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Improved Decomposition Methods for Solving a Class of Variational Inequalities

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Abstract. To solve a class of variational inequalities(VI) with linear equality constraint, this paper presents two new descent methods to improve the decomposition method proposed by Gabay and Mercier[4,5] in the following senses: the sub-VI in the improved decomposition methods is strongly monotone; the iterate generated by the original decomposition method is utilized to generate descent direction, and the new iterate is generated along the descent direction. Under mild conditions, the global convergence of the improved methods is proved. Preliminary numerical experiments illustrate that the improved methods are efficient.

Keywords: Decomposition method; Projection; Descent direction; Global convergence.

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1 Introduction

The variational inequalities, denoted by VI(f, S), is to find a vector $x^* \in S$, such that

$$(x - x^*)^\top f(x^*) \geq 0, \quad \forall x \in S, \tag{1}$$

where S is a nonempty, closed convex subset of $\mathcal{R}^n$, $f(\cdot)$ is a continuous mapping from $S \subset \mathcal{R}^n$ to itself. VI(f, S) reduces to nonlinear complementarity problems when S is nonnegative orthant $\mathcal{R}^n_+$ and nonlinear equations when S is the Euclidean space $\mathcal{R}^n$. Variational inequality problems have important applications in many fields such as elasticity, optimization, economics, transportation and structural analysis, and various numerical methods have been studied by many researchers.

In this paper, we are concerned with the variational inequality problem that S has the following structure

$$S = \{x \in \mathcal{R}^n | Ax = b, x \in \mathcal{X}\}, \tag{2}$$

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where $A \in \mathcal{R}^{m\times n}$, $b \in \mathcal{R}^m$ and $\mathcal{X}$ is a simple closed convex subset of $\mathcal{R}^n$. This problem has several important applications in many fields, such as the capacitated transportation problem[1], the capacitated traffic assignment problem[2] and the packet routing in telecommunication with path and flow restrictions[3].

By appending a Lagrangian multiplier vector $y \in \mathcal{R}^m$ to the linear equality constraint $Ax = b$, an equivalent form of VI(f, S) can be expressed as follows, denoted by VI$(F,\mathcal{U})$: Find a vector $u^* \in \mathcal{U}$, such that

$$(u - u^*)^\top F(u^*) \geq 0 \quad \forall u \in \mathcal{U}, \tag{3}$$

where

$$u = \begin{pmatrix} x \\ y \end{pmatrix}, F(u) = \begin{pmatrix} f(x) - A^\top y \\ Ax - b \end{pmatrix}, \mathcal{U} = \mathcal{X} \times \mathcal{R}^m.$$

For solving VI$(F,\mathcal{U})$ problem, Gabay[4] and Gabay and Mercier[5] proposed the following decomposition method. In their method, the new iterate $\tilde{u}^k = (\tilde{x}^k, \tilde{y}^k) \in \mathcal{X} \times \mathcal{R}^m$ is generated from a given vector $u^k = (x^k, y^k) \in \mathcal{X} \times \mathcal{R}^m$ via the following procedure:

Given $(x^k, y^k) \in \mathcal{X} \times \mathcal{R}^m$, find $\tilde{x}^k \in \mathcal{X}$, such that

$$(x' - \tilde{x}^k)^\top \{f(\tilde{x}^k) - A^\top [y^k - \beta(A\tilde{x}^k - b)]\} \geq 0, \quad \forall x' \in \mathcal{X}, \tag{4}$$

then update y^k via

$$\tilde{y}^k = y^k - \beta(A\tilde{x}^k - b).$$

Here $\beta > 0$ is a given penalty parameter for the linear constraint $Ax = b$.

Note that the involved auxiliary VI(4) may not be well-conditioned, without strongly monotone assumption on f. Motivated by proximal-based alternating direction method[9,10], in this paper, the new iterate is generated via:

$$(x' - \tilde{x}^k)^\top \{f(\tilde{x}^k) - A^\top [y^k - H(Ax^k - b)] + R(\tilde{x}^k - x^k)\} \geq 0, \quad \forall x' \in \mathcal{X}, \tag{5}$$

and

$$\tilde{y}^k = y^k - H(A\tilde{x}^k - b), \tag{6}$$

where the symmetric positive definite matrix $R \in \mathcal{R}^{n\times n}$ is the proximal parameter, and $H \in \mathcal{R}^{m\times m}$ is a given symmetric positive definite matrix that can be regarded as the penalty parameter for the linear constraint $Ax = b$. Compared with the original monotone VI(f, S) (1)-(2), (5) is strongly monotone VI with lower dimension.

