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Wavelet-Galerkin Finite Difference Solutions of ODEs

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Abstract - Obtaining solutions of ordinary differential equations through fictitious boundary or other approach of wavelet-Galerkin requires use of connection coefficients or fast Fourier transform which involves computational complexities and time consumption. In this paper, finite difference based wavelet-Galerkin method has been developed for ordinary differential equations which is rather simple and avoids above complexities. The method is testified using numerical problems. The solutions are comparably good, oftentimes better in comparison to known numerical techniques such as Euler and finite difference at a suitably chosen higher value of the scale.

Keywords: Multiresolution Analysis, Finite Difference Technique, Wavelet-Galerkin Technique, Boundary Value Problem.

1. Introduction

Since the contribution of orthogonal bases of compactly supported wavelet by Daubechies (1988) and multiresolution analysis based fast wavelet transform algorithm by Belkin (1992)[2], wavelet based approximation of ordinary and partial differential equations gained momentum in attractive way. Wavelets have the capability of representing the solutions at different levels of resolutions, which make them particularly useful for developing hierarchical solutions to engineering problems. Among the approximations, wavelet-Galerkin technique is the most frequently used scheme these days. Daubechies

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wavelets as bases in a Galerkin method - to solve differential equations - require a computational domain of simple shape. This has become possible due to the remarkable work by Latto et al. (1992), Xu et al. (1994), Williams et al. (1993&1994), Amartunga et al. (1994) and Dianfeng (1996) (see [1], [4-8]). Yet there is difficulty in dealing with boundary conditions. So far problems with periodic boundary conditions or periodic distribution have been dealt successfully. A good attempt was made by Beylkin (1994)[3] to find the solution of boundary value problem by applying wavelet to finite difference scheme.

2. Wavelet-Galerkin Method

In order to find the solution of differential equation

$$Lu = f, \tag{1}$$

u,*f* are real valued and continuous functions of *x* on $L^2(R)$. L is a uniformly elliptic differential operator.

The solution *u* is approximated by an approximate function

$$\tilde{u} = \sum_{j=1}^{n} c_j \, \varphi_j \tag{2}$$

which is a combination of trial functions. The approach is based on projection method since approximate solution is a projection of exact solution onto subspace spanned by the trial functions. It was a Russian Engineer V.I. Galerkin who proposed a projection method in 1917 based on the weak form or equation of virtual work. Accordingly, the residual be considered orthogonal to the test functions, i.e.

$$< v, L\tilde{u} - f > = 0 \tag{3}$$

where *v* should be of the same form as $\tilde{u}$, i.e.

$$v = \sum_{i=1}^{n} a_i \, \varphi_{i.} \tag{4}$$

Eq. (3) using (4) gives

$$\sum_{j=1}^{n} c_j < \varphi_i, \varphi_j > = < \varphi_i, f >, \; i = 1, 2, \dots, n \tag{5}$$

In matrix form this can be written as

$$Sc = f \tag{6}$$

where *S* is an nxn matrix with elements $s_{i,j} = < \varphi_i, \varphi_j >$, $c = \left(c_{1,} \dots, c_n\right)^T$,

$c = (< \varphi_i, f >, \dots, < \varphi_n, f >)^T$.

When the base function in Galerkin method are wavelets, it is called wavelet-Galerkin method[WGM]. This method is limited to the cases for unbounded domain or periodic boundary conditions.

Theorem 2. Let $V_j,\ \ j \in Z$ be a given MRA with scaling function φ and $P_j f$, projection of $f \in L^2(R)$ onto V_j such that

$$P_j f = \sum_k c_k 2^{j/2} \varphi\left(2^j x - k\right),$$

then for j sufficiently large

$$c_k \cong 2^{-j/2} f(k2^{-j}) \ \text{ with } \int \overline{\varphi(x)}\ dx = 1.$$

3. Development of Method

Here we develop a wavelet-Galerkin finite difference method [WGFDM], based on finite difference approach, to find wavelet solution of certain ODEs.

