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SOLVING LINEAR-QUADRATIC BILEVEL PROGRAMMING PROBLEM USING KUHN-TUCKER CONDITIONS

Ritu Arora*
Department of Mathematics, Keshav Mahavidyalaya,
University of Delhi, India
and
Dr. S.R. Arora
Department of Mathematics, Hansraj College, Delhi University, India

ABSTRACT

In this paper, an algorithm is proposed to solve a bilevel programming problem in which the upper level objective function is linear fractional and the lower level is quadratic. The variables associated with both the level problems are related by linear constraints. On applying the Kuhn-Tucker conditions to the lower level problem and by making use of the complementarity condition and the upper level objective function, a fractional programming problem is formed. The optimum solution of this problem determines the optimum solution of the given bilevel programming problem. It is illustrated with the help of an example.

Keywords: Kuhn-Tucker conditions, Bilevel programming, Quadratic Programming, fractional programming.

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INTRODUCTION

The Bilevel Programming Problem (BLPP) is defined as

(BLPP) $\underset{X}{\text{Max}}\, f(X, Y)$

where Y solves

$\underset{Y}{\text{Max}}\, F(X, Y)$

subject to $(X, Y) \in S$.

where $S = \{(X, Y) : AX + BY \leq b,\ X, Y \geq 0\}$

In (BLPP), two decision makers are located at two different hierarchical levels, upper level and lower level, each controlling independently a separate set of decision variables. Both the levels are interested in optimizing their own benefits. The objectives at each level are conflicting in nature.

The problem (BLPP) can be thought of as a static version of the Stackelberg's leader-follower game in which a Stackelberg strategy is used by the higher level decision maker called the leader given the rational reaction of the lower level decision maker called the follower. Different solution methodologies have been proposed to solve (BLPP) such as parametric complimentary pivot algorithm, grid search algorithm and penalty function approach.

(BLPP) has been used by researchers in several fields ranging from economics to transportation engineering. (BLPP) is also used to model problems involving multiple decision makers. These problems include traffic signal optimization [3,4], and genetic algorithms [9]. (BLPP) has been developed and studied by Bialas and Karwan [6, 7] in the year 1982, 1984; Candler and Townsley [8] in 1982; Bard [2, 3, 4] in the year 1983, 84, 92 developed different techniques for solving (BLPP).

Based on the Kuhn-Tucker conditions and the duality theory, Wang et al. [12] in the year 1994 has derived necessary and sufficient optimality conditions for linear-quadratic bilevel programs.

A parametric method for solving bilevel programming problem has been discussed by Faisca, Dua, Rustem, Saraiva and Pistikopoulos [10] in the year 2007.

Here, in this paper, we have taken linear fractional function at the upper level and quadratic function at the lower level. A method is developed in which the lower level problem is replaced using the Kuhn-Tucker conditions, which when combined with the upper level problem forms a fractional programming problem with complementarity conditions. The solution of this problem satisfying the compementarity conditions gives the optimal solution of the Bilevel Programming Problem.

MATHEMATICAL FORMULATION

The linear fractional quadratic bilevel programming problem (LFQBPP) is given by

(LFQBPP) $$\underset{X_1}{\text{Max}}\, Z_1(X) = \frac{c_1X_1 + c_2X_2 + \alpha}{d_1X_1 + d_2X_2 + \beta}$$

where for a given X_1, X_2 solves

$$\underset{X_2}{\text{Max}}\, Z_2(X) = e^TX + \frac{1}{2}X^TQX$$

subject to $X \in S$,

where $S = \{X = (X_1, X_2) \in \mathbb{R}^{n_1+n_2} : A_1X_1 + A_2X_2 \leq b;\ X_1, X_2 \geq 0\}$

$c_1, d_1 \in \mathbb{R}^{n_1}$; $c_2, d_2 \in \mathbb{R}^{n_2}$; $\alpha, \beta \in \mathbb{R}$;

$A_1 \in \mathbb{R}^{m\times n_1}$; $A_2 \in \mathbb{R}^{m\times n_2}$; $b \in \mathbb{R}^m$; $e = (e_1, e_2) \in \mathbb{R}^{n_1+n_2}$

Q is an $((n_1 + n_2) \times (n_1 + n_2))$ symmetric positive semi-definite matrix.