Of course, if we set $u^{k+1} = \tilde{u}^k$, a new decomposition method is generated. However, are there other kind of u^{k+1} be more beneficial for the next iteration? To enhance the efficiency of the iterative method generated by (5)-(6), motivated by Tao's idea[8], this paper updates the new iterate by

$$u_{\mathrm{I}}^{k+1}(\alpha) = P_{\mathcal{U}}[u^k - \alpha d_1(u^k, \tilde{u}^k)], \tag{7}$$

and

$$u_{\mathrm{II}}^{k+1}(\alpha) = P_{\mathcal{U}}[u^k - \alpha d_2(u^k, \tilde{u}^k)], \tag{8}$$

where

$$d_1(u^k, \tilde{u}^k) = \begin{pmatrix} R(x^k - \tilde{x}^k) \\ H^{-1}(y^k - \tilde{y}^k) \end{pmatrix}, \tag{9}$$

and

$$d_2(u^k, \tilde{u}^k) = F(\tilde{u}^k) + \begin{pmatrix} A^\top HA(x^k - \tilde{x}^k) \\ 0 \end{pmatrix}. \tag{10}$$

In the following, the method generated by (5)(6)(7)(9) is referred to as IDM-I, and the method generated by (5)(6)(8)(10) is referred to as IDM-II.

The rest of this paper is organized as follows. In the next section, we summarize some basic concepts we will use in the following nanlysis. In Section 3, we describe the IDM-I and IDM-II in details, and their global convergence is also analyzed. We report some preliminary computational results in Section 4 and some conclusions are given in Section 5.

2 Preliminaries

In this section, we summarize some definitions and related properties which will be used in the following discussions. The projection of a point $x \in \mathcal{R}^n$ onto the closed convex set $\mathcal{K}$, denoted by $P_{\mathcal{K}}[x]$, is defined as the unique solution of the problem

$$\min \|x - y\|, \quad \text{subject to} \quad y \in \mathcal{K}.$$

A basic property of the projection operator $P_{\mathcal{K}}[\cdot]$ is

$$(x - P_{\mathcal{K}}[x])^\top (z - P_{\mathcal{K}}[x]) \le 0, \quad \forall x \in \mathcal{R}^n, z \in \mathcal{K}. \tag{11}$$

It follows from (11) that

$$\|P_{\mathcal{K}}[x] - P_{\mathcal{K}}[y]\|^2 \le \|x - y\|^2 - \|P_{\mathcal{K}}[x] - x + y - P_{\mathcal{K}}[y]\|^2, \quad \forall x, y \in \mathcal{R}^n, \tag{12}$$

and

$$\|P_K[x] - P_K[y]\| \le \|x - y\|, \quad \forall x, y \in \mathcal{R}^n. \tag{13}$$

For any $\beta > 0$, it is well-known[6] that the problem VI(F,$\mathcal{U}$) is equivalent to the projection equation

$$u = P_{\mathcal{U}}[u - \beta F(u)].$$

Let

$$e(u,\beta) := u - P_{\mathcal{U}}[u - \beta F(u)], \tag{14}$$

denote the residual error of the projection equation, then solving VI(F,$\mathcal{U}$) is equivalent to finding zero points of the residual function $e(u,\beta)$.

Let $G \in \mathcal{R}^{n\times n}$ be a symmetric positive definite, then the G-norm of a vector $z \in \mathcal{R}^n$ is denoted by $\|z\|_G$, i.e., $\|z\|_G^2 = z^\top Gz$.

Definition. Let g be a mapping from $\mathcal{R}^l$ into itself and $\Omega \subset \mathcal{R}^l$. Then,

a) g is said to be monotone if

$$(s-t)^\top(g(s)-g(t)) \geq 0, \quad \forall s,t \in \Omega.$$

b) g is strongly monotone if there exists a constant $\mu > 0$ such that

$$(s-t)^\top(g(s)-g(t)) \geq \mu\|s-t\|^2, \quad \forall s,t \in \Omega.$$

We make the following standard assumptions throughout this paper:

Assumptions.

- f is a monotone mapping on $\mathcal{X}$.
- The solution set of problem VI(F,$\mathcal{U}$), denoted by $\mathcal{U}^*$, is nonempty.
- $\mathcal{X}$ is a simple closed convex set. That is, the projection onto the set is simple to carry out(e.g., $\mathcal{X}$ is the nonnegative orthant $\mathcal{R}^n_+$, or more generally, a box).

Note that F is monotone on $\mathcal{U}$ whenever f is monotone on $\mathcal{X}$. In fact, for any $u_1 = (x_1, y_1), u_2 = (x_2, y_2) \in \mathcal{U}$, we have

$$\begin{aligned} & (u_1-u_2)^\top(F(u_1)-F(u_2)) \\ = & (x_1-x_2)^\top(f(x_1)-A^\top y_1 - f(x_2) + A^\top y_2) + (y_1-y_2)^\top(Ax_1 - Ax_2) \\ = & (x_1-x_2)^\top(f(x_1)-f(x_2)) \geq 0. \end{aligned}$$

Because f is monotone and $\mathcal{X}$ is closed convex, the solution set $\mathcal{U}^*$ of VI(F,$\mathcal{U}$) is closed and convex.