Lemma. For large $j \in Z_+$, $f^{(n)}(x) = 2^{nj} \sum_{i=0}^{n} (-1)^{n+i} \binom{n}{i} f\left(x + \frac{i}{2^j}\right)$.

Proof. For $\frac{1}{2^j}$ small,

$$f'(x) = \frac{f\left(x+\frac{1}{2^j}\right) - f(x)}{\frac{1}{2^j}} = 2^j \left[f\left(x + \frac{1}{2^j}\right) - f(x)\right]$$

using forward difference Taylor's expansion.

$$f''(x) = 2^j \left[f'\left(x + \frac{1}{2^j}\right) - f'(x)\right]$$

$$= 2^{2j} \left[\left\{f\left(x + \frac{2}{2^j}\right) - 2f\left(x + \frac{1}{2^j}\right) + f(x)\right\}\right]$$

$$f'''(x) = 2^{2j} \left[\left\{f'\left(x + \frac{2}{2^j}\right) - 2f'\left(x + \frac{1}{2^j}\right) + f'(x)\right\}\right]$$

$$= 2^{3j} \left[f\left(x + \frac{3}{2^j}\right) - 3f\left(x + \frac{2}{2^j}\right) + 3f\left(x + \frac{1}{2^j}\right) - f(x)\right]$$

Proceeding in this way,

$$f^{(n)}(x) = \quad 2^{nj}\left[f\left(x+\frac{n}{2^j}\right)-\binom{n}{1}f\left(x+\frac{n-1}{2^j}\right)+\binom{n}{2}f\left(x+\frac{n-2}{2^j}\right)+\right.$$
$$\left.\ldots\ldots+\binom{n}{n-1}f\left(x+\frac{1}{2^j}\right)-f(x)\right]$$

$$= 2^{nj}\sum_{i=0}^{n}(-1)^{n+i}\binom{n}{i}f\left(x+\frac{i}{2^j}\right), \quad j\in Z_+$$

Remark. For large $j \in Z_+$, $f^{(n)}(x) = 2^{nj}\sum_{i=0}^{n}(-1)^i\binom{n}{n-i}f\left(x-\frac{i}{2^j}\right)$.

This can easily be proved by letting $f'(x) = 2^j\left[f(x)-f\left(x-\frac{1}{2^j}\right)\right]$.

4. Applications in ODEs

To solve nth order linear ODE

$$\sum_{l=0}^{n} A_l f^{(l)}(x) = B(x). \tag{7}$$

Let

$$f(x) = \sum \alpha_{j,k}\,\varphi_{jk}(x) = \sum_k \alpha_{j,k}2^{j/2}\,\varphi\left(2^jx-k\right). \tag{8}$$

Using above Lemma,

$$f^l(x) = 2^{lj}\sum_{i=0}^{l}(-1)^{l+i}\binom{l}{i}\sum_k \alpha_{j,k}2^{j/2}\,\varphi\left(2^j(x+\frac{i}{2^j})-k\right)$$
$$= 2^{lj}\sum_{i=0}^{l}(-1)^{l+i}\binom{l}{i}\sum_k \alpha_{j,k}2^{j/2}\,\varphi\left(2^jx+i-k\right)$$
$$= 2^{lj}\sum_{i=0}^{l}(-1)^{l+i}\binom{l}{i}\sum_k \alpha_{j,k+i}\,\varphi_{jk}.$$

Substituting in (7),

$$\sum_{l=0}^{n} A_l 2^{lj}\sum_{i=0}^{l}(-1)^{l+i}\binom{l}{i}\sum_k \alpha_{j,k+i}\,\varphi_{jk} = B(x).$$

Taking inner product with $\varphi_{j,m}$,

$$\sum_{l=0}^{n} A_l 2^{lj}\sum_{i=0}^{l}\binom{l}{i}(-1)^{l+i}\alpha_{j,m+i} = C(x), \tag{9}$$

where $C(x) = \int_R B(x)\,\varphi_{j,m}dx$.

5. Test Problems

Test Problem 1: First order ODE

Let us consider the first order linear ODE

$$f'(x) + af(x) = b, \quad x \in [0,1] \tag{10}$$

with initial boundary condition $f(0) = c$.