Here, $S \subset \mathbb{R}^{n_1+n_2}$, defines the common constraint region and it is assumed that the feasible region S is closed and bounded. It is also assumed that $(d_1X_1 + d_2X_2 + \beta) > 0 \quad \forall (X_1, X_2) \in S$.

Define, $S(X_1) = \{X_2 \in \mathbb{R}^{n_2} ; (X_1, X_2) \in S\}$ gives the feasible region of the lower level problem, for a given X_1.

Consider the lower level problem, for a given X_1

$$\underset{X_2}{\text{Max}}\, Z_2(X) = e^T X + \frac{1}{2} X^T Q X$$

$$= e_1^T X_1 + e_2^T X_2 + \frac{1}{2}[X_1\, X_2]^T \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

subject to $A_2X_2 \le b - A_1X_1$

$X_2 \ge 0$.

Here, $f(X) = e_1^T X_1 + e_2^T X_2 + \frac{1}{2} X_1^T Q_{11} X_1 + X_1^T Q_{12} X_2 + \frac{1}{2} X_2^T Q_{22} X_2$.

$g(X) = b - A_1X_1 - A_2X_2 \ge 0$.

Define, the Lagrangian function $L(X, \lambda)$ as

$L(X, \lambda) = f(X) + \lambda^T g(X)$,

where $\lambda \ge 0$ is the vector of Lagrange's multipliers.

Applying the Kuhn Tucker conditions, we have

$$\frac{\partial L}{\partial X_2} \le 0 \Rightarrow e_2 + Q_{12}X_1 + Q_{22}X_2 - A_2^T\lambda \le 0 \tag{1}$$

$$\frac{\partial L}{\partial \lambda} \ge 0 \Rightarrow g(X) \ge 0 \Rightarrow b - A_1X_1 - A_2X_2 \ge 0 \tag{2}$$

$$\lambda^T \frac{\partial L}{\partial \lambda} = 0 \Rightarrow \lambda^T g(X) = 0 \Rightarrow \lambda^T(b - A_1X_1 - A_2X_2) = 0 \tag{3}$$

$$X_2^T \frac{\partial L}{\partial X_2} = 0 \Rightarrow X_2^T(e_2 + Q_{12}X_1 + Q_{22}X_2 - A_2^T\lambda) = 0 \tag{4}$$

In equation (2), introducing the surplus variable and converting the inequality into equality, we get

$$b - A_1X_1 - A_2X_2 - Iy = 0$$

or $$A_1X_1 + A_2X_2 + Iy = b \quad (5)$$

$$y \geq 0 \quad (6)$$

From equation (3) and (5), we get

$$\lambda^T y = 0 \quad (7)$$

In equation (1), introducing the slack variable, we get

$$e_2 + Q_{12}X_1 + Q_{22}X_2 - A_2^T\lambda + Iu = 0$$

or $$-Q_{12}X_1 - Q_{22}X_2 + A_2^T\lambda - Iu = e_2 \quad (8)$$

$$u \geq 0 \quad (9)$$

Using equations (4) and (8), we get

$$X_2^T u = 0 \quad (10)$$

Equations (1), (5), (7) and (10) gives the Kuhn-Tucker conditions corresponding to the lower level objective function. Thus, the given (LFQBPP) Problem becomes a Linear Fractional Programming Problem (LFPP) given by

(LFPP) $$\underset{X_1}{\text{Max}}\, Z_1(X) = \frac{c_1X_1 + c_2X_2 + \alpha}{d_1X_1 + d_2X_2 + \beta}$$

subject to $A_1X_1 + A_2X_2 + Iy = b$

$$-Q_{12}X_1 - Q_{22}X_2 + A_2^T\lambda - Iu = e_2 \quad (11)$$

$$X_1, X_2, \lambda, y, u \geq 0$$

with the condition $\lambda^T y = 0$ and $X_2^T u = 0$.

Here, I is the identity matrix of appropriate dimension and the condition $\lambda^T y = 0$ and $X_2^T u = 0$ represents the complementary condition.

As the objective function is linear fractional, therefore, it is both pseudoconcave and pseudoconvex and thus, its optimal solution will be at an extreme point. We are interested in finding that extreme point which satisfies the condition $\lambda^T y = 0$ and $X_2^T u = 0$.