3 Algorithm and global convergence

In this section, we present two new decomposition methods for solving VI(F,$\mathcal{U}$) and show their global convergence. For convenience, we denote

$$M_{(n+m)\times(n+m)} = \begin{pmatrix} R & 0 \\ 0 & H^{-1} \end{pmatrix}. \tag{15}$$

Since R and H is positive definite, we have that the matrix M defined in (15) is also positive definite. Note that the search direction $d_1(u^k, \tilde{u}^k)$ can be rewritten as $M(u^k - \tilde{u}^k)$.

The Proposed IDM-I

Step 0. Given $\varepsilon > 0$, choose $u^0 = (x^0, y^0)^\top \in \mathcal{U}$, $\gamma \in [1, 2)$ and set k:=0.

Step 1. Produced $\tilde{u}^k = (\tilde{x}^k, \tilde{y}^k)$ by (5)-(6).

Step 2. If $\|u^k - \tilde{u}^k\| < \varepsilon$, then stop; otherwise, go to Step 3.

Step 3. Update the next iterate $u^{k+1} = (x^{k+1}, y^{k+1})$ via

$$u^{k+1} = P_{\mathcal{U}}[u^k - \gamma\alpha_k^1 d_1(u^k, \tilde{u}^k)], \tag{16}$$

where

$$\alpha_k^1 = \frac{\Phi(u^k, \tilde{u}^k)}{\|d_1(u^k, \tilde{u}^k)\|^2}, \tag{17}$$

and

$$\Phi(u^k, \tilde{u}^k) = \|u^k - \tilde{u}^k\|_M^2 + (y^k - \tilde{y}^k)^\top A(x^k - \tilde{x}^k). \tag{18}$$

Set $k := k + 1$ and goto Step 1.

The Proposed IDM-II

Step 0. Given $\varepsilon > 0$, choose $u^0 = (x^0, y^0)^\top \in \mathcal{U}$, $\gamma \in [1, 2)$, and choose R such that $R - A^\top A/2$ is positive definite, H such that $H^{-1} - E/2$ is positive definite,and set k:=0.

Step 1. Produced $\tilde{u}^k = (\tilde{x}^k, \tilde{y}^k)$ by (5)-(6).

Step 2. If $\|u^k - \tilde{u}^k\| < \varepsilon$, then stop; otherwise, go to Step 3.

Step 3. Update the next iterate $u^{k+1} = (x^{k+1}, y^{k+1})$ via

$$u^{k+1} = P_{\mathcal{U}}[u^k - \gamma\alpha_k^2 d_2(u^k, \tilde{u}^k)], \tag{19}$$

where

$$\alpha_k^2 = \frac{\Phi(u^k, \tilde{u}^k)}{\|u^k - \tilde{u}^k\|_M^2}. \tag{20}$$

Set $k := k + 1$ and goto Step 1.

Note that $\|u^k - \tilde{u}^k\| = 0$ if and only if $x^k = \tilde{x}^k, y^k = \tilde{y}^k$. Then, from (5)-(6), $u^k = \tilde{u}^k$ implies that u^k is actually a solution, which means the iteration will be terminated. Thus, the stopping condition in Step 2 is reasonable.

Remark 3.1. The variational inequality (5)-(6) is equivalent to the following variational inequality:

$$\begin{pmatrix} x' - \tilde{x}^k \\ y' - \tilde{y}^k \end{pmatrix}^\top \{F(\tilde{u}^k) + \begin{pmatrix} A^\top HA(x^k - \tilde{x}^k) \\ 0 \end{pmatrix} - M(u^k - \tilde{u}^k)\} \geq 0, \quad \forall u' \in \mathcal{U}. \tag{21}$$

From the definition of $d_1(u^k, \tilde{u}^k)$ and $d_2(u^k, \tilde{u}^k)$, inequality (21) can be written as

$$(u' - \tilde{u}^k)^\top \{d_2(u^k, \tilde{u}^k) - d_1(u^k, \tilde{u}^k)\} \geq 0, \quad \forall u' \in \mathcal{U}. \tag{22}$$

The following lemma shows that $\Phi(u^k, \tilde{u}^k)$ is lower bounded away from zero whenever $u^k \neq \tilde{u}^k$.

