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# AN ITERATIVE METHOD FOR A FINITE FAMILY OF VARIATIONAL INCLUSIONS, EQUILIBRIUM PROBLEMS AND FIXED POINT PROBLEMS IN HILBERT SPACES*

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Abstract. In this paper, we propose an iterative algorithm approximating a common element of the set of solutions of a finite family of variational inclusions, of solutions of equilibrium problems and of the set of fixed points of nonexpansive mappings in a Hilbert space. We prove the strong convergence of the proposed iterative algorithm to the unique solution of a variational inequality, which is the optimality condition for a minimization problem. Our results extend and generalize related results.

**Key words.** Equilibrium problem, Variational inclutions, Maximal monotone mapping, $W$-mapping, Inverse-strongly monotone.

**AMS Subject Classifications:** 47H05; 47H07; 47H10.

## 1. Introduction

Let $H$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. $2^H$ denotes the family of all the nonempty subsets of $H$. Let $C$ be a nonempty closed convex subset of $H$.

Let $F$ be a bifunction from $C \times C$ to $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem for $F : C \times C \to \mathbb{R}$ is to find $x \in C$ such that

$$F(x, y) \geq 0, \quad \forall y \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $EP(F)$. The problem (1.1) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, Nash equilibrium problem in noncooperative games, and others.

Recall that a mapping $S$ of a closed convex subset $C$ into itself is nonexpansive if there holds that $\|Sx - Sy\| \leq \|x - y\|, \forall x, y \in C$. $F(S) = \{x \in H : Sx = x\}$ is the set of fixed points of mapping $S$. A mapping $f : C \to C$ is called contractive if there exists a constant $\alpha \in (0, 1)$ such that $\|fx - fy\| \leq \alpha \|x - y\|, \forall x, y \in C$.

Let $A : H \to H$ be a single-valued nonlinear mapping and $M : H \to 2^H$ be a set-valued mapping. The variational inclusion is to find a point $u \in H$ such that

$$\theta \in A(u) + M(u) \tag{1.2}$$

where $\theta$ is the zero vector in $H$. The set of solutions of problem (1.2) is denoted by $I(A, M)$. If $H = \mathbb{R}^m$, then problem (1.2) becomes the generalized equation introduced by Robinson [16]. If $A = 0$, then problem (1.2) becomes the inclusion problem introduced by Rockafellar [17]. If

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$M = \partial\phi : H \to 2^H$, where $\phi : H \to \mathbb{R} \cup \{+\infty\}$ is a proper convex lower semicontinuous function and $\partial\phi$ is the sub-differential of $\phi$, then the variational inclusion problem (1.2) is equivalent to called the mixed quasi-variational inequality which is to find $u \in H$ such that

$$\langle Au, v - u\rangle + \phi(y) - \phi(u) \geq 0, \quad \forall y \in H, \tag{1.3}$$

(see, e.g., [13]). If $M = \partial\delta_C$, where $C$ is a nonempty closed convex subset of $H$ and $\delta_C : H \to [0, \infty]$ is the indicator function of $C$, then the variational inclusion problem (1.2) is equivalent to variational inequality problem

$$\langle Au, v - u\rangle \geq 0, \quad \forall v \in C, \tag{1.4}$$

(see, e.g., [11]). More generally, we can have a finite family of variational inclusions

$$\text{find}\, u \in H \,\text{such that}\, \theta \in A_i(u) + M_i(u), \quad \forall i = 1, 2, \ldots, N. \tag{1.5}$$

In [1], it is shown that in the case of a single variational inclusion, the formulation provides a convenient framework for the unified study of optimal solutions in many optimization-related areas covering mathematical programming, complementarity, variational inequalities, optimal control, mathematical economics, equilibria, game theory, and so forth.

The formulation (1.5) extends this formalism to a finite family of variational inclusions covering, in particular, various forms of feasibility problems (see, e.g., [3]).

Mapping $W_n$ has been intensively studied and applied to develop various iterative algorithms for finding common solutions of fixed points of a finite family of nonexpansive mappings and of other problems (see, e.g., [2, 7, 21, 22]). Since under suitable conditions (to be stated precisely in Section 2), $J_{M,\lambda}(I - \lambda A)$ is a nonexpansive mapping, we can introduce following mapping $W_n$ for a finite family of variational inclusions (1.5).

$$\begin{aligned}
&U_{n,1} = t_{n,1} J^1_{\lambda_{1,n}}(I - \lambda_{1,n}A_1) + (1 - t_{n,1})I,\\
&U_{n,2} = t_{n,2} J^2_{\lambda_{2,n}}(I - \lambda_{2,n}A_2)U_{n,1} + (1 - t_{n,2})I,\\
&\vdots\\
&U_{n,N-1} = t_{n,N-1} J^{N-1}_{\lambda_{N-1,n}}(I - \lambda_{N-1,n}A_{N-1})U_{n,N-2} + (1 - t_{n,N-1})I,\\
&W_n = U_{n,N} = t_{n,N} J^N_{\lambda_{N,n}}(I - \lambda_{N,n}A_N)U_{n,N-1} + (1 - t_{n,N})I,
\end{aligned} \tag{1.6}$$

where $A_i : H \to H$ is an $\alpha_i$-inverse-strongly monotone mapping, $M_i : H \to 2^H$ is a maximal monotone mapping and the resolvent operator $J^i_{\lambda_{i,n}}$ associated with $M_i$ is defined by

$$J^i_{\lambda_{i,n}}(u) = (I + \lambda_{i,n}M_i)^{-1}(u), \quad u \in H, \quad i \in \{1, \ldots, N\},$$

for $\lambda_{i,n} > 0$. Such a mapping $W_n$ is called the $W_n$-mapping generated by $\{J^k_{\lambda_{k,n}}(I - \lambda_{k,n}A_k)\}_{k=1}^N$ and $\{t_{n,k}\}_{k=1}^N$. Nonexpansivity of $J^k_{\lambda_{k,n}}(I - \lambda_{k,n}A_k)$ yields the nonexpansivity of mapping $W_n$.

Recently, Zhang et al. [23], Peng et al. [14], Cholamjiak and Suantai [5], and Plubtieng and Sriprad [15] studied variational inclusions and presented strong convergent results. Plubtieng and Sriprad [15] proposed the following iterative scheme for finding a common element of the set of solutions to the problem (1.2), the set of solutions of an equilibrium problem, and the set of fixed points of nonexpansive mappings $S_n$ in Hilbert space. Starting with $x_1 \in H$, define sequence $\{x_n\}$, $\{y_n\}$, and $\{u_n\}$ by

$$\begin{aligned}
&F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad y \in H,\\
&x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n B)S_n y_n,\\
&y_n = J_{M,\lambda}(u_n - \lambda A u_n), \quad \forall n \geq 0
\end{aligned}$$

where $B$ be a strongly bounded linear operator on $H$. They proved that under certain appropriate conditions imposed on $\{\alpha_n\}$ and $\{r_n\}$, the sequences $\{x_n\}$, $\{y_n\}$, and $\{u_n\}$ converge strongly to $z \in (\bigcap_{i=1}^{\infty} F(S_i)) \cap EP(F) \cap I(A, M)$.