Take $n = 1, A_0 = a,\ A_1 = 1\ \ C(x) = b =$ constant in (9)

For j sufficiently large, $\alpha_{j,0} = \langle f, \varphi_{j,0} \rangle = 2^{-j/2} f(0) = c2^{-j/2}$ for $k = 0$ and $\alpha_{j,k} = 2^{-j/2} f(2^{-j}k)$.

The eq.(9) reduces to

$$\sum_{l=0}^{1} A_l 2^{lj} \sum_{i=0}^{l} \binom{l}{i} (-1)^{l+i} \alpha_{j,m+i} = b.$$

$$A_0 \sum_{i=0}^{0} \binom{0}{i} (-1)^{0+i} \alpha_{j,m+i} + A_1 2^j \sum_{i=0}^{1} \binom{1}{i} (-1)^{1+i} \alpha_{j,m+i} = b \tag{11}$$

That is

$$\alpha_{j,m+1} - B\alpha_{j,m} = A, \text{ where } A = \frac{b}{2^{3j/2}}, B = 1 - \frac{a}{2^j}. \tag{12}$$

Let $m = 0, 1, 2 \ldots\ldots, k - 1$ in (12). The system (11) converts to

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & - & 0 & 0 & 0 \\ -B & 1 & 0 & 0 & 0 & - & 0 & 0 & 0 \\ 0 & -B & 1 & 0 & 0 & - & 0 & 0 & 0 \\ - & - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - & - \\ 0 & 0 & 0 & 0 & 0 & - & -B & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & - & 0 & 0 & -B \end{bmatrix} \begin{bmatrix} \alpha_{j,1} \\ \alpha_{j,2} \\ \alpha_{j,3} \\ - \\ - \\ - \\ \alpha_{j,k-2} \\ \alpha_{j,k-1} \end{bmatrix} = \begin{bmatrix} A + B\alpha_{j,0} \\ A \\ A \\ - \\ - \\ A \\ A \\ A - \alpha_{j,k} \end{bmatrix}$$

Solving to find $\alpha_{j,k}$.

Certainly j has to be raised to a comfortably higher value in order to obtain better solution.

Alternatively

$$\alpha_{j,1} = A + B\alpha_{j,0}$$

$$\alpha_{j,2} = A + B\alpha_{j,1}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\alpha_{j,k} = A + B\alpha_{j,k-1}$$

This leads to

$$\alpha_{j,k} = A\left[1 + \left(1 - \frac{\mathrm{a}}{2^j}\right) + \left(1 - \frac{\mathrm{a}}{2^j}\right)^2 + \cdots + \left(1 - \frac{\mathrm{a}}{2^j}\right)^{k-1}\right] + \left(1 - \frac{\mathrm{a}}{2^j}\right)^k \alpha_{j,0}.$$

That is,

$$2^{-j/2} f\left(2^{-j}k\right) = \frac{b}{2^{3j/2}}\left[\frac{1 - \left(1 - \frac{\mathrm{a}}{2^j}\right)^k}{1 - \left(1 - \frac{\mathrm{a}}{2^j}\right)}\right] + c2^{-j/2}\left(1 - \frac{\mathrm{a}}{2^j}\right)^k.$$

This leads to

$$f(2^{-j}k) = \frac{b}{a}\left[1 - \left(1 - \frac{\mathrm{a}}{2^j}\right)^k\right] + c\left(1 - \frac{\mathrm{a}}{2^j}\right)^k.$$

The exact solution is: $f(x) = \frac{b}{a} + \left(c - \frac{b}{a}\right)e^{-ax}$.