Lemma 3.1. Let $\tilde{u}^k = (\tilde{x}^k, \tilde{y}^k)$ be generated by (4)-(5). Then, there exists a constant $\lambda_2 > 0$, such that

$$\Phi(u^k, \tilde{u}^k) \geq \lambda_2 \|u^k - \tilde{u}^k\|_M^2, \tag{23}$$

Proof. From the definition (18), we have

$$\begin{aligned}
& \Phi(u^k, \tilde{u}^k) \\
= \quad & \|x^k - \tilde{x}^k\|_R^2 + \|\tilde{y}^k - y^k\|_{H^{-1}}^2 + (y^k - \tilde{y}^k)^\top A(x^k - \tilde{x}^k) \\
\geq \quad & \|x^k - \tilde{x}^k\|_R^2 + \|\tilde{y}^k - y^k\|_{H^{-1}}^2 - \frac{1}{2}(\|A(x^k - \tilde{x}^k)\|^2 + \|\tilde{y}^k - y^k\|^2) \\
= \quad & (x^k - \tilde{x}^k)^\top (R - \frac{1}{2}A^\top A)(x^k - \tilde{x}^k) + (y^k - \tilde{y}^k)^\top (H^{-1} - \frac{1}{2}E)(y^k - \tilde{y}^k) \\
= \quad & \lambda_2 \|u^k - \tilde{u}^k\|_M^2,
\end{aligned}$$

where the first inequality follows from the Cauchy-Schwartz Inequality and the last inequality follows from the equivalence of the matrix norm. This completes the proof.

Remark 3.2. Since all vector norms are equivalent, from the definition of $d_1(u^k, \tilde{u}^k)$ and (23), there exists a constant $\lambda_1 > 0$, such that

$$\alpha_k^1 \geq \lambda_1, \quad \forall k \geq 0. \tag{24}$$

Remark 3.3. From (20) and (23), it follows that

$$\alpha_k^2 \geq \lambda_2, \quad \forall k \geq 0. \tag{25}$$

Now, we are ready to prove that $-d_1(u^k, \tilde{u}^k)$ defined in (9) and $-d_2(u^k, \tilde{u}^k)$ defined in (10) are two descent directions of $\|u^k - u^*\|^2$ at $u = u^k$.

Lemma 3.2. Let $\tilde{u}^k = (\tilde{x}^k, \tilde{y}^k)$ be generated by (4)-(5). Then, for any $u^* = (x^*, y^*) \in \mathcal{U}^*$, we have

$$(u^k - u^*)^\top d_1(u^k, \tilde{u}^k) \geq \Phi(u^k, \tilde{u}^k). \tag{26}$$

Proof. From $u^* = (x^*, y^*) \in \mathcal{U}^*$ and $\tilde{x}^k \in \mathcal{X}$, we have

$$(\tilde{x}^k - x^*)^\top (f(x^*) - A^\top y^*) \geq 0, \tag{27}$$

and

$$Ax^* = b. \tag{28}$$

On the other hand, from (5) (6) and $x^* \in \mathcal{X}$, it follows that

$$(x^* - \tilde{x}^k)^\top \{f(\tilde{x}^k) - A^\top \tilde{y}^k + A^\top HA(x^k - \tilde{x}^k) + R(\tilde{x}^k - x^k)\} \geq 0, \tag{29}$$

Adding (27) and (29), and using (28) and the monotonicity of f, we get

$$(A\tilde{x}^k - b)^\top(\tilde{y}^k - y^*) + (\tilde{x}^k - x^*)^\top R(x^k - \tilde{x}^k) \geq (y^k - \tilde{y}^k)^\top A(x^k - \tilde{x}^k).$$

From (5) again, we obtain

$$(\tilde{y}^k - y^*)^\top H^{-1}(y^k - \tilde{y}^k) + (\tilde{x}^k - x^*)^\top R(x^k - \tilde{x}^k) \geq (y^k - \tilde{y}^k)^\top A(x^k - \tilde{x}^k). \tag{30}$$

Therefore, we have

$$\begin{aligned} & (u^k - u^*)^\top d_1(u^k, \tilde{u}^k) \\ = \;& (u^k - \tilde{u}^k + \tilde{u}^k - u^*)^\top M(u^k - \tilde{u}^k) \\ \geq \;& \|u^k - \tilde{u}^k\|_M^2 + (y^k - \tilde{y}^k)^\top A(x^k - \tilde{x}^k)(\text{using } (30)) \\ = \;& \Phi(u^k, \tilde{u}^k). \end{aligned}$$

The proof is completed.

Lemma 3.3. Let $\tilde{u}^k = (\tilde{x}^k, \tilde{y}^k)$ be generated by (4)-(5). Then, for any $u^* = (x^*, y^*) \in \mathcal{U}^*$, we have

$$(u^k - u^*)^\top d_2(u^k, \tilde{u}^k) \geq \Phi(u^k, \tilde{u}^k). \tag{31}$$

Proof. From the monotonicity of F and $u^* \in \mathcal{U}^*$, we have

$$(\tilde{u}^k - u^*)^\top F(\tilde{u}^k) \geq (\tilde{u}^k - u^*)^\top F(u^*) \geq 0. \tag{32}$$

From (22) and $u^k \in \mathcal{U}$, we have

$$(u^k - \tilde{u}^k)^\top \{d_2(u^k, \tilde{u}^k) - d_1(u^k, \tilde{u}^k)\} \geq 0. \tag{33}$$