In this paper, inspired and motivated by [23, 14, 5, 15], we introduce an iterative scheme for finding a common element of the set of solutions of a finite family of variational inclusions problems (1.5) with multi-valued maximal monotone mappings and inverse-strongly monotone mappings, the set of solutions of an equilibrium problem and the set of fixed points of nonexpansive mappings in Hilbert space. Starting with an arbitrary point $x_1 \in H$, define sequences $\{x_n\}$ and $\{u_n\}$ by

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad y \in H,$$
$$x_{n+1} = \epsilon_n \gamma f(x_n) + \beta x_n + ((1-\beta)I - \epsilon_n B)S_n W_n u_n,$$

for all $n \in \mathbb{N}$, where $\epsilon_n \in (0, 1)$, $\{r_n\} \subset (0, \infty)$, $B$ be a strongly bounded linear operator on $H$, $\{S_n\}$ is a sequence of nonexpansive mappings on $H$ and mapping $W_n$ is defined by (1.6). Under suitable conditions, we prove that the sequences $\{x_n\}$, $\{u_n\}$ and $\{W_n u_n\}$ converge strongly to $x \in \Omega := (\bigcap_{k=1}^{N} I(A_k, M_k)) \cap EP(F) \cap (\bigcap_{i=1}^{\infty} F(S_i))$ which is the unique solution of the variational inequality

$$\langle (B - \gamma f)x^*, x - x^*\rangle \geq 0 \quad \forall x \in \Omega, \tag{1.7}$$

Variational inequality (1.7) is the optimality condition for the minimization problem.

$$\min_{x\in\Omega} \frac{1}{2}\langle Bx, x\rangle - h(x),$$

where $h$ is a potential function for $\gamma f$. Our results extend and improve some corresponding results in [23, 14, 5, 15].

## 2. Preliminaries

Let $C$ be a closed convex subset of $H$. Recall that the (nearest point) projection $P_C$ from $H$ onto $C$ assigns to each $x \in H$ the unique point $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \min_{y\in C} \|x - y\|.$$

The following lemma characterizes the projection $P_C$.

**Lemma 2.1.** ([19]) Given $x \in H$ and $y \in C$. Then $P_C x = y$ if and only if there holds the inequality

$$\langle x - y, y - z\rangle \geq 0 \quad \forall z \in C.$$

Recall that a mapping $A : H \to H$ is called $\alpha$-inverse strongly monotone, if there exists an $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2, \quad \forall x, y \in H.$$

Let $I$ be the identity mapping on $H$. It is well known that if $A : H \to H$ is an $\alpha$-inverse strongly monotone and $0 < \lambda \leq 2\alpha$, then $I - \lambda A$ is a nonexpansive mapping.

A set-valued mapping $M : H \to 2^H$ is called monotone if for all $x, y \in H$, $f \in Mx$, and $g \in My$ imply $\langle x - y, f - g\rangle \geq 0$. A monotone mapping $M : H \to 2^H$ is maximal if its graph $G(M) := \{(f, x) \in H \times H | f \in M(x)\}$ of $M$ is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping $M$ is maximal if and only if for $(x, f) \in H \times H$, $\langle x - y, f - g\rangle \geq 0$ for every $(y, g) \in G(M)$ implies $f \in Mx$.

Let the set-valued mapping $M : H \to 2^H$ be maximal monotone. It is worth mentioning that the resolvent operator $J_{M,\lambda}$ associated with $M$ is single-valued, nonexpansive, and 1-inverse strongly monotone, (see, e.g., [4]), and that a solution of problem (1.2) is a fixed point of the operator $J_{M,\lambda}(I - \lambda A)$ for all $\lambda > 0$ (see, e.g., [10]). Therefore, a solution of problem (1.5) is an element of common set of fixed points of the operators $J_{M_i,\lambda}(I - \lambda A_i)$, $i \in \{1, \dots, N\}$, for all $\lambda > 0$.

**Lemma 2.2.** ([18]) Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space $X$ and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n\to\infty} \beta_n$ and $\limsup_{n\to\infty} \beta_n < 1$. Suppose

$$x_{n+1} = \beta_n x_n + (1 - \beta_n) z_n$$

for all integers $n \geq 0$ and

$$\limsup_{n\to\infty} (\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Then $\lim_{n\to\infty} \|x_n - z_n\| = 0$.

**Lemma 2.3.** ([20]) Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the property:

$$a_{n+1} \leq (1 - \gamma_n) a_n + \gamma_n \beta_n, \quad n \geq 0$$

where $\{\gamma_n\}$ is a sequence in (0,1) and $\{\beta_n\}$ is a sequence in $\mathbb{R}$ such that

(i) $\sum_{n=1}^{\infty} \gamma_n = +\infty$;

(ii) $\limsup_{n\to\infty} \beta_n \leq 0$ or $\sum_{n=1}^{\infty} |\gamma_n \beta_n| < +\infty$.

Then $\lim_{n\to\infty} a_n = 0$.

**Lemma 2.4.** *(The Resolvent Identity) Let $M$ be a maximal monotone operator. For $\lambda > 0$, $\mu > 0$ and $x \in H$,*

$$J_{M,\lambda} x = J_{M,\mu} \left( \frac{\mu}{\lambda} x + \left(1 - \frac{\mu}{\lambda}\right) J_{M,\lambda} x \right).$$

**Lemma 2.5.** *Let $A$ be $\alpha$-inverse-strongly-monotone and $M$ be a maximal monotone operator. For $\lambda > 0$, $\mu > 0$ and $x \in H$,*

$$\|J_{M,\lambda}(I - \lambda A)x - J_{M,\mu}(I - \mu A)x\| \leq \left|1 - \frac{\mu}{\lambda}\right| (\|J_{M,\lambda}(I - \lambda A)x\| + \|x\|).$$

*Proof.* From Lemma 2.4, we have

$$\begin{aligned}
&\|J_{M,\lambda}(I - \lambda A)x - J_{M,\mu}(I - \mu A)x\| \\
&= \|J_{M,\mu} \left( \frac{\mu}{\lambda} I + \left(1 - \frac{\mu}{\lambda}\right) J_{M,\lambda} \right) (I - \lambda A)x - J_{M,\mu}(I - \mu A)x\| \\
&\leq \|\frac{\mu}{\lambda}(I - \lambda A)x + \left(1 - \frac{\mu}{\lambda}\right) J_{M,\lambda}(I - \lambda A)x - (I - \mu A)x\| \\
&\leq \left|1 - \frac{\mu}{\lambda}\right| (\|J_{M,\lambda}(I - \lambda A)x\| + \|x\|).
\end{aligned}$$

□

**Lemma 2.6.** ([12]) Assume that $A$ is a strongly positive linear bounded operator on a Hilbert space $H$ with coefficient $\bar{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho\bar{\gamma}$.

**Lemma 2.7.** ([8]) Let $C$ be a nonempty closed convex subset of $H$ and $F : C \times C \to \mathbb{R}$ satisfy following conditions:

(A1) $F(x, x) = 0$, $\forall x \in C$;

(A2) $F$ is monotone, i.e., $F(x,y)+F(y,x)\le 0$, $\forall x,y\in C$;
(A3) $\limsup_{t\to 0^+} F(tz+(1-t)x,y)\le F(x,y)$, $\forall x,y,z\in C$;
(A4) for each $x\in C$, $F(x,\cdot)$ is convex and lower semicontinuous.