Numerical Example 1

Let $a = 1, b = 1, c = 0$ in problem (10)

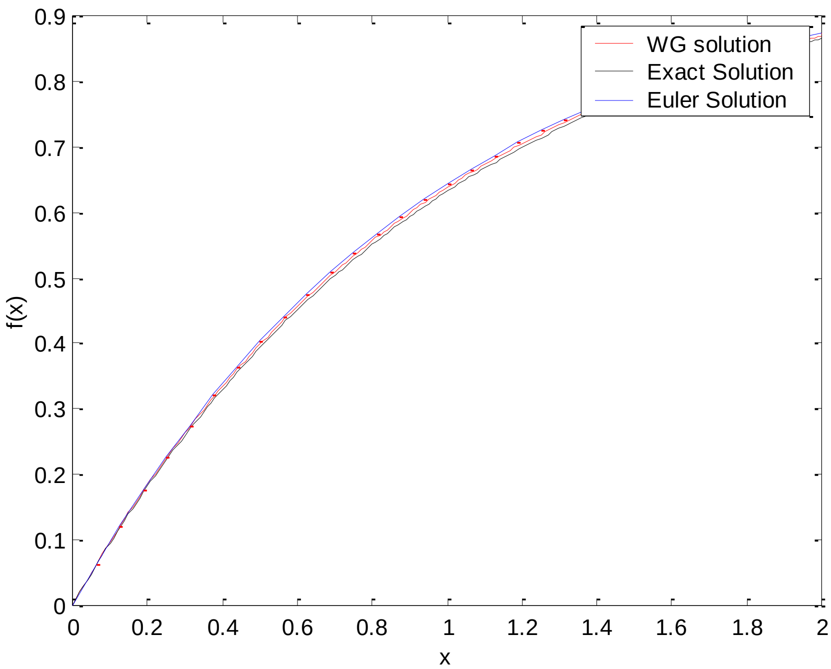

Fig.1: Solution for $j = 5$

Test Problem2: Second order ODE

$$\sum_{l=0}^{2} A_l \ f^l(x) = F(x) \tag{13}$$

with boundary conditions $f(0),\ f(1)$.

We choose $A_2 = 1\ n = 2$ in (9). The solution *f* is chosen as in (8). Symbolize $F(2^{-j}k) = r(k)$.

Eq.(9) reduces to

$$\sum_{l=0}^{2} A_l 2^{lj} \sum_{i=0}^{l} \binom{l}{i} (-1)^{l+i} \sum_k \alpha_{j,m+i} \quad = F(x).$$

That is,

$$A_0 2^{0j} \sum_{i=0}^{0} \binom{0}{i} (-1)^{0+i} \alpha_{j,m+i} \quad + \ A_1 2^j \sum_{i=0}^{1} \binom{1}{i} (-1)^{1+i} \alpha_{j,m+i}$$
$$+ \, 2^{2j} \sum_{i=0}^{2} \binom{2}{i} (-1)^{2+i} \alpha_{j,m+i} \ = \frac{1}{2^{5j/2}} r(k).$$

That is,

$$\alpha_{j,m+2} = -\left(p\alpha_{j,m} + q\alpha_{j,m+1}\right) + \frac{1}{2^{5j/2}} r(k), \tag{14}$$

where $p = \frac{A_0}{2^{2j}} - \frac{A_1}{2^j} + 1,\ \ q = \frac{A_1}{2^j} - 2.$

Letting $m = 0, 1, 2, \dots . k-2$, the system reduces to

$$\begin{bmatrix} q & 1 & 0 & 0 & 0 & - & 0 & 0 & 0 \\ p & q & 1 & 0 & 0 & - & 0 & 0 & 0 \\ 0 & p & q & 1 & 0 & - & 0 & 0 & 0 \\ - & - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - & - \\ 0 & 0 & 0 & 0 & 0 & - & p & q & 1 \\ 0 & 0 & 0 & 0 & 0 & - & 0 & p & q \end{bmatrix} \begin{bmatrix} \alpha_{j,1} \\ \alpha_{j,2} \\ \alpha_{j,3} \\ - \\ - \\ - \\ - \\ \alpha_{j,k-2} \\ \alpha_{j,k-1} \end{bmatrix} = \begin{bmatrix} 2^{-j/2}F(0) - p\alpha_{j,0} \\ F(2^{-j}) \\ F(2^{-j}2) \\ - \\ - \\ - \\ - \\ F(2^{-j}(k-1)) \\ -2^{-j/2}F\left(2^{-j}(k-2)\right) + \alpha_{j,k} \end{bmatrix}$$

Solving to find $\alpha_{j,k}{}'s$.