Maximization of (LFPP)

The (LFPP) problem defined above by (11) can be re-written as

$$\underset{X_1}{\text{Max}}\, Z_1(X) = \frac{cX + \alpha}{dX + \beta}$$

subject to

$$\begin{bmatrix} A_1 & A_2 & I & O & O \\ -Q_{12} & -Q_{22} & O & A_2^T & -I \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ y \\ \lambda \\ u \end{bmatrix} = \begin{bmatrix} b \\ e_2 \end{bmatrix}$$

$X_1, X_2, y, \lambda, u \geq 0.$

where $c = (c_1, c_2) \in \mathbb{R}^{n_1+n_2}$; $d = (d_1, d_2) \in \mathbb{R}^{n_1+n_2}$;

$X = (X_1, X_2) \in \mathbb{R}^{n_1+n_2}$; $\alpha, \beta \in \mathbb{R}$, $e_2 \in \mathbb{R}^{n_2}$.

The above problem becomes

(LFBPP) $\underset{X_1}{\text{Max}}\, Z_1(X) = \dfrac{cX + \alpha}{dX + \beta}$

subject to

$UX = V$

$X \geq 0$

where $U = \begin{bmatrix} A_1 & A_2 & I & O & O \\ -Q_{12} & -Q_{22} & O & A_2^T & -I \end{bmatrix}$, $X = \begin{bmatrix} X_1 \\ X_2 \\ y \\ \lambda \\ u \end{bmatrix}$, $V = \begin{bmatrix} b \\ e_2 \end{bmatrix}$.

Here, $U \in \mathbb{R}^{(m+n_2)\times(n_1+2n_2+2m)}$, $X \in \mathbb{R}^{n_1+2n_2+2m}$, $V \in \mathbb{R}^{m+n_2}$

Suppose that we are given an extreme point of the feasible region with basis B, such that $X_B = B^{-1}V > 0$ and $X_N = 0$, where X_B is the basis vector and X_N is the non-basic vector. B is $(m + n_2)\times(m + n_2)$ invertible matrix.

Since the current point is an extreme point with $X_N = 0$, the non-basic variable cannot be decreased further, as it would violate the non-negativity restriction.

Letting

$$\begin{aligned} \mathbf{r}^T &= (r_B^T, r_N^T) = \nabla f(X)^T - \nabla_B f(X)^T B^{-1}U \\ &= [\nabla_B f(X)^T, \nabla_N f(X)^T - \nabla_B f(X)^T B^{-1}(B, N)] \\ &= [\nabla_B f(X)^T - \nabla_B f(X)^T B^{-1}B,\ \nabla_N f(X)^T - \nabla_B f(X)^T B^{-1}N)] \\ &= [0,\ \nabla_N f(X)^T - \nabla_B f(X)^T B^{-1}N] \end{aligned}$$

Here, r_N denote the non-basic components of the reduced gradient vector $\mathbf{r}^T$. so that $\mathbf{r}_N^T = \nabla_N f(X)^T - \nabla_B f(X)^T B^{-1}N$.

Entering Variable

Find r_s = Max $\{r_1 : r_1 \geq 0\}$, where r_i is the i-th component of r_N. The non-basic variable X_s is increased, and the basic variables are modified to maintain feasibility.

Departing Variable:

Determine the basic variable X_B, to lease the basis by the following minimum ratio test:

$$\frac{V_{B_r}}{Y_{r_j}} = \underset{1\leq i\leq m+n_2}{\mathrm{Min}} \left\{ \frac{V_{B_i}}{y_{ij}} :\quad Y_{ij} > 0 \right\}.$$

where $Y_j = B^{-1}u_j$, u_j is the j-th column of U.

Replace the variable X_B, by the variable X_j. Update the table correspondingly by pivoting at Y_{rj}.

Optimality Criterion

The optimal solution is obtained when $r_N \leq 0$ resulting in a Kuhn-Tucker point. The optimal solution of (LFPP) problem will maximize the objective function only if $\lambda^T y = 0$ and $X_2^T u = 0$. If $\lambda^T y \neq 0$ or $X_2^T u \neq 0$ find the next solution to the above problem, with the restricted basis entry, till $\lambda^T y = 0$ and $X_2^T u = 0$.

Convergence

Here, we have assumed that $X_B > 0$ for each extreme point. The above method moves from one extreme point to another extreme point. By the non-degeneracy assumption, the objective function strictly increases at each iteration so that the extreme points generated are distinct. There is only a finite number of these points and hence, the procedure stops in a finite number of steps.