Therefore, we have

$$\begin{aligned} & (u^k - u^*)^\top d_2(u^k, \tilde{u}^k) \\ = \;& (u^k - \tilde{u}^k)^\top d_2(u^k, \tilde{u}^k) + (\tilde{u}^k - u^*)^\top F(\tilde{u}^k) \\ & +(\tilde{u}^k - u^*)^\top \begin{pmatrix} A^\top HA(x^k - \tilde{x}^k) \\ 0 \end{pmatrix} \\ \geq \;& (u^k - \tilde{u}^k)^\top d_2(u^k, \tilde{u}^k) + 0 + (y^k - \tilde{y}^k)^\top A(x^k - \tilde{x}^k)(\text{using } (32)) \\ \geq \;& (u^k - \tilde{u}^k)^\top d_1(u^k, \tilde{u}^k) + (y^k - \tilde{y}^k)^\top A(x^k - \tilde{x}^k)(\text{using } (33)) \\ \geq \;& \|u^k - \tilde{u}^k\|_M^2 + (y^k - \tilde{y}^k)^\top A(x^k - \tilde{x}^k)(\text{using } (26)) \\ = \;& \Phi(u^k, \tilde{u}^k). \end{aligned} \tag{34}$$

The proof is completed.

Note that, from the first inequality of (34), we have

$$(\tilde{u}^k - u^*)^\top d_2(u^k, \tilde{u}^k) \geq (y^k - \tilde{y}^k)^\top A(x^k - \tilde{x}^k). \tag{35}$$

In the following, we analyze why α_k^1 defined by (17) is the optimal step length along the descent direction $-d_1(u^k, \tilde{u}^k)$ and why α_k^2 defined by (20) is the optimal step length along the descent direction $-d_2(u^k, \tilde{u}^k)$. For this purpose, we denote the new iterate with the step length α along the two descent directions by

$$u_{\mathrm{I}}^{k+1}(\alpha) = P_{\mathcal{U}}[u^k - \alpha d_1(u^k, \tilde{u}^k)],$$

and

$$u_{\mathrm{II}}^{k+1}(\alpha) = P_{\mathcal{U}}[u^k - \alpha d_2(u^k, \tilde{u}^k)],$$

respectively. Then

$$\Theta_{\mathrm{I}}^k(\alpha) := \|u^k - u^*\|^2 - \|u_{\mathrm{I}}^{k+1}(\alpha) - u^*\|^2 \tag{36}$$

measures the progress of the iterate $u_{\mathrm{I}}^{k+1}(\alpha)$, and

$$\Theta_{\mathrm{II}}^k(\alpha) := \|u^k - u^*\|^2 - \|u_{\mathrm{II}}^{k+1}(\alpha) - u^*\|^2 \tag{37}$$

measures the progress of the iterate $u_{\mathrm{II}}^{k+1}(\alpha)$. Since u^* is unknown, we can not maximize $\Theta_{\mathrm{I}}^k(\alpha)$ (or $\Theta_{\mathrm{II}}^k(\alpha)$) directly. In the following, we will provide a lower bound function of $\Theta_{\mathrm{I}}^k(\alpha)$ (or $\Theta_{\mathrm{II}}^k(\alpha)$) which does not contain u^*.

Theorem 3.1. Let u^* be an arbitrary point in $\mathcal{U}^*$. For given u^k, let $\tilde{u}^k$ be generated by (4)-(5), $\Phi(u^k, \tilde{u}^k)$ be defined by (15) and $\Theta_{\mathrm{I}}^k(\alpha)$ be defined by (36). Then, we have

$$\Theta_{\mathrm{I}}^k(\alpha) \geq \Upsilon_{\mathrm{I}}^k(\alpha) + \|u^k - \alpha d_1(u^k, \tilde{u}^k) - u_{\mathrm{I}}^{k+1}(\alpha)\|^2, \tag{38}$$

where

$$\Upsilon_{\mathrm{I}}^k(\alpha) = 2\alpha\Phi(u^k, \tilde{u}^k) - \alpha^2\|d_1(u^k, \tilde{u}^k)\|^2.$$

Proof. Since $u^* \in \mathcal{U}^*$, it follows from (12) that

$$\|u_{\mathrm{I}}^{k+1}(\alpha) - u^*\|^2 \leq \|u^k - \alpha d_1(u^k, \tilde{u}^k) - u^*\|^2 - \|u^k - \alpha d_1(u^k, \tilde{u}^k) - u_{\mathrm{I}}^{k+1}(\alpha)\|^2.$$