For $x\in C$ and $r>0$, set $T_r: H\to C$ to be

$$T_r(x)=\Big\{z\in C: F(z,y)+\frac{1}{r}\langle y-z,z-x\rangle\ge 0,\quad \forall y\in C\Big\}.$$

Then $T_r$ is well defined and the following hold:

1. $T_r$ is single-valued;
2. $T_r$ is firmly nonexpansive [9], i.e., for any $x,y\in E$,
$$\|T_rx-T_ry\|^2\le\langle T_rx-T_ry,x-y\rangle;$$
3. $F(T_r)=EP(F)$;
4. $EP(F)$ is closed and convex.

By the proof of Lemma 5 in [6], we have following lemma.

**Lemma 2.8.** *Let $C$ be a nonempty closed convex subset of a Hilbert space $H$ and $F: C\times C\to\mathbb{R}$ be a bifunction. Let $x\in C$ and $r_1,r_2\in(0,\infty)$. Then*

$$\|T_{r_1}x-T_{r_2}x\|\le\left|1-\frac{r_2}{r_1}\right|(\|T_{r_1}x\|+\|x\|). \tag{2.1}$$

From the definition 2.6 given by Colao, Marino and Xu [7], we can give following definition.

**Definition 2.9.** Let $C$ be a nonempty convex subset of a Hilbert space $H$. Let $A_i: H\to H$, $i\in\{1,\dots,N\}$ be $\alpha_i$-inverse-strongly monotone mappings and $M_i: H\to 2^H$, $i\in\{1,\dots,N\}$ be maximal monotone mappings. Let $t_1,\cdots,t_N$ be real numbers such that $0\le t_i\le 1$ and $\lambda_i\in(0,2\alpha_i]$, $i\in\{1,\dots,N\}$. We define a mapping $W$ of $C$ into itself as follows:

$$\begin{aligned}
U_1&=t_1J^1_{\lambda_1}(I-\lambda_1A_1)+(1-t_1)I,\\
U_2&=t_2J^2_{\lambda_2}(I-\lambda_2A_2)U_1+(1-t_2)I,\\
&\vdots\\
U_{N-1}&=t_{N-1}J^{N-1}_{\lambda_{N-1}}(I-\lambda_{N-1}A_{N-1})U_{N-2}+(1-t_{N-1})I,\\
W=U_N&=t_NJ^N_{\lambda_N}(I-\lambda_NA_N)U_{N-1}+(1-t_N)I.
\end{aligned} \tag{2.2}$$

Such a mapping $W$ is called the $W$-mapping generated by $J^1_{\lambda_1}(I-\lambda_1A_1),\dots,J^N_{\lambda_N}(I-\lambda_NA_N)$ and $t_1,\dots,t_N$.

**Lemma 2.10.** *Let $C$ be a nonempty convex set of a Hilbert space, $A_i: H\to H$, $i\in\{1,2,\dots,N\}$ be $\alpha_i$-inverse-strongly monotone mappings and $M_i: H\to 2^H$, $i\in\{1,2,\dots,N\}$ be maximal monotone mappings. Let $\{t_{n,i}\}_{i=1}^N$ be sequences in $[0,1]$ such that $t_{n,i}\to t_i$ and $\{\lambda_{i,n}\}$ be sequences such that $\lambda_{i,n}\in(0,2\alpha_i]$ and $\lambda_{i,n}\to\lambda_i$, $\lambda_i\in(0,2\alpha_i]$, $(i=1,\dots,N)$. Moreover for every $n\in\mathbb{N}$, let $W$ be the $W$-mappings generated by $\{J^i_{\lambda_i}(I-\lambda_iA_i)\}_{i=1}^N$ and $t_1,\dots,t_N$ and $W_n$ be the $W_n$-mappings generated by $\{J^i_{\lambda_{i,n}}(I-\lambda_{i,n}A_i)\}_{i=1}^N$ and $t_{n,1},\dots,t_{n,N}$. Then for every $x\in C$, it follows that*

$$\lim_{n\to\infty}\|W_nx-Wx\|=0. \tag{2.3}$$

*Proof.* Let $x \in C$. For $k \in \{1, \dots, N\}$, $U_k$ and $U_{n,k}$ be generated by $J^1_{\lambda_1}(I-\lambda_1 A_1), \dots, J^N_{\lambda_N}(I-\lambda_N A_N)$ and $t_1, \dots, t_N$ and $J^1_{\lambda_{1,n}}(I-\lambda_{1,n}A_1), \dots, J^N_{\lambda_{N,n}}(I-\lambda_{N,n}A_N)$ and $t_{n,1}, \dots, t_{n,N}$ respectively, as in Definition 2.8. From Lemma 2.5, we have

$$\begin{aligned}
&\|U_{n,1}x - U_1 x\| \\
&= \|t_{n,1}J^1_{\lambda_{1,n}}(I-\lambda_{1,n}A_1)x + (1-t_{n,1})x - t_1 J^1_{\lambda_1}(I-\lambda_1 A_1)x - (1-t_1)x\| \\
&= \|t_{n,1}(J^1_{\lambda_{1,n}}(I-\lambda_{1,n}A_1)x - J^1_{\lambda_1}(I-\lambda_1 A_1)x) + (t_{n,1}-t_1)(J^1_{\lambda_1}(I-\lambda_1 A_1)x - x)\| \\
&\le \|J^1_{\lambda_{1,n}}(I-\lambda_{1,n}A_1)x - J^1_{\lambda_1}(I-\lambda_1 A_1)x\| + |t_{n,1}-t_1|\|J^1_{\lambda_1}(I-\lambda_1 A_1)x - x\| \\
&\le \left|1-\frac{\lambda_{1,n}}{\lambda_1}\right|(\|J^1_{\lambda_1}(I-\lambda_1 A_1)x\| + \|x\|) + |t_{n,1}-t_1|\|J^1_{\lambda_1}(I-\lambda_1 A_1)x - x\|.
\end{aligned}$$

Let $k \in \{2, ..., N\}$, then

$$\begin{aligned}
&\|U_{n,k}x - U_k x\| \\
&= \|t_{n,k}J^k_{\lambda_{k,n}}(I-\lambda_{k,n}A_k)U_{n,k-1}x + (1-t_{n,k})x - t_k J^k_{\lambda_k}(I-\lambda_k A_k)U_{k-1}x \\
&\quad - (1-t_k)x\| \\
&= \|t_{n,k}\left(J^k_{\lambda_{k,n}}(I-\lambda_{k,n}A_k)U_{n,k-1}x - J^k_{\lambda_{k,n}}(I-\lambda_{k,n}A_k)U_{k-1}x\right) \\
&\quad + t_{n,k}\left(J^k_{\lambda_{k,n}}(I-\lambda_{k,n}A_k)U_{k-1}x - J^k_{\lambda_k}(I-\lambda_k A_k)U_{k-1}x\right) \\
&\quad + (t_{n,k}-t_k)\left(J^k_{\lambda_k}(I-\lambda_k A_k)U_{k-1}x - x\right)\| \\
&\le \|U_{n,k-1}x - U_{k-1}x\| + \left|1-\frac{\lambda_{k,n}}{\lambda_k}\right|(\|J^k_{\lambda_k}(I-\lambda_k A_k)U_{k-1}x\| + \|U_{k-1}x\|) \\
&\quad + |t_{n,k}-t_k|\|J^k_{\lambda_k}(I-\lambda_k A_k)U_{k-1}x - x\|.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\|W_n x - Wx\| = \|U_{n,N}x - U_N x\| \\
&\le \sum_{k=2}^{N}\Big(\left|1-\frac{\lambda_{k,n}}{\lambda_k}\right|(\|J^k_{\lambda_k}(I-\lambda_k A_k)U_{k-1}x\| + \|U_{k-1}x\|) \\
&\quad + |t_{n,k}-t_k|\|J^k_{\lambda_k}(I-\lambda_k A_k)U_{k-1}x - x\|\Big) \\
&\quad + \left|1-\frac{\lambda_{1,n}}{\lambda_1}\right|(\|J^1_{\lambda_1}(I-\lambda_1 A_1)x\| + \|x\|) + |t_{n,1}-t_1|\|J^1_{\lambda_1}(I-\lambda_1 A_1)x - x\|.
\end{aligned}$$

