1 or 0 as being in the domination set or not. Also, we denote by x_v^* the sum of the variables x_w corresponding to the adjacent vertices w of v. For a vertex v corresponding to a square we have (see Figure 6)

$$2x_v + x_v^* \leq 4. \tag{4}$$

We denote by T the sum of all x_v over all vertices corresponding to triangles and by S the sum of all x_v over all vertices corresponding to squares. If we sum up all equalities (4) over all the squares we get:

$$2S + \sum_{\text{v for a squares}} x_v^* \leq 4mn \Rightarrow 2S + (2S + T) \leq 4mn \text{ or } 4S + T \leq 4mn.$$

For a vertex v corresponding to an arbitrary triangle we have

$$2x_v + x_v^* \leq 3, \tag{5}$$

which gives, as before, if summed up over all of the triangles:

$$2T + \sum_{\text{v for a trianlge}} x_v^* = 6mn \Rightarrow 2T + (2S + 2T) \leq 6mn \text{ or } 2S + 4T \leq 6mn.$$

If we let $x = \frac{S}{3mn}$ and $y = \frac{T}{3mn}$, we need to maximize $x+y$ and, as we have seen above, x and y are subjected to the restrictions

$$\begin{cases} 4x + y \le \frac{4}{3} \\ \\ 2x + 4y \le 2. \end{cases}$$

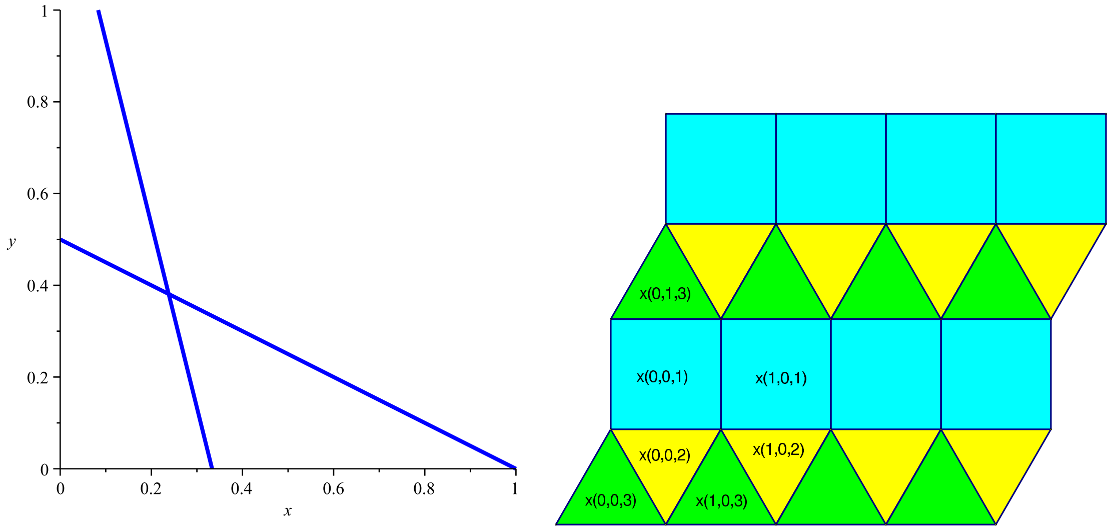


FIGURE 7. The two lines and the variables convention

The usual maximization technique (see Figure 7) gives that for real numbers x and y, the maximum of $x+y$ is attained for $x = \frac{5}{21}$ and $y = \frac{8}{21}$ with $x + y = \frac{13}{21}$. This proves that

$$\rho_{(3^3,4^2)} = \limsup_{m,n\to\infty} \frac{S(m,n) + T(m,n)}{3mn} \le \frac{13}{21}. \qquad \blacksquare$$