Algorithm for Solving (LFQBPP) Problem

Step 1: Consider (LFQBPP) problem

For a given value of X_1, take

$$f(X) = e_1^T X_1 + e_2^T X_2 + \frac{1}{2} X_1^T Q_{11} X_1 + X_1^T Q_{12} X_2 + \frac{1}{2} X_2^T Q_{22} X_2 \text{ and}$$

$$g(X) = b - A_1X_1 - A_2X_2 \geq 0.$$

Define the Lagrangian function $L(X, \lambda) = f(X) + \lambda^T g(X)$. Apply the Kuhn Tucker conditions and convert (LFQBPP) problem to (LFPP) problem with the condition that $\lambda^T y = 0$ and $X_2^T u = 0$.

Step 2 : Remove the condition $\lambda^T y = 0$ and $X_2^T u = 0$ and solve the (LFPP). To solve the (LFPP) problem, find an initial basic feasible solution and express the basic variables in terms of non-basic variables.

Step 3: Compute the vector $\mathbf{r}_N^T = \nabla_N f(X_s)^T - \nabla_B f(X_s)^T B^{-1} N$.

If $r_N \leq 0$, the current point X_s is an optimal solution of (LFPP). Go to step 6. Otherwise, go to step 4.

Step 4 : Let $\mathbf{r}_s = \max \{r_i : r_i \geq 0\}$, where r_i is the i-th component of $\mathbf{r}_N$. Determine the basic variable X_{B_r} to leave the basis by the minimum ratio test defined as

$$\frac{V_{B_r}}{Y_{r_j}} = \underset{1 \leq i \leq m+n_2}{\text{Min}} \left\{ \frac{V_{B_i}}{Y_{ij}} : Y_{ij} > 0 \right\}$$

where $Y_j = B^{-1} u_j$; u_j is the jth column of U.

Go to Step 5.

Step 5 Replace the variable X_{B_r} by the variable X_j. Update the table correspondingly by pivoting at Y_{rj}. Let the current solution be X_{s+1}. Replace s by s + 1.

Go to step 3.

Step 6: (a) Check $\lambda^T y = 0$ and $X_2^T u = 0$.

If $\lambda^T y = 0$ and $X_2^T u = 0$, then this solution will be the optimal solution of (LFQBPP).

(b) If $\lambda^T \neq 0$ or $X_2^T u \neq 0$, find such a solution with restricted basis entry, such that the condition $\lambda^T y = 0$ and $X_2^T u = 0$ is satisfied. It will be the optimal solution of (LFQBPP).

Example: Consider the following linear-quadratic bilevel programming problem:

(LFQBPP): $\underset{x_1}{\text{Max}}\, Z_1(x_1, x_2, x_3) = \dfrac{2 - x_1 - x_2 + 2x_3}{4 + x_1 + 3x_3}$

where (x_2, x_3) solves

$$\underset{x_2, x_3}{\text{Max}}\, Z_2(x_1, x_2, x_3) = x_1^2 + 2x_2^2 + x_3^2 + 2x_1x_2 + 2x_2x_3 + 2x_1 - x_2 - 8x_3$$

subject to

$$x_1 + 2x_2 + x_3 \le 10$$

$$x_1 - x_3 \le 2$$

$$3x_1 + 5x_2 \le 4$$

$$x_1, x_2, x_3 \ge 0.$$

Solution: For a given value of x_1, define the Lagrangian function,

$$L(x_1, x_2, x_3, \lambda) = f(x_1, x_2, x_3) + \lambda^T g(x_1, x_2, x_3) \qquad (1)$$

where $f(x_1, x_2, x_3) = x_1^2 + 2x_2^2 + x_3^2 + 2x_1x_2 + 2x_2x_3 + 2x_1 - x_2 - 8x_3$

$$g(x_1, x_2, x_3) = \begin{cases} 10 - x_1 - 2x_2 - x_3 & \ge 0 \\ 2 - x_1 + x_3 & \ge 0 \\ 4 - 3x_1 - 5x_2 & \ge 0 \end{cases} \qquad (2)$$