Since for every $k \in \{1, \dots, N\}$, $\lim_{n\to\infty}|t_{n,k}-t_k| = 0$ and $\lim_{n\to\infty}|\lambda_{k,n}-\lambda_k| = 0$, the result follows. ☐

**Lemma 2.11.** *Let $C$ be a nonempty closed convex set of a Hilbert space $H$. Let $A_i : H \to H$, $i \in \{1, \dots, N\}$ be $\alpha_i$-inverse-strongly monotone mappings and $M_i : H \to 2^H$, $i \in \{1, \dots, N\}$ be maximal monotone mappings with $\bigcap_{i=1}^N I(A_i, M_i) \neq \emptyset$. Assume $\lambda_i \in (0, 2\alpha_i], i \in \{1, \dots, N\}$, and $\{\lambda_{i,n}\}_{i=1}^N$ be sequences such that $\lambda_{i,n} \in (0, 2\alpha_i]$ and $\lambda_{i,n} \to \lambda_i, i \in \{1, \dots, N\}, \forall n \ge 1$. Let $t_1, \dots, t_N$ be real numbers such that $0 < t_i < 1$ for every $i = 1, \dots, N-1$ and $0 < t_N \le 1$, and $\{t_{n,i}\}_{i=1}^N$ be sequences in (0,1) and satisfy $t_{n,i} \to t_i$. For every $n \in \mathbb{N}$, let $W$ be the $W$-mappings generated by $\{J^i_{\lambda_i}(I-\lambda_i A_i)\}_{i=1}^N$ and $t_1, \dots, t_N$ and $W_n$ be the $W_n$-mappings generated by $\{J^i_{\lambda_{i,n}}(I-\lambda_{i,n}A_i)\}_{i=1}^N$ and $t_{n,1}, \dots, t_{n,N}$. Then $\bigcap_{i=1}^N I(A_i, M_i) = F(W) = \bigcap_{n=1}^\infty F(W_n)$.*

*Proof.* Following Colao et al. [7] and using $F(J^i_{\lambda_i}(I-\lambda_i A_i)) = I(A_i, M_i)$, $i \in \{1, \dots, N\}$, we have $F(W) = \bigcap_{i=1}^N I(A_i, M_i)$.

Next we show $\bigcap_{n=1}^{\infty} F(W_n) = \bigcap_{i=1}^{N} I(A_i, M_i)$. Take $p \in \bigcap_{i=1}^{N} I(A_i, M_i)$ arbitrarily, then $J^i_{\lambda_{i,n}}(I - \lambda_{i,n}A_i)p = p$, $i \in \{1, \dots, N\}$ and $n \geq 1$. From (1.6), it follows $W_n p = p$, $\forall n \geq 1$ and consequently, $p \in \bigcap_{n=1}^{\infty} F(W_n)$. So we have $\bigcap_{i=1}^{N} I(A_i, M_i) \subseteq \bigcap_{n=1}^{\infty} F(W_n)$. On the other hand, put $p \in \bigcap_{n=1}^{\infty} F(W_n)$ and $q \in \bigcap_{i=1}^{N} I(A_i, M_i)$. Assume $U_{n,0} = I$, by (1.6), we get, for $k \in \{1, \dots, N\}$,

$$
\begin{aligned}
&\|W_n p - W_n q\| = \|U_{n,N} p - U_{n,N} q\| \\
&= \|[t_{n,N} J^N_{\lambda_{N,n}}(I - \lambda_{N,n}A_N)U_{n,N-1}p + (1 - t_{n,N})p] \\
&\quad - [t_{n,N} J^N_{\lambda_{N,n}}(I - \lambda_{N,n}A_N)U_{n,N-1}q + (1 - t_{n,N})q]\| \\
&\leq t_{n,N}\|J^N_{\lambda_{N,n}}(I - \lambda_{N,n}A_N)U_{n,N-1}p - J^N_{\lambda_{N,n}}(I - \lambda_{N,n}A_N)U_{n,N-1}q\| \\
&\quad + (1 - t_{n,N})\|p - q\| \\
&\leq t_{n,N}\|U_{n,N-1}p - U_{n,N-1}q\| + (1 - t_{n,N})\|p - q\| \\
&\leq \dots \\
&\leq \prod_{i=k+1}^{N} t_{n,i}\|U_{n,k}p - U_{n,k}q\| + \left(1 - \prod_{i=k+1}^{N} t_{n,i}\right)\|p - q\|
\end{aligned}
$$

$$
\begin{aligned}
&= \prod_{i=k+1}^{N} t_{n,i}\big\|[t_{n,k} J^k_{\lambda_{k,n}}(I - \lambda_{k,n}A_k)U_{n,k-1}p + (1 - t_{n,k})p] \\
&\quad - [t_{n,k} J^k_{\lambda_{k,n}}(I - \lambda_{k,n}A_k)U_{n,k-1}q + (1 - t_{n,k})q]\big\| + \left(1 - \prod_{i=k+1}^{N} t_{n,i}\right)\|p - q\| \\
&= \prod_{i=k+1}^{N} t_{n,i}\big\|t_{n,k}[J^k_{\lambda_{k,n}}(I - \lambda_{k,n}A_k)U_{n,k-1}p - J^k_{\lambda_{k,n}}(I - \lambda_{k,n}A_k)U_{n,k-1}q] \\
&\quad + (1 - t_{n,k})(p - q)\| + \left(1 - \prod_{i=k+1}^{N} t_{n,i}\right)\|p - q\| \\
&\leq \prod_{i=k}^{N} t_{n,i}\|J^k_{\lambda_{k,n}}(I - \lambda_{k,n}A_k)U_{n,k-1}p - J^k_{\lambda_{k,n}}(I - \lambda_{k,n}A_k)U_{n,k-1}q\| \\
&\quad + \left(1 - \prod_{i=k}^{N} t_{n,i}\right)\|p - q\| \\
&\leq \prod_{i=k}^{N} t_{n,i}\|U_{n,k-1}p - U_{n,k-1}q\| + \left(1 - \prod_{i=k}^{N} t_{n,i}\right)\|p - q\| \\
&\leq \|p - q\|.
\end{aligned}
$$

From Lemma 2.10, we have, as $n \to \infty$.