Therefore, from (36), we have

$$\begin{aligned}
& \Theta_{\mathrm{I}}^k(\alpha) \\
\geq\ & \|u^k - u^*\|^2 - \|u^k - \alpha d_1(u^k, \tilde{u}^k) - u^*\|^2 + \|u^k - \alpha d_1(u^k, \tilde{u}^k) - u_{\mathrm{I}}^{k+1}(\alpha)\|^2 \\
=\ & 2\alpha(u^k - u^*)^\top d_1(u^k, \tilde{u}^k) - \alpha^2\|d_1(u^k, \tilde{u}^k)\|^2 \\
& + \|u^k - \alpha d_1(u^k, \tilde{u}^k) - u_{\mathrm{I}}^{k+1}(\alpha)\|^2 \\
\geq\ & 2\alpha\Phi(u^k, \tilde{u}^k) - \alpha^2\|d_1(u^k, \tilde{u}^k)\|^2 + \|u^k - \alpha d_1(u^k, \tilde{u}^k) - u_{\mathrm{I}}^{k+1}(\alpha)\|^2.
\end{aligned}$$

This completes the proof.

Theorem 3.1 tells us that $\Upsilon_{\rm I}^k(\alpha)$ is a lower bound of $\Theta_{\rm I}^k(\alpha)$ for any α. Note that $\Upsilon_{\rm I}^k(\alpha)$ is a quadratic function of α, and it reaches its maximus at α_k^1.

Theorem 3.2. Let u^* be an arbitrary point in $\mathcal{U}^*$. For given u^k, let $\tilde{u}^k$ be generated by (4)-(5), $\Phi(u^k,\tilde{u}^k)$ be defined by (15) and $\Theta_{\rm II}^k(\alpha)$ be defined by (37). Then, we have

$$\Theta_{\rm II}^k(\alpha) \ge \Upsilon_{\rm II}^k(\alpha) + \|u^k - \alpha M(u^k-\tilde{u}^k) - u_{\rm II}^{k+1}(\alpha)\|^2, \tag{39}$$

where

$$\Upsilon_{\rm II}^k(\alpha) = 2\alpha\Phi(u^k,\tilde{u}^k) - \alpha^2\|u^k-\tilde{u}^k\|_M^2.$$

Proof. Note that $d_2(u^k,\tilde{u}^k)$ can be written as

$$\begin{aligned} & d_2(u^k,\tilde{u}^k) \\ = & \begin{pmatrix} f(\tilde{x}^k) - A^\top[y^k - H(Ax^k-b)] + R(\tilde{x}^k - x^k) \\ 0 \end{pmatrix} + \begin{pmatrix} R(x^k-\tilde{x}^k) \\ H^{-1}(y^k-\tilde{y}^k) \end{pmatrix}. \end{aligned}$$

Therefore, from $u_{\rm II}^{k+1}(\alpha)\in\mathcal{U}$, we have

$$\begin{aligned} & (u_{\rm II}^{k+1}(\alpha)-\tilde{u}^k)^\top d_2(u^k,\tilde{u}^k) \\ = & (x_{\rm II}^{k+1}(\alpha)-\tilde{x}^k)^\top\{f(\tilde{x}^k) - A^\top[y^k - H(Ax^k-b)] \\ & +R(\tilde{x}^k-x^k)\} + (u_{\rm II}^{k+1}(\alpha)-\tilde{u}^k)^\top M(u^k-\tilde{u}^k) \\ \ge & (u_{\rm II}^{k+1}(\alpha)-\tilde{u}^k)^\top M(u^k-\tilde{u}^k) (\text{using (4) and (5)}). \end{aligned} \tag{40}$$

Since $u^*\in\mathcal{U}^*$, it follows from (12) that

$$\|u_{\rm II}^{k+1}(\alpha)-u^*\|^2 \le \|u^k-\alpha d_2(u^k,\tilde{u}^k)-u^*\|^2 - \|u^k-\alpha d_2(u^k,\tilde{u}^k)-u_{\rm II}^{k+1}(\alpha)\|^2. \tag{41}$$