3.2. **Toroidal examples for $\mathcal{T} = (3^3, 4^2)$.** The number of variables grows rapidly with $m = n$. For $n = 7$ we have $3n^2 = 147$ variables and this seems to be a good bound for what one can obtain with LPSolve. We use variables $x_{i,j,k}$ (having values 0 or 1) with $i = 0, 1, 2, ..., n-1$, $j = 0, 1, 2, ..., m-1$ and $k = 1, 2, 3$ with $x_{i,j,1}$ for a square, $x_{i,j,2}$ and $x_{i,j,3}$ for the triangles in the cluster of minimal tiles as in the Figure 7. The system of inequalities can be written as follows

$$\begin{cases} 2x_{i,j,1} + x_{i,j,2} + x_{u,j,2} + x_{v,j,1} + x_{i,t,3} \le 4, \ \textit{where}\ u \equiv i-1\ (\textit{mod}\ n),\ v \equiv i+1\ (\textit{mod}\ n), \\ \\ w \equiv j-1\ (\textit{mod}\ m),\ t \equiv j+1\ (\textit{mod}\ m),\ u, v \in \{0,1,2,...,n-1\}, w, t \in \{0,1,2,...,m-1\} \\ \\ 2x_{i,j,2} + x_{i,j,1} + x_{u,j,3} + x_{v,j,3} \le 3,\ \text{and}\ 2x_{i,j,3} + x_{i,j,2} + x_{u,j,2} + x_{i,w,1} \le 3. \end{cases}$$

The best densities we have gotten are listed next:

n	1	2	3	4	5	6	7
$\rho_{3^3 4^2,n}$	0	$\frac{1}{2}$	$\frac{5}{9}$	$\frac{7}{12}$	$\frac{3}{5}$	$\frac{11}{18}$	$\ge \frac{4}{7}$

Of these arrangements, the case $m = n = 6$ (Figure 6 (b)) gives the highest density which is within $\frac{1}{126}$ to the upper bound shown above.

The same method used before for finding upper bounds does not give a better bound as the one we have proved in Theorem 3.2. For $m = n = 7$ we let $x_{i,0,2} = x_{i,0,3} = 0$ and $x_{0,j,1} = x_{0,j,2} = x_{0,j,3} = 0$ with $i \in \{0, 1, 2, ..., n-1\}$, $j \in \{0, 1, 2, ..., m-1\}$ and LPSolve gives a maximum of 72. So, the upper bound is $\frac{72}{147-(7(2)+7(3)-2)} = \frac{12}{19} > \frac{13}{21}$.

In [7] we have introduced the concept of deficiency function, $\delta_{i,j,k}$, and global deficiency Δ of an arrangement. Let us see how this works in this situation. We define

$$\delta_{i,j,k} = \begin{cases} \text{if } x_{i,j,k} = 1 & \begin{cases} x^*_{i,j,1} - 2 \text{ if } k = 1 \\ x^*_{i,j,k} - 1 \text{ if } k = 1 \text{ or } 2, \end{cases} \\ \\ \text{if } x_{i,j,k} = 0 & \begin{cases} x^*_{i,j,1} - 4 \text{ if } k = 1 \\ x^*_{i,j,k} - 3 \text{ if } k = 1 \text{ or } 2, \end{cases} \end{cases} \qquad \text{and}$$

$$\Delta = \frac{1}{|V|} \sum_{(i,j,k) \in V} \delta_{i,j,k}, \text{where } V \text{ is the set of vertices.}$$

We observe that the arrangement in Figure 6 (a) has $\Delta = \frac{-2}{12} = -\frac{1}{6}$ and the arrangement in Figure 6 (b) has $\Delta = -\frac{1}{18}$. We point out that the closer the global deficiency, Δ, is to zero, the bigger the density of an arrangement is. It seems like an arrangement in which $\Delta = 0$ is not possible.

3.3. **The case $\mathcal{T} = (3, 6, 3, 6)$.** In Figure 8 we see that the density in this case is at least $\frac{2}{3}$.