$$
\begin{aligned}
\|Wp - Wq\| \leq &\prod_{i=k+1}^{N} t_{n,i}\big\|t_{n,k}[J^k_{\lambda_{k,n}}(I - \lambda_{k,n}A_k)U_{n,k-1}p - J^k_{\lambda_{k,n}}(I - \lambda_{k,n}A_k)U_{n,k-1}q] \\
&+ (1 - t_{n,k})(p - q)\| + \left(1 - \prod_{i=k+1}^{N} t_{n,i}\right)\|p - q\|
\end{aligned}
$$

$$\leq \prod_{i=k}^{N} t_{n,i} \| J^k_{\lambda_{k,n}}(I - \lambda_{k,n}A_k)U_{n,k-1}p - J^k_{\lambda_{k,n}}(I - \lambda_{k,n}A_k)U_{n,k-1}q\| + \left(1 - \prod_{i=k}^{N} t_{n,i}\right) \|p - q\|$$
$$\leq \|p - q\|.$$

Since

$$\|Wp - Wq\| = \|p - q\|$$

and $0 < t_{n,i} < 1$ for all $i \in \mathbb{N}$, we have, for every $k \in \mathbb{N}$,

$$\|t_{n,k}[J^k_{\lambda_{k,n}}(I - \lambda_{k,n}A_k)U_{n,k-1}p - J^k_{\lambda_{k,n}}(I - \lambda_{k,n}A_k)U_{n,k-1}q] + (1 - t_{n,k})(p - q)\|$$
$$= \|J^k_{\lambda_{k,n}}(I - \lambda_{k,n}A_k)U_{n,k-1}p - J^k_{\lambda_{k,n}}(I - \lambda_{k,n}A_k)U_{n,k-1}q\|$$
$$= \|p - q\|.$$

Since Hilbert space $H$ is strictly convex and $q \in \bigcap_{i=1}^{N} I(A_i, M_i)$, we have

$$p - q = J^k_{\lambda_{k,n}}(I - \lambda_{k,n}A_k)U_{n,k-1}p - J^k_{\lambda_{k,n}}(I - \lambda_{k,n}A_k)U_{n,k-1}q$$
$$= J^k_{\lambda_{k,n}}(I - \lambda_{k,n}A_k)U_{n,k-1}p - q$$

and hence

$$p = J^k_{\lambda_{k,n}}(I - \lambda_{k,n}A_k)U_{n,k-1}p, \quad k = 1, \dots, N.$$

On the other hand, from

$$U_{n,k}p = t_{n,k}J^k_{\lambda_{k,n}}(I - \lambda_{k,n}A_k)U_{n,k-1}p + (1 - t_{n,k})p = p, \quad \forall n \in \mathbb{N},\ k = 1, \dots, N,$$

we have

$$Wp = \lim_{n\to\infty} W_n p = \lim_{n\to\infty} U_{n,N}p = p,$$

which implies $p \in F(W)$. Hence we obtain $\bigcap_{n=1}^{\infty} F(W_n) \subseteq F(W) = \bigcap_{i=1}^{N} I(A_i, M_i)$ and then $\bigcap_{n=1}^{\infty} F(W_n) = \bigcap_{i=1}^{N} I(A_i, M_i)$. Thus the proof is completed.

□

**Lemma 2.12.** *For all $x, y \in H$, there holds the inequality*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle.$$

## 3. Main result

**Theorem 3.1.** *Let $H$ be a real Hilbert space, $F$ be a bifunction from $C \times C \to \mathbb{R}$ satisfying $(A1) - (A4)$ and $\{S_n\}$ be a sequence of nonexpansive mappings on $H$. For $i = \{1, \dots, N\}$, let $A_i : H \to H$ be $\alpha_i$-inverse-strongly monotone mappings, $M_i : H \to 2^H$ be maximal monotone mappings such that $\Omega := (\bigcap_{k=1}^{N} I(A_k, M_k)) \cap EP(F) \cap (\bigcap_{i=1}^{\infty} F(S_i))$. Let $f$ be a contraction of $H$ into itself with a constant $\alpha$ and $B$ be a strongly bounded linear operator on $H$ with coefficient $\bar{\gamma} > 0$ and $0 < \gamma < \bar{\gamma}/\alpha$. Moreover, let $\{\epsilon_n\}$ be a sequence in $(0, 1)$, $\{t_{n,i}\}_{i=1}^{N}$ sequences in $[a, b]$ with $0 < a \leq b < 1$, $\{r_n\}$ a sequence in $(0, \infty)$, and $\{\lambda_{i,n}\}_{i=1}^{N}$ sequences such that $\lambda_{i,n} \in (0, 2\alpha_i]$. Assume*

- (B1) $\lim_n \epsilon_n = 0$;
- (B2) $\sum_{n=1}^{\infty} \epsilon_n = \infty$;
- (C1) $\liminf_n r_n > 0$*;*
- (C2) $\lim_n |1 - \frac{r_{n+1}}{r_n}| = 0$*;*
- (D1) $\lim_n |1 - \frac{\lambda_{j,n+1}}{\lambda_{j,n}}| = 0$*, for every $j \in \{1, \dots, N\}$;*
- (E1) $\lim_n |t_{n,j} - t_{n-1,j}| = 0$*, for every $j \in \{1, \dots, N\}$.*

*Let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_1 \in H$ and*

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad y \in H, \tag{3.1}$$
$$x_{n+1} = \epsilon_n \gamma f(x_n) + \beta x_n + ((1-\beta)I - \epsilon_n B)S_n W_n u_n,$$

*for all $n \in \mathbb{N}$. Suppose that $\sum_{n=1}^{\infty} \sup\{\|S_{n+1}z - S_n z\|, z \in K\} < \infty$ for any bounded subset $K$ of $H$. Let $S$ be a mapping of $H$ into itself defined by $Sx = lim_{n\to\infty} S_n x$, for all $x \in H$ and suppose that $F(S) = \bigcap_{n=1}^{\infty} F(S_n)$. Then, $\{x_n\}$, $\{u_n\}$ and $\{W_n u_n\}$ converge strongly to $z$, where $z = P_\Omega(I - B + \gamma f)(z)$ is a unique solution of the variational inequalities (1.7).*

*Proof.* Since $B$ is a strongly positive bounded linear operator with coefficient $\bar{\gamma}$, $\frac{B}{1-\beta}$ is a strongly positive bounded linear operator with coefficient $\frac{\bar{\gamma}}{1-\beta}$. By $\epsilon_n \to 0$, we may assume, with no loss of generality, that $\epsilon_n \leq (1-\beta)\|B\|^{-1}$. From Lemma 2.6, we know that

$$\|(1-\beta)I - \epsilon_n B\| = (1-\beta)\|I - \frac{\epsilon_n B}{1-\beta}\| \leq (1-\beta)(1 - \frac{\epsilon_n \bar{\gamma}}{1-\beta}) = 1 - \beta - \epsilon_n \bar{\gamma}. \tag{3.2}$$

