AMO - Advanced Modeling and Optimization, Volume 13, Number 2, 2011

New Approach Based on Relatively Inexact Proximal Point Algorithms to Linear Convergence Analysis and Applications

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Abstract

A new approach based the *relatively maximal* $(m)-$*relaxed monotonicity* is introduced and then it is applied to the linear convergence analysis in the context of the approximation solvability of a general