Certainly j has to be taken higher for a better solution.

Alternatively

Putting $m = 0, 1, 2, \dots \dots . k-2$ in (14), we find

$$\alpha_{j,2} = -p\alpha_{j,0} - q\alpha_{j,1} + \frac{r(0)}{2^{5j/2}}.$$

$$\alpha_{j,3} = -p\alpha_{j,1} - q\alpha_{j,2} + \frac{r(1)}{2^{5j/2}}$$
$$= pq\alpha_{j,0} + (-p+q^2)\alpha_{j,1} + \frac{1}{2^{\frac{5j}{2}}}(r(1) - qr(0)).$$

$$\alpha_{j,4} = -p\alpha_{j,2} - q\alpha_{j,3} + \frac{r(2)}{2^{5j/2}}$$
$$= (p^2 - pq^2)\alpha_{j,0} + (2pq - q^3)\alpha_{j,1} + \frac{1}{2^{\frac{5j}{2}}}[r(2) - qr(1) - (p - q^2)r(0)].$$

$$\alpha_{j,5} = -p\alpha_{j,3} - q\alpha_{j,4} + \frac{r(3)}{2^{5j/2}}$$
$$= (-2p^2q + pq^3)\alpha_{j,0} + (p^2 - 3pq^2 + q^4)\alpha_{j,1}$$
$$+ \frac{1}{2^{\frac{5j}{2}}}[(r(3) - (p - q^2)r(1) - qr(2) + (2pq - q^3)r(0)].$$

$$\alpha_{j,6} = -p\alpha_{j,4} - q\alpha_{j,5} + \frac{r(4)}{2^{5j/2}}$$
$$= (-p^3 + 3p^2q^2 - pq^4)\alpha_{j,0} + (-3p^2q + 4pq^3 - q^5)\alpha_{j,1} + \frac{1}{2^{\frac{5j}{2}}}[r(4) - qr(3)$$
$$- (p - q^2)r(2) - qr(3) + (2pq - q^3)r(1) + (p^2 - 3pq^2 + q^4)r(0)].$$

$$\alpha_{j,7} = -p\alpha_{j,5} - q\alpha_{j,6} + \frac{r(5)}{2^{5j/2}}$$
$$= (3p^3q - 4p^2q^3 + pq^5)\alpha_{j,0} + (-p^3 + 6p^2q^2 - 5pq^4 + q^6)\alpha_{j,1}$$
$$+ \frac{1}{2^{\frac{5j}{2}}}[r(5) - qr(4) - (p - q^2)r(3) + (2pq - q^3)r(2)$$
$$+ (p^2 - 3pq^2 + q^4)r(1) + (4pq^3 - 3p^2q - q^5)r(0)].$$

$$\alpha_{j,8} = -p\alpha_{j,6} - q\alpha_{j,7} + \frac{r(6)}{2^{5j/2}}$$

$$= (p^4 - 6p^3q^2 + 5p^2q^4 - pq^6)\alpha_{j,0} + (4p^3q - 10p^2q^3 + 6pq^5 - q^4)\alpha_{j,1}$$
$$+ \frac{1}{2^{\frac{5j}{2}}}[r(6) - qr(5) - (p - q^2)r(4) + (2pq - q^3)r(3)$$
$$+ (p^2 - 3pq^2 + q^4)r(2) + (4pq^3 - 3p^2q - q^5)r(1) + (6p^2q^2 - p^3$$
$$- 5pq^4 + q^6)r(0)].$$

And so on.

Here $r(0) = F(0), r(1) = F(2^{-j}1), r(2) = F(2^{-j}2), r(3) = F(2^{-j}3), r(4) = F(2^{-j}4), r(5) = F(2^{-j}5),\ r(6) = F(2^{-j}6)$.