Here, $\lambda = (\lambda_1, \lambda_2, \lambda_3) \ge 0$ is the vector of Lagrange's multipliers, therefore, equation (1) becomes

$$L(x_1, x_2, x_3, \lambda) = (x_1^2 + 2x_2^2 + x_3^2 + 2x_1x_2 + 2x_2x_3 + 2x_1 - x_2 - 8x_3)$$
$$+\lambda_1(10 - x_1 - 2x_2 - x_3) + \lambda_2(2 - x_1 + x_3) + \lambda_3(4 - 3x_1 - 5x_2)$$

Applying the Kuhn Tucker conditions, we have

$$\begin{aligned} &\frac{\partial L}{\partial x_2} \le 0 \Rightarrow 2x_1 + 4x_2 + 2x_3 - 1 - 2\lambda_1 - 5\lambda_3 \le 0 \\ &\frac{\partial L}{\partial x_3} \le 0 \Rightarrow 2x_2 + 2x_3 - \lambda_1 + \lambda_2 - 8 \le 0 \end{aligned} \qquad (3)$$

$$\frac{\partial L}{\partial \lambda_1} \geq 0 \Rightarrow 10 - x_1 - 2x_2 - x_3 \geq 0$$

$$\frac{\partial L}{\partial \lambda_2} \geq 0 \Rightarrow 2 - x_1 + x_3 \geq 0 \qquad (4)$$

$$\frac{\partial L}{\partial \lambda_3} \geq 0 \Rightarrow 4 - 3x_1 - 5x_2 \geq 0$$

$$x_2 \frac{\partial L}{\partial x_2} = 0 \Rightarrow x_2(2x_1 + 4x_2 + 2x_3 - 2\lambda_1 - 5\lambda_3 - 1) = 0$$

$$x_3 \frac{\partial L}{\partial x_3} = 0 \Rightarrow x_3(2x_2 + 2x_3 - \lambda_1 + \lambda_2 - 8) = 0$$

$$\lambda_1 \frac{\partial L}{\partial \lambda_1} = 0 \Rightarrow \lambda_1(10 - x_1 - 2x_2 - x_3) = 0 \qquad (5)$$

$$\lambda_2 \frac{\partial L}{\partial \lambda_2} = 0 \Rightarrow \lambda_2(2 - x_1 + x_3) = 0$$

$$\lambda_3 \frac{\partial L}{\partial \lambda_3} = 0 \Rightarrow \lambda_3(4 - 3x_1 - 5x_2) = 0$$

Introducing slack variables in equations (3) and (4) and using equations (5), (LFPP) problem can be written as

(LFPP) $\quad \underset{x_1}{\text{Max}}\, Z_1(x_1, x_2, x_3) = \dfrac{2 - x_1 - x_2 + 2x_3}{4 + x_1 + 3x_3}$

subject to

$$x_1 + 2x_2 + x_3 + y_1 = 10$$

$$x_1 - x_3 + y_2 = 2$$

$$3x_1 + 5x_2 + y_3 = 4$$

$$2x_1 + 4x_2 + 2x_3 - 2\lambda_1 - 5\lambda_3 + u_1 = 1$$

$$2x_2 + 2x_3 - \lambda_1 + \lambda_2 + u_2 = 8$$

$x_1, x_2, x_3, y_1, y_2, y_3, \lambda_1, \lambda_2, \lambda_3, u_1, u_2 \geq 0$,

with $x_2u_1 = 0$, $x_3u_2 = 0$, $\lambda_1y_1 = 0$, $\lambda_2y_2 = 0$, $\lambda_3y_3 = 0$

Relax the given problem, the initial feasible solution of the above problem is given by

	$\nabla f(X^1)$	–2/3	–1/3	2/9	0	0	0	0	0	0	0	0
V_B	X_B	x_1	x_2	x_3	y_1	y_2	y_3	λ_1	λ_2	λ_3	u_1	u_2
y_1	10	1	2	1	1	0	0	0	0	0	0	0
y_2	2	1	0	–1	0	1	0	0	0	0	0	0
y_3	4	3	5	0	0	0	1	0	0	0	0	0
u_1	1	2	4	2	0	0	0	–2	0	–5	1	0
u_2	8	0	2	2	0	0	0	–1	1	0	0	1
	r	–2/3	–1/3	2/9	0	0	0	0	0	0	0	0

We have X^1 = (0, 0, 0, 10,2, 4, 0, 0, 0, 1, 8).