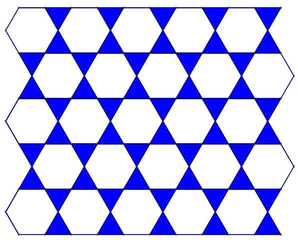


FIGURE 8. "Trivial" arrangement

For a vertex v corresponding to a hexagon we get $3x_v + x^*_v \leq 6$, and for v corresponding to a triangle we have $2x_v + x^*_v \leq 3$. We denote by H the sum of all x_v over all vertices corresponding to hexagons and by T the sum of all x_v over all vertices corresponding to triangles. These inequalities imply

$$3H + 3T \leq 6mn \text{ and } 2T + 6H \leq 6mn.$$

The system in $x = \frac{T}{3mn}$ and $y = \frac{H}{3mn}$ becomes

$$\begin{cases} x+y\le 2 \\ \\ x+3y\le 3,\ x\le \frac{2}{3},\ y\le \frac{1}{3}. \end{cases}$$

This system does not give anything new for $max(x+y)$. There is no clear good upper bound for this situation but it seems like $\frac{2}{3}$ is a good choice and one can use a reasonable short analysis to show that in a maximum configuration no hexagon can be in the domination set.

3.4. **The case $\mathcal{T}=(3,4,6,4)$.** The system in this case becomes a little more complicated since we have three different type of tilings, but the gives an upper bound which is strictly less than $\frac{2}{3}$.

Theorem 3.2. *The half-domination density for the tessellation $\mathcal{T}=(3,4,6,4)$ satisfies*

$$\rho_{(3,4,6,4)}\le \frac{19}{30}.$$

Proof. The inequalities defining the problem are given by

$$\begin{cases} 2x_v+x_v^*\le 3 \text{ for v corresponding to a triangle} \\ \\ 2x_v+x_v^*\le 4 \text{ for v corresponding to a square} \\ \\ 3x_v+x_v^*\le 6 \text{ for v corresponding to a hexagon.} \end{cases}$$

As before we introduce $T=\sum_{\text{v for triangle}} x_v$, $S=\sum_{\text{v for square}} x_v$, and $H=\sum_{\text{v for hexagon}} x_v$. The inequalities above give

$$\begin{cases} 2T+2S\le 3(2mn) \\ \\ 2S+(3T+6H)\le 4(3mn) \\ \\ 3H+2S\le 6mn, \end{cases} \quad or \quad \begin{cases} 2x+2y\le 1 \\ \\ 3x+2y+6z\le 2 \\ \\ 2y+3z\le 1, \end{cases}$$

where $x=\frac{T}{6mn}$, $y=\frac{S}{6mn}$ and $z=\frac{H}{6mn}$. The usual optimization methods give the maximum for $x+y+z$, under the above constrains and $x,y,z\in[0,1)$, to be attained for $x=\frac{1}{5}$, $y=\frac{3}{10}$ and $z=\frac{2}{15}$ and a value of $\frac{19}{30}$. ■

The variables we are going to use for writing the system for LPSolve are indexed as before $x_{i,j,k}$, with $i\in\{0,1,2,...,n-1\}$, $j\in\{0,1,2,...,m-1\}$ and $k\in\{1,2,3,4,5,6\}$ (see Figure 9). Given (i,j) as above, we let $u,v\in\{0,1,2,...,n-1\}$, and $w,t\in\{0,1,2,...,m-1\}$ such that $u\equiv i-1\ (mod\ n)$, $v\equiv i+1\ (mod\ n)$, $w\equiv j-1\ (mod\ m)$, and $t\equiv j+1\ (mod\ m)$. Then the system can be written in the following way (see Figure 9):