In this way it is difficult to proceed for higher j. This in turn may be supplicated by the convenient matrix algorithm below:

I. $$\alpha_{j,k}I = A\begin{bmatrix}\alpha_{j,k-2}\\ \alpha_{j,k-1}\end{bmatrix} + r(0)I$$

$$\begin{bmatrix}\alpha_{j,k-2}\\ \alpha_{j,k-1}\end{bmatrix} = A^{-1}\left[(\alpha_{j,k} - r(0))I\right] \quad \text{as } \alpha_{j,k} \text{is known}$$

Wherein

$$A = \begin{bmatrix}-p & 0\\ 0 & -q\end{bmatrix}, \quad A^{-1} = \begin{bmatrix}-\frac{1}{p} & 0\\ 0 & -\frac{1}{q}\end{bmatrix}$$

II. Solve the matrix system ( ) for each $k = k, k-2, k-4, \dots\dots, 6$

Finally for $k = 4$

$$\begin{bmatrix}\alpha_{j,2}\\ \alpha_{j,3}\end{bmatrix} = A^{-1}\left[(\alpha_{j,4} - r(0))I\right].$$

III. The solution $\alpha_{j,2}$ obtained above is used to get the value of $\alpha_{j,1}$ from the first relation amongst$\alpha_{j,0}, \alpha_{j,1}, \alpha_{j,2}$.

Hence all $\alpha_{j,k}{}'s$ are evaluated.

Numerical Example 2

Let $A_0 = 1, A_1 = 0, F = \sin(4\pi x)$ in eq.(12). The BCs are $f(0) = 1,\ f(1) = 0$.

Exact solution is: $cosx + \frac{1}{sin1}\left[\frac{sin4\pi}{16\pi^2-1} - cos1\right]sinx - \frac{sin4\pi x}{16\pi^2-1}$.

The following figure shows the comparison between the exact, FDM and WGFDM.

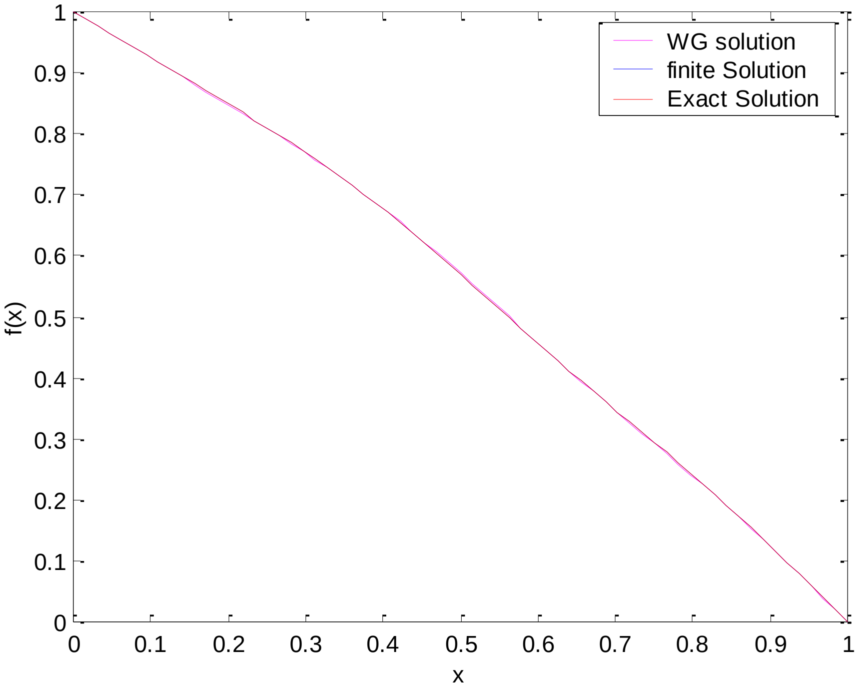

Fig.2: Solution for $j = 6$

6. Conclusion

Wavelet-Galerkin finite difference technique for ODEs has been shown through numerical problems to be effective and oftentimes better than numerical techniques such as of Euler and finite difference for scale j large enough. In order to find solution of ODEs through wavelet-Galerkin method: (1) one needs to find connection coefficients for desired scale, and (2) use of scaling functions corresponding to that value of the scale. Whereas in the present technique the comparably good solution is obtained with less computational cost and without involvement of scaling functions at the chosen scale.

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