To find the entering variable, find $\nabla f(X)$ by calculating the partial derivatives of the objective function as,

$$\nabla f(X) = \begin{pmatrix} \dfrac{x_2 - 5x_3 - 6}{(3 + x_1 + 3x_3)^2} \\ \dfrac{-1}{3 + x_1 + 3x_3} \\ \dfrac{5x_1 + 3x_2 + 2}{(3 + x_1 + 3x_3)^2} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \nabla f(X') = \begin{pmatrix} -2/3 \\ -1/3 \\ 2/9 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$r_N^T = (r_1, r_2, r_3, r_7, r_8, r_9)$$

$$= \nabla_N f(X')^T - \nabla_B f(X')^T B^{-1} N$$

$$= \left(\frac{-2}{3}, \frac{-1}{3}, \frac{2}{9}, 0, 0, 0\right) - (0,0,0,0,0) \begin{bmatrix} 1 & 2 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 3 & 5 & 0 & 0 & 0 & 0 \\ 2 & 4 & 2 & -2 & 0 & -5 \\ 0 & 2 & 2 & -1 & 1 & 0 \end{bmatrix}$$

$$= \left(\frac{-2}{3}, \frac{-1}{3}, \frac{2}{9}, 0, 0, 0\right)$$

Entering variable is given by

$$r_s = \max\{r_i : r_i \geq 0\} = \max\left\{\frac{2}{9}, 0, 0, 0\right\} = \frac{2}{9}$$

$\therefore$ Enter x_3.

Departing variable is given by $\text{Min}\left\{\frac{10}{1}, \frac{1}{2}, \frac{8}{2}\right\} = \frac{1}{2}$

$\therefore$ Depart u_1.

The next basic feasible solution is given by

	$\nabla f(X^2)$	–34/81	–2/9	8/81	0	0	0	0	0	0	0	0
V_B	X_B	x_1	x_2	x_3	y_1	y_2	y_3	λ_1	λ_2	λ_3	u_1	u_2
y_1	19/2	0	–4	0	1	0	0	1	0	5/2	–1/2	0
y_2	5/2	2	2	0	0	1	0	–1	0	–5/2	½	0
y_3	4	3	5	0	0	0	1	0	0	0	0	0
x_3	1/2	1	2	1	0	0	0	–1	0	–5/2	1/2	0
u_2	7	–2	–2	0	0	0	0	1	1	5	–1	1
	r	–42/81	–34/81	0	0	0	0	8/81	0	20/81	–4/81	0

$$X^2 = \left(0, 0, \frac{1}{2}, \frac{19}{2}, \frac{5}{2}, 4, 0, 0, 0, 0, 7\right)$$

$$\nabla f(X^2) = \left(\frac{-34}{81}, \frac{-2}{9}, \frac{8}{81}, 0, 0, 0, 0, 0, 0, 0, 0\right)$$

By the same method as above, enter λ_3 and depart u_2.

Proceed as above, the final feasible solution is given by

	$\nabla f(X^5)$	–56/1089	–1/33	2/1089	0	0	0	0	0	0	0	0
V_B	X_B	x_1	x_2	x_3	y_1	y_2	y_3	λ_1	λ_2	λ_3	u_1	u_2
u_1	5	4	–4	0	2	0	0	0	–2	–5	1	–2
y_2	12	2	–2	0	1	1	0	0	0	0	0	0
y_3	4	3	5	0	0	0	1	0	0	0	0	0
x_3	10	1	–2	1	1	0	0	0	0	0	0	0
λ_1	12	2	–6	0	2	0	0	1	–1	0	0	–1
	r	–58/1089	–319/11979	0	0	0	0	0	–2/1089	0	0	0

X^5 = (0, 0, 10, 5, 12, 4, 12, 0, 0 0, 0).

$$\nabla f(X^5) = \left(\frac{-56}{1089}, \frac{-1}{33}, \frac{2}{1089}, 0, 0, 0, 0, 0, 0, 0, 0\right)$$

Since $r_N^T \le 0$, the above solution is an optimal solution. Also $x_2u_1 = 0$, $x_3u_2 = 0$,$\lambda_1 y_1 = 0$, $\lambda_2 y_2 = 0$ and $\lambda_3 y_3 = 0$, therefore, the optimal solution will maximize the objective function.

The solution of the (LFQBPP) problem is given by

$x_1 = 0$, $x_2 = 0$, $x_3 = 10$ with Max $Z_1 = \frac{11}{17}$ = and Max $Z_2 = 20$.

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