**Step 1.** The sequence $\{x_n\}$ is bounded.

Put $p \in \Omega$. Then, from $u_n = T_{r_n} x_n$, we have

$$\|u_n - p\| = \|T_{r_n} x_n - T_{r_n} p\| \leq \|x_n - p\|. \tag{3.3}$$

From Lemma 2.11, it follows $W_n p = p$. Due to nonexpansivity of $W_n$ and (3.3), we have

$$\|W_n u_n - p\| \leq \|u_n - p\| \leq \|x_n - p\|. \tag{3.4}$$

Combining (3.1), (3.2) and (3.4), we get

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\epsilon_n(\gamma f(x_n) - Bp) + \beta(x_n - p) + ((1-\beta)I - \epsilon_n B)(S_n W_n u_n - p)\| \\
&\leq \epsilon_n(\gamma\|f(x_n) - f(p)\| + \|\gamma f(p) - Bp\|) + \beta\|x_n - p\| \\
&\quad + (1 - \beta - \epsilon_n\bar{\gamma})\|S_n W_n u_n - p)\| \\
&\leq (1 - \epsilon_n(\bar{\gamma} - \alpha\gamma))\|x_n - p\| + \epsilon_n(\bar{\gamma} - \alpha\gamma)\frac{\|\gamma f(p) - Bp\|}{\bar{\gamma} - \alpha\gamma}
\end{aligned}$$

which implies that

$$\|x_n - p\| \leq \max\left\{\|x_1 - p\|, \frac{\|\gamma f(p) - Bp\|}{\bar{\gamma} - \alpha\gamma}\right\}, \quad \forall n \geq 1.$$

Hence $\{x_n\}$ is bounded and therefore $\{u_n\}, \{f(x_n)\}$ and $\{S_n W_n u_n\}$ are also bounded.
**Step 2.** Let $\{w_n\}$ be a bounded sequence in $H$. Then

$$\lim_{n\to\infty} \|S_{n+1} W_{n+1} w_n - S_n W_n w_n\| = 0. \tag{3.5}$$

Let $j \in \{0, \ldots, N-2\}$ and set

$$\begin{aligned}
M : = \sup_{n\in\mathbb{N}} \Big\{ &\|w_n\| + \|J^1_{\lambda_{1,n}}(I - \lambda_{1,n}A_1)w_n\| \\
&+ \sum_{j=2}^{N} \Big(\|J^j_{\lambda_{j,n}}(I - \lambda_{j,n}A_j)U_{n,j-1}w_n\| + \|U_{n,j-1}w_n\|\Big)\Big\} < \infty.
\end{aligned}$$

It follows from (1.6) and Lemma 2.5 that

$$\begin{aligned}
&\|U_{n+1,N-j}w_n - U_{n,N-j}w_n\| \\
&= \|t_{n+1,N-j}J^{N-j}_{\lambda_{N-j,n+1}}(I-\lambda_{N-j,n+1}A_{N-j})U_{n+1,N-j-1}w_n + (1-t_{n+1,N-j})w_n \\
&\quad - t_{n,N-j}J^{N-j}_{\lambda_{N-j,n}}(I-\lambda_{N-j,n}A_{N-j})U_{n,N-j-1}w_n - (1-t_{n,N-j})w_n\| \\
&\le t_{n+1,N-j}\|J^{N-j}_{\lambda_{N-j,n+1}}(I-\lambda_{N-j,n+1}A_{N-j})U_{n+1,N-j-1}w_n \\
&\quad - J^{N-j}_{\lambda_{N-j,n+1}}(I-\lambda_{N-j,n+1}A_{N-j})U_{n,N-j-1}w_n\| \\
&\quad + t_{n+1,N-j}\|J^{N-j}_{\lambda_{N-j,n+1}}(I-\lambda_{N-j,n+1}A_{N-j})U_{n,N-j-1}w_n \\
&\quad - J^{N-j}_{\lambda_{N-j,n}}(I-\lambda_{N-j,n}A_{N-j})U_{n,N-j-1}w_n\| \\
&\quad + |t_{n+1,N-j} - t_{n,N-j}|\|J^{N-j}_{\lambda_{N-j,n}}(I-\lambda_{N-j,n}A_{N-j})U_{n,N-j-1}w_n - w_n\| \\
&\le \|U_{n+1,N-j-1}w_n - U_{n,N-j-1}w_n\| + \left|1-\frac{\lambda_{N-j,n+1}}{\lambda_{N-j,n}}\right| \\
&\quad \Big(\|J^{N-j}_{\lambda_{N-j,n}}(I-\lambda_{N-j,n}A_{N-j})U_{n,N-j-1}w_n\| + \|U_{n,N-j-1}w_n\|\Big) \\
&\quad + |t_{n+1,N-j} - t_{n,N-j}|(\|J^{N-j}_{\lambda_{N-j,n}}(I-\lambda_{N-j,n}A_{N-j})U_{n,N-j-1}w_n\| + \|w_n\|) \\
&\le \|U_{n+1,N-j-1}w_n - U_{n,N-j-1}w_n\| + M\left(\left|1-\frac{\lambda_{N-j,n+1}}{\lambda_{N-j,n}}\right| + |t_{n+1,N-j} - t_{n,N-j}|\right).
\end{aligned}$$

Thus, repeatedly using the above recursive inequalities, we deduce

$$\begin{aligned}
&\|W_{n+1}w_n - W_nw_n\| = \|U_{n+1,N}w_n - U_{n,N}w_n\| \\
&\le M\sum_{j=2}^{N}\left(\left|1-\frac{\lambda_{j,n+1}}{\lambda_{j,n}}\right| + |t_{n+1,j}-t_{n,j}|\right) + \left|1-\frac{\lambda_{1,n+1}}{\lambda_{1,n}}\right| \\
&\quad (\|J^1_{\lambda_{1,n}}(I-\lambda_{1,n}A_1)w_n\| + \|w_n\|) + |t_{n+1,1}-t_{n,1}|(\|J^1_{\lambda_{1,n}}(I-\lambda_{1,n}A_1)w_n\| + \|w_n\|) \\
&\le M\sum_{j=1}^{N}\left(\left|1-\frac{\lambda_{j,n+1}}{\lambda_{j,n}}\right| + |t_{n+1,j}-t_{n,j}|\right) \to 0,
\end{aligned} \tag{3.6}$$

by condition (D1), (E1). From (3.6) and property of $S_n$, it follows that

$$\begin{aligned}
\|S_{n+1}W_{n+1}w_n - S_nW_nw_n\| &\le \|S_{n+1}W_{n+1}w_n - S_nW_{n+1}w_n\| + \|S_nW_{n+1}w_n - S_nW_nw_n\| \\
&\le \|S_{n+1}W_{n+1}w_n - S_nW_{n+1}w_n\| + \|W_{n+1}w_n - W_nw_n\| \\
&\to 0,
\end{aligned}$$

and Step 2 is proven.