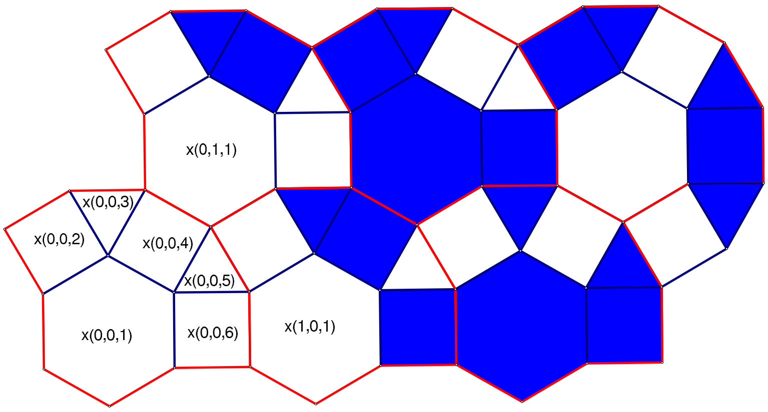
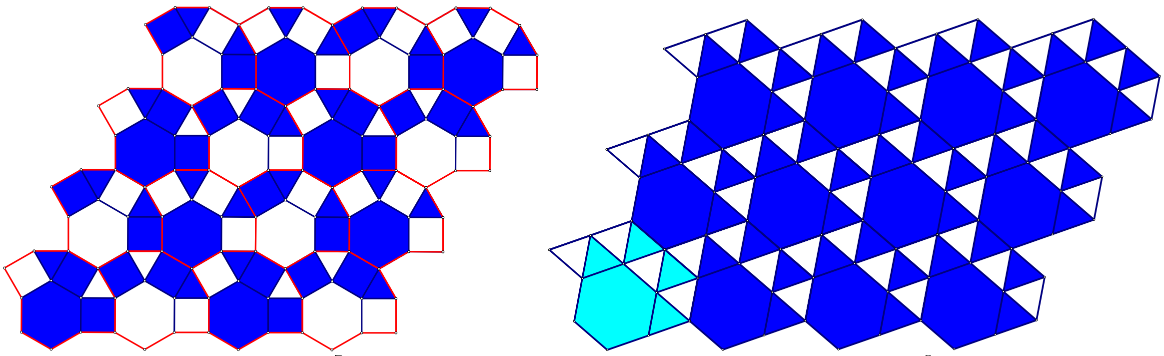


FIGURE 9. Variables convention

$$(6)\quad \begin{cases} 3x_{i,j,1}+x_{i,j,2}+x_{i,j,4}+x_{i,j,6}+x_{u,j,6}+x_{i,w,4}+x_{v,w,2}\leq 6 \\ 2x_{i,j,3}+x_{i,j,2}+x_{u,j,4}+x_{u,t,6}\leq 3, \quad 2x_{i,j,5}+x_{i,j,4}+x_{i,j,6}+x_{v,j,2}\leq 3, \\ 2x_{i,j,2}+x_{i,j,1}+x_{i,j,3}+x_{u,j,5}+x_{u,t,1}\leq 4, \quad 2x_{i,j,4}+x_{i,j,1}+x_{i,j,3}+x_{i,j,5}+x_{v,j,1}\leq 4, \text{ and} \\ 2x_{i,j,6}+x_{i,j,1}+x_{i,j,5}+x_{v,j,1}+x_{v,w,3}\leq 4. \end{cases}$$

Contrary to what is expected LPSolve takes more time to solve these systems even under one hundred variables. The best densities we have gotten are listed next:

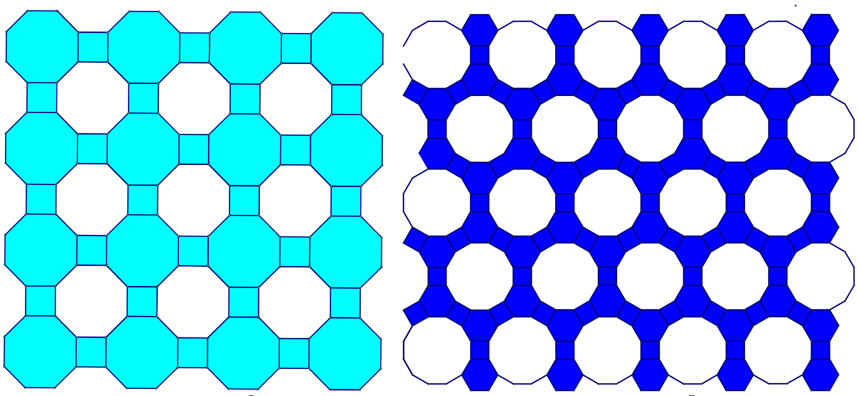
(m,n)	(1,1)	(2,2)	(3,3)	(4,4)
$\rho_{(3,4,6,4),(m,n)}$	$\frac{1}{2}$	$\frac{7}{12}$	$\frac{31}{54}$	$\frac{7}{12}$