Substitute (41) into (37), we have

$$\begin{aligned} & \Theta_{\rm II}^k(\alpha) \\ \ge & \|u^k-u^*\|^2 - \|u^k-\alpha d_2(u^k,\tilde{u}^k)-u^*\|^2 \\ & +\|u^k-\alpha d_2(u^k,\tilde{u}^k)-u_{\rm II}^{k+1}(\alpha)\|^2 \\ = & \|u^k-u_{\rm II}^{k+1}(\alpha)\|^2 + 2\alpha(u^k-\tilde{u}^k+\tilde{u}^k-u^*)^\top d_2(u^k,\tilde{u}^k) \\ & +2\alpha(u_{\rm II}^{k+1}(\alpha)-u^k)^\top d_2(u^k,\tilde{u}^k) \\ \ge & \|u^k-u_{\rm II}^{k+1}(\alpha)\|^2 + 2\alpha(u^k-\tilde{u}^k)^\top d_2(u^k,\tilde{u}^k) \\ & +2\alpha(y^k-\tilde{y}^k)^\top A(x^k-\tilde{x}^k) + 2\alpha(u_{\rm II}^{k+1}(\alpha)-u^k)^\top d_2(u^k,\tilde{u}^k)(\text{using (35)}) \\ = & \|u^k-u_{\rm II}^{k+1}(\alpha)\|^2 + 2\alpha(u_{\rm II}^{k+1}(\alpha)-\tilde{u}^k)^\top d_2(u^k,\tilde{u}^k) \\ & +2\alpha(y^k-\tilde{y}^k)^\top A(x^k-\tilde{x}^k) \\ \ge & \|u^k-u_{\rm II}^{k+1}(\alpha)\|^2 + 2\alpha(u_{\rm II}^{k+1}(\alpha)-\tilde{u}^k)^\top M(u^k-\tilde{u}^k) \\ & +2\alpha(y^k-\tilde{y}^k)^\top A(x^k-\tilde{x}^k)(\text{using (40)}) \\ = & \|u^k-u_{\rm II}^{k+1}(\alpha)\|^2 + 2\alpha(u_{\rm II}^{k+1}(\alpha)-u^k)^\top M(u^k-\tilde{u}^k) + 2\alpha\Phi(u^k,\tilde{u}^k) \\ = & \|u^k-\alpha M(u^k-\tilde{u}^k)-u_{\rm II}^{k+1}(\alpha)\|^2 + 2\alpha\Phi(u^k,\tilde{u}^k) - \alpha^2\|u^k-\tilde{u}^k\|_M^2. \end{aligned} \tag{42}$$

Then the assertion of the theorem is proved. This completes the proof.

Theorem 3.2 shows that $\Upsilon_{\rm II}^k(\alpha)$ is a lower bound of $\Theta_{\rm II}^k(\alpha)$ for any $\alpha \geq 0$. Note that $\Theta_{\rm II}^k(\alpha)$ is a quadratic function of α and it reaches its maximum at α_k^2.

In the following, we assume that the proposed IDM-I or IDM-II method generates an infinite sequence $\{u^k\}$, otherwise, an approximate solution $u^k \in \mathcal{U}$ is obtained. We are now in the position to prove the global convergence of the proposed methods.

Theorem 3.3. Suppose that the function f is monotone on $\mathcal{X}$, and let the sequence $\{u^k\}$ be generated by the IDM-I method. Then, we have

(1). The sequence $\{u^k\}$ is bounded.

(2). The sequence $\{u^k\}$ converges to some $u^* \in \mathcal{U}^*$.

Proof. (1) Using (36) and (38), we obtain

$$\begin{array}{rl} & \|u^{k+1}-u^*\|^2 \\ \leq & \|u^k-u^*\|^2 - \Upsilon_{\rm I}^k(\gamma\alpha_k^1) \\ = & \|u^k-u^*\|^2 - 2\gamma\alpha_k^1\Phi(u^k,\tilde{u}^k) + \gamma^2(\alpha_k^1)^2\|d_1(u^k,\tilde{u}^k)\|^2 \\ = & \|u^k-u^*\|^2 - \gamma(2-\gamma)\alpha_k^1\Phi(u^k,\tilde{u}^k)(\text{using } (17)) \\ \leq & \|u^k-u^*\|^2 - \gamma(2-\gamma)\lambda_1\lambda_2\|u^k-\tilde{u}^k\|_M^2(\text{using } (23)(24)). \end{array} \tag{43}$$

Thus $\{u^k\}$ is bounded.

(2) Since $\{u^k\}$ is bounded, it has at least one cluster point, denoted as $u^* = (x^*, y^*)$. It follows from (43) that

$$\lim_{k\to\infty} \|u^k - \tilde{u}^k\|_M = 0. \tag{44}$$

Thus $\{\tilde{u}^k\}$ is also bounded and $u^* = (x^*, y^*)$ is also a cluster point of $\{\tilde{u}^k\}$ and the subsequence $\{\tilde{u}^{k_j}\}$ converges to u^*. Moreover, (4)-(5) and (44) imply that

$$\left\{ \begin{array}{l} \lim_{j\to\infty}(x-\tilde{x}^{k_j})^\top\{f(\tilde{x}^{k_j}) - A^\top\tilde{y}^{k_j}\} \geq 0, \quad \forall x \in \mathcal{X}; \\ \lim_{j\to\infty} A\tilde{x}^{k_j} - b = 0 \end{array} \right. \tag{45}$$

and consequently

$$\left\{ \begin{array}{l} (x-x^*)^\top\{f(x^*) - A^\top y^*\} \geq 0, \quad \forall x \in \mathcal{X}; \\ Ax^* - b = 0 \end{array} \right.$$

which implies that $u^* \in \mathcal{U}^*$. Since (44) and $\{u^{k_j}\} \to u^\infty$, for any given $\varepsilon > 0$, there is an integer l, such that

$$\|u^{k_j} - \tilde{u}^{k_j}\| < \frac{\varepsilon}{2}, \quad \text{and} \quad \|\tilde{u}^{k_j} - u^*\| < \frac{\varepsilon}{2}.$$

Therefore, for any $k \geq k_l$, it follows from (43) that

$$\|u^k - u^*\| \leq \|u^{k_j} - u^*\| \leq \|u^{k_j} - \tilde{u}^{k_j}\| + \|\tilde{u}^{k_j} - u^*\| < \varepsilon.$$

Thus, the sequence $\{u^k\}$ converges to u^*, which is a solution of VI(F,$\mathcal{U}$). This completes the proof.