**Step 3.** $\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0$.

From Lemma 2.8, we have

$$\begin{aligned}
\|u_{n+1}-u_n\| &= \|T_{r_{n+1}}x_{n+1} - T_{r_n}x_n\| \\
&\le \|T_{r_{n+1}}x_{n+1} - T_{r_{n+1}}x_n\| + \|T_{r_{n+1}}x_n - T_{r_n}x_n\| \\
&\le \|x_{n+1}-x_n\| + \left|1-\frac{r_{n+1}}{r_n}\right|(\|T_{r_n}x_n\| + \|x_n\|).
\end{aligned} \tag{3.7}$$

Rewrite the iterative process (3.1) as follows:

$$\begin{aligned} x_{n+1} &= \epsilon_n\gamma f(x_n) + \beta x_n + (1-\beta I - \epsilon_n B)S_nW_nu_n \\ &= \beta x_n + (1-\beta)\frac{\epsilon_n\gamma f(x_n) + (1-\beta I - \epsilon_n B)S_nW_nu_n}{1-\beta} \\ &= \beta x_n + (1-\beta)y_n \end{aligned}$$

where

$$y_n = \frac{\epsilon_n\gamma f(x_n) + (1-\beta I - \epsilon_n B)S_nW_nu_n}{1-\beta}. \tag{3.8}$$

Since $\{x_n\}$ is bounded, we have, for some big enough constant $M > 0$,

$$\begin{aligned} \|y_{n+1} - y_n\| &= \Big\|\frac{\epsilon_{n+1}\gamma f(x_{n+1}) - \gamma\epsilon_n f(x_n)}{1-\beta} + (S_{n+1}W_{n+1}u_{n+1} - S_nW_nu_n) \\ &\quad - \frac{\epsilon_{n+1}BS_{n+1}W_{n+1}u_{n+1} - \epsilon_nBS_nW_nu_n}{1-\beta}\Big\| \\ &\le \frac{\gamma}{1-\beta}\left(\epsilon_{n+1}\|f(x_{n+1})\| + \epsilon_n\|f(x_n)\|\right) + \|S_{n+1}W_{n+1}u_{n+1} - S_nW_nu_n\| \\ &\quad + \frac{1}{1-\beta}\left(\epsilon_{n+1}\|BS_{n+1}W_{n+1}u_{n+1}\| + \epsilon_n\|BS_nW_nu_n\|\right) \\ &\le \|S_{n+1}W_{n+1}u_{n+1} - S_{n+1}W_{n+1}u_n\| + \|S_{n+1}W_{n+1}u_n - S_nW_nu_n\| + M(\epsilon_{n+1}+\epsilon_n) \\ &\le \|u_{n+1} - u_n\| + \|S_{n+1}W_{n+1}u_n - S_nW_nu_n\| + M(\epsilon_{n+1}+\epsilon_n) \\ &\le \|x_{n+1} - x_n\| + \left|1 - \frac{r_{n+1}}{r_n}\right|(\|T_{r_n}x_n\| + \|x_n\|) + \|S_{n+1}W_{n+1}u_n - S_nW_nu_n\| \\ &\quad + M(\epsilon_{n+1}+\epsilon_n). \end{aligned} \tag{3.9}$$

By conditions on $\{\epsilon_n\}$ and $\{r_n\}$, and Steps 2, we immediately conclude from (3.9)

$$\begin{aligned} &\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \\ &\le \limsup_{n\to\infty}\left(\left|1 - \frac{r_{n+1}}{r_n}\right|(\|T_{r_n}x_n\| + \|x_n\|) + \|S_{n+1}W_{n+1}u_n - S_nW_nu_n\| + M(\epsilon_{n+1}+\epsilon_n)\right) \\ &= 0. \end{aligned}$$

By Lemma 2.2, we obtain

$$\lim_{n\to\infty}\|x_n - y_n\| = 0,$$

which implies

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = \lim_{n\to\infty}(1-\beta)\|x_n - y_n\| = 0.$$

**Step 4.** $\lim_{n\to\infty}\|x_n - S_nW_nu_n\| = 0.$

We have

$$\begin{aligned} \|x_n - S_nW_nu_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - S_nW_nu_n\| \\ &= \|x_n - x_{n+1}\| + \|\epsilon_n(\gamma f(x_n) - BS_nW_nu_n) + \beta(x_n - S_nW_nu_n)\| \\ &\le \|x_n - x_{n+1}\| + \epsilon_n\|\gamma f(x_n) - BS_nW_nu_n\| + \beta\|x_n - S_nW_nu_n\|. \end{aligned}$$

It follows from Step 3 and condition (B1) that

$$\|x_n - S_nW_nu_n\| \le \frac{1}{1-\beta}(\|x_n - x_{n+1}\| + \epsilon_n\|\gamma f(x_n) - BS_nW_nu_n\|) \to 0.$$

**Step 5.** $\lim_{n\to\infty}\|x_n - u_n\| = 0.$

Let $v \in \Omega$. Since $T_{r_n}$ is firmly nonexpansive, we obtain

$$\begin{aligned}\|v - T_{r_n}x_n\|^2 &= \|T_{r_n}v - T_{r_n}x_n\|^2 \\ &\le \langle T_{r_n}x_n - T_{r_n}v, x_n - v\rangle = \langle T_{r_n}x_n - v, x_n - v\rangle \\ &= \frac{1}{2}\left(\|T_{r_n}x_n - v\|^2 + \|x_n - v\|^2 - \|x_n - T_{r_n}x_n\|^2\right),\end{aligned}$$

which implies

$$\|T_{r_n}x_n - v\|^2 \le \|x_n - v\|^2 - \|x_n - T_{r_n}x_n\|^2. \tag{3.10}$$

Set $y_n = \gamma f(x_n) - BS_nW_nT_{r_n}x_n$ and let $\lambda > 0$ be a constant such that

$$\lambda > \sup_{n,k}\{\|y_n\|, \|x_k - v\|\}.$$

Using Lemma 2.12 and noting that $\|\cdot\|^2$ is convex, we derive, using (3.10)

$$\begin{aligned}\|x_{n+1} - v\|^2 &= \|(1-\beta)(S_nW_nT_{r_n}x_n - v) + \beta(x_n - v) + \epsilon_n(\gamma f(x_n) - BS_nW_nT_{r_n}x_n)\|^2 \\ &\le \|(1-\beta)(S_nW_nT_{r_n}x_n - v) + \beta(x_n - v)\|^2 + 2\epsilon_n\langle y_n, x_{n+1} - v\rangle \\ &\le (1-\beta)\|S_nW_nT_{r_n}x_n - v\|^2 + \beta\|x_n - v\|^2 + 2\lambda^2\epsilon_n \\ &\le (1-\beta)\|T_{r_n}x_n - v\|^2 + \beta\|x_n - v\|^2 + 2\lambda^2\epsilon_n \\ &\le (1-\beta)(\|x_n - v\|^2 - \|x_n - T_{r_n}x_n\|^2) + \beta\|x_n - v\|^2 + 2\lambda^2\epsilon_n \\ &= \|x_n - v\|^2 - (1-\beta)\|x_n - T_{r_n}x_n\|^2 + 2\lambda^2\epsilon_n.\end{aligned}$$

It follows that

$$\begin{aligned}\|x_n - T_{r_n}x_n\|^2 &\le \frac{1}{1-\beta}\left(\|x_n - v\|^2 - \|x_{n+1} - v\|^2 + 2\lambda^2\epsilon_n\right) \\ &\le \frac{1}{1-\beta}\left(\|x_n - x_{n+1}\|(\|x_n - v\| + \|x_{n+1} - v\|) + 2\lambda^2\epsilon_n\right) \\ &\to 0,\end{aligned}$$

by Step 3 and condition (B1). From $u_n = T_{r_n}x_n$, it follows $\|x_n - u_n\| \to 0$.