(a) $\rho_{(6,4,3,4)}=\frac{7}{12}$ (b) $\rho_{(6,3,3,3,3)}=\frac{5}{9}$

FIGURE 10. Some distributions

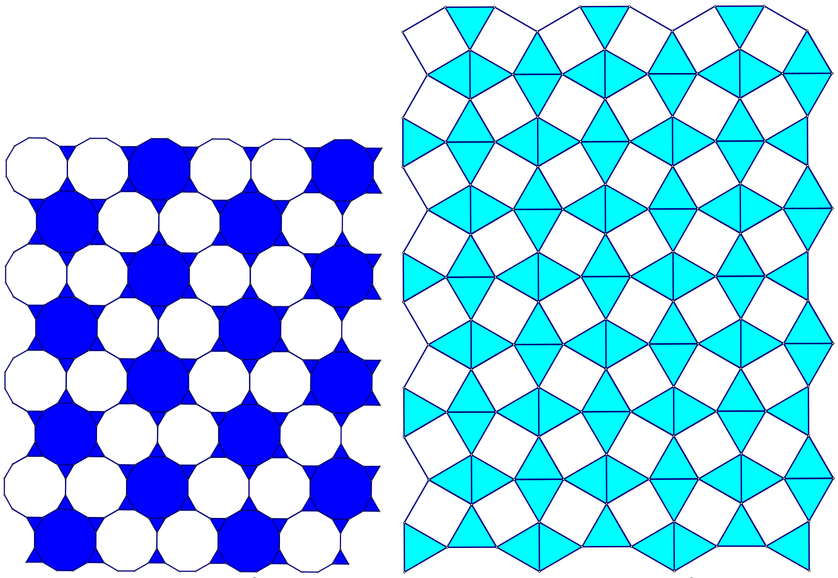
3.5. **The rest of the semi-regular tessellations.** We are going to include for the rest of the semi-regular tessellations only the most significant facts found but without proves. One case use the same methods to check them. For the semi-regular tessellations $(8^2, 4)$ and $(12, 6, 4)$ the following arrangements (Figure 11(a) and (b)) gives the best densities.



(a) $\rho_{(8^2,4)}=\frac{3}{4}$ (b) $\rho_{(12,6,4)}=\frac{5}{6}$

FIGURE 11. Best arrangements

We notice that the deficiency for each of the arrangements in Figure 11(a) and (b) is equal to zero.



(a) $\rho_{(12^2,3)}=\frac{3}{4}$ (b) $\rho_{(4,3,3,4,3)}=\frac{2}{3}$

FIGURE 12. Best arrangements

For the semi-regular tessellation $(12^2, 3)$, we have gotten the theoretical upper bound of $\frac{7}{9}$ and one of the arrangements as in Figure 12(a) which we think it is actually the sharp. In Figure 12(b), we have a sharp arrangement for tessellation $(4, 3, 3, 4, 3)$. Finally, for $\mathcal{T} = (6, 3, 3, 3, 3)$ we see see an arrangement of density $\frac{5}{9}$ in Figure 10 (b), but one can show that the best density is actually $\frac{2}{3}$ given by the distribution shown in Figure 2.

4. Conjectures and other comments

From what we have seen so far, there are some patterns that emerge. Given a *vertex transitive* infinite graph (for every two vertices, there exists a graph isomorphism mapping one vertex into the other) have half-domination arrangements which have rational best densities. It is not clear if such arrangements are *unique* (up to the isomorphisms of the graph) or there exist essentially different variations. In any case, we see that if the deficiency is zero, then the solution seems to be unique. If the deficiency is positive, one may expect to have more solutions and we have such an example in the case of the King's Graph (see [6]). The bigger the deficiency the higher the number of combinatorial possibilities that can result in maximum arrangements but, we conjecture that there are only finitely many of them. Results that show the exact number of such maximum arrangements are, nevertheless, at our interest in further investigations. However, we believe that the right methods to approach these questions successfully, even with the assistance of powerful computers, are yet to be discovered. Another path of investigations is to look into finding similar answers to k-dependence problems in all of the graphs studied here.

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