Theorem 3.4. Suppose that the function f is monotone on $\mathcal{X}$, and let the sequence $\{u^k\}$ be generated by the IDM-II method. Then, we have

(1). The sequence $\{u^k\}$ is bounded.

(2). The sequence $\{u^k\}$ converges to some $u^* \in \mathcal{U}^*$.

Proof. Its proof is similar to that of Theorem 3.3, so is omitted.

4 Preliminary Computational Results

In this section, we illustrate the efficiency of our method. The example used here is the test problem in paper[10], which constraint set S and the mapping f are taken, respectively, as

$$S = \{x \in R^5_+ | \sum_{i=1}^{5} x_i = 10\},$$

and

$$f(x) = Mx + \rho C(x) + q,$$

where M is an $R^{5\times5}$ asymmetric positive matrix and $C_i(x) = \arctan(x_i - 2), i = 1, 2, \cdots, 5$. The parameter ρ is used to vary the degree of asymmetry and nonlinearity. The data of example are illustrate as follows:

$$M = \begin{pmatrix} 0.726 & -0.949 & 0.266 & -1.193 & -0.504 \\ 1.645 & 0.678 & 0.333 & -0.217 & -1.443 \\ -1.016 & -0.225 & 0.769 & 0.943 & 1.007 \\ 1.063 & 0.587 & -1.144 & 0.550 & -0.548 \\ -0.256 & 1.453 & -1.073 & 0.509 & 1.026 \end{pmatrix}$$

and

$$q = (5.308,\ 0.008\ , -0.938,\ 1.024,\ \ -1.312)^\top.$$

In the experiment, we take the stopping criterion $\varepsilon = 10^{-6}$ as the initial point. All programs are coded in Matlab 7.1. 'IN' denotes the number of iterations and 'CPU' denotes the CPU time in seconds. 'ODM' denotes the decomposition method proposed by Gabay and Mercier.

The results in the Table 1 and Table 2 indicate that the performance of the improved methods are better than the original decomposition method.

Table 1: Numerical results for $\rho = 10$.

Starting point	Method	IN	CPU	Error
(0 2.5 2.5 2.5 2.5)	ODM	97	0.02	8.4920×10^{-7}
	IDM-I	51	0.02	8.5978×10^{-7}
	IDM-II	32	0.02	8.4052×10^{-7}
(10 0 0 0 0)	ODM	95	0.02	9.2592×10^{-7}
	IDM-I	51	0.02	9.1903×10^{-7}
	IDM-II	30	0.01	9.1051×10^{-7}
(25 0 0 0 0)	ODM	83	0.03	9.5720×10^{-7}
	IDM-I	58	0.02	8.5539×10^{-7}
	IDM-II	40	0.02	8.9207×10^{-7}
(10 0 10 0 10)	ODM	93	0.03	7.6515×10^{-7}
	IDM-I	69	0.03	8.6384×10^{-7}
	IDM-II	41	0.01	8.5981×10^{-7}

5 Conclusions

In this paper, we observe two new descent directions at each iteration, and present two descent decomposition methods for monotone variational inequalities with linear equality constraint. Under some mild conditions, we proved the global convergence of the two new methods. Some preliminary computational results illustrated the efficiency of the proposed methods.

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Table 2: Numerical results for $\rho = 20$.

Starting point	Method	IN	CPU	Error
(0 2.5 2.5 2.5 2.5)	ODM	170	0.03	9.3854×10^{-7}
	IDM-I	107	0.06	8.1140×10^{-7}
	IDM-II	52	0.02	9.7452×10^{-7}
(10 0 0 0 0)	ODM	170	0.03	9.3004×10^{-7}
	IDM-I	104	0.05	8.2548×10^{-7}
	IDM-II	52	0.01	9.7551×10^{-7}
(25 0 0 0 0)	ODM	163	0.03	9.5615×10^{-7}
	IDM-I	107	0.06	8.1140×10^{-7}
	IDM-II	54	0.02	9.7538×10^{-7}
(10 0 10 0 10)	ODM	170	0.03	8.9971×10^{-7}
	IDM-I	111	0.05	7.6826×10^{-7}
	IDM-II	55	0.02	7.9127×10^{-7}

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