**Step 6.** The weak $\omega$-limit set of $\{x_n\}$, $\omega(x_n)$, is a subset of $\Omega$.

Let $z \in \omega(x_n)$ and $\{x_{n_m}\}$ be a subsequence of $\{x_n\}$ weakly converging to $z$. Noticing Step 5, we have $u_{n_m} \rightharpoonup z$. We will show that $z \in \Omega$. By (A2), we have

$$\frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge F(y, u_n), \quad y \in C.$$

In particular,

$$\left\langle y - u_{n_m}, \frac{u_{n_m} - x_{n_m}}{r_{n_m}}\right\rangle \ge F(y, u_{n_m}). \tag{3.11}$$

Step 5 and condition (C1) imply $(u_{n_m} - x_{n_m})/r_{n_m} \to 0$ in norm. By condition (A4), $F(y,\cdot)$ is lower semicontinuous and convex, and thus weakly semicontinuous. Therefore, letting $m \to \infty$ in (3.11) yields

$$F(y, z) \le \lim_{m\to\infty} F(y, u_m) \le 0,$$

for all $y \in H$. Replacing $y$ with $y_t := ty + (1-t)z$ with $t \in (0,1)$ and using (A1) and (A4), we obtain

$$0 = F(y_t, y_t) \le tF(y_t, y) + (1-t)F(y_t, z) \le tF(y_t, y).$$

Hence $F(ty + (1-t)z, y) \ge 0$, for all $t \in (0,1)$ and $y \in H$. Letting $t \to 0^+$ and using (A3), we obtain

$$F(z, y) \ge 0,$$

for all $y \in H$. Therefore $z \in EP(F)$.

Next, we show that $z \in \left(\bigcap_{n=1}^{N} I(A_n, M_n)\right) \cap (\bigcap_{n=1}^{\infty} F(S_n))$ . Assume that $z \notin \left(\bigcap_{n=1}^{N} I(A_n, M_n)\right) \cap (\bigcap_{n=1}^{\infty} F(S_n))$, by Lemma 2.11, then $z \neq SWz$. Since Step 4 and Step 5, and using Opials property of a Hilbert space, we have

$$\begin{aligned}
\liminf_m \|x_{n_m} - z\| &< \liminf_m \|x_{n_m} - SWz\| \\
&\leq \liminf_m \big(\|x_{n_m} - S_{n_m}W_{n_m}u_{n_m}\| + \|S_{n_m}W_{n_m}u_{n_m} - S_{n_m}W_{n_m}x_{n_m}\| \\
&\quad + \|S_{n_m}W_{n_m}x_{n_m} - S_{n_m}W_{n_m}z\| + \|S_{n_m}W_{n_m}z - SW_{n_m}z\| \\
&\quad + \|SW_{n_m}z - SWz\|\big) \\
&\leq \liminf_m \big(\|x_{n_m} - S_{n_m}W_{n_m}u_{n_m}\| + \|u_{n_m} - x_{n_m}\| \\
&\quad + \|x_{n_m} - z\| + \|S_{n_m}W_{n_m}z - SW_{n_m}z\| + \|W_{n_m}z - Wz\|\big) \\
&\leq \liminf_m \|x_{n_m} - z\|.
\end{aligned}$$

This is a contradiction. Therefore, $z$ must belong to $\left(\bigcap_{n=1}^{N} I(A_n, M_n)\right) \cap (\bigcap_{n=1}^{\infty} F(S_n))$ . Proof is completed.

**Step 7.** Let $x^*$ be the unique solution of the variational inequality (1.7). That is,

$$\langle (B - \gamma f)x^*, x - x^* \rangle \geq 0, \quad x \in \Omega. \tag{3.12}$$

Then

$$\limsup_n \langle (\gamma f - B)x^*, x_n - x^* \rangle \leq 0, \quad x \in \Omega. \tag{3.13}$$

Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that

$$\lim_k \langle (\gamma f - B)x^*, x_{n_k} - x^* \rangle = \limsup_n \langle (\gamma f - B)x^*, x_n - x^* \rangle. \tag{3.14}$$

Without loss of generality, we can assume that $\{x_{n_k}\}$ weakly converges to some $z$ in $C$. By Step 6, $z \in \Omega$. Thus combining (3.14) and (3.12), we get

$$\limsup_n \langle (\gamma f - B)x^*, x_n - x^* \rangle = \langle (\gamma f - B)x^*, z - x^* \rangle \leq 0$$

as required.

**Step 8.** The sequences $\{x_n\}$, $\{u_n\}$ and $\{W_n u_n\}$ converge strongly to $x^*$.

By the definition (3.1) of $\{x_n\}$, and using Lemma 2.6 and Lemma 2.12, we have

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &= \|[((1-\beta)I-\epsilon_n B)(S_nW_nx_n-x^*)+\beta(x_n-x^*)]+\epsilon_n(\gamma f(x_n)-Bx^*)\|^2\\
&\le \|((1-\beta)I-\epsilon_n B)(S_nW_nx_n-x^*)+\beta(x_n-x^*)\|^2\\
&\quad +2\epsilon_n\langle \gamma f(x_n)-Bx^*, x_{n+1}-x^*\rangle\\
&= \|(1-\beta)\frac{(1-\beta)I-\epsilon_n B}{1-\beta}(S_nW_nx_n-x^*)+\beta(x_n-x^*)\|^2\\
&\quad +2\epsilon_n\gamma\langle f(x_n)-f(x^*), x_{n+1}-x^*\rangle+2\epsilon_n\langle \gamma f(x^*)-Bx^*, x_{n+1}-x^*\rangle\\
&\le \frac{\|(1-\beta)I-\epsilon_n B\|^2}{1-\beta}\|S_nW_nx_n-x^*\|^2+\beta\|x_n-x^*\|^2\\
&\quad +\epsilon_n\gamma\alpha(\|x_n-x^*\|^2+\|x_{n+1}-x^*\|^2)+2\epsilon_n\langle \gamma f(x^*)-Bx^*, x_{n+1}-x^*\rangle\\
&\le \left(\frac{((1-\beta)-\bar\gamma\epsilon_n)^2}{1-\beta}+\beta+\epsilon_n\gamma\alpha\right)\|x_n-x^*\|^2+\epsilon_n\gamma\alpha\|x_{n+1}-x^*\|^2\\
&\quad +2\epsilon_n\langle \gamma f(x^*)-Bx^*, x_{n+1}-x^*\rangle.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &\le \left(1-\frac{2(\bar\gamma-\alpha\gamma)\epsilon_n}{1-\alpha\gamma\epsilon_n}\right)\|x_n-x^*\|^2\\
&\quad +\frac{\epsilon_n}{1-\alpha\gamma\epsilon_n}\left(2\langle \gamma f(x^*)-Ax^*, x_{n+1}-x^*\rangle+\frac{\bar\gamma^2\epsilon_n}{1-\beta}\|x_n-x^*\|^2\right).
\end{aligned}$$

Now, from conditions (B1) and (B2), Step 7 and Lemma 2.3, we get $\|x_n-x^*\|\to 0$. Namely, $x_n\to x^*$ in norm. Finally, noticing $\|u_n-x^*\|\le\|x_n-x^*\|$ and $\|W_nu_n-x^*\|\le\|x_n-x^*\|$, we also conclude that $u_n\to x^*$ and $W_nu_n\to x^*$ in norm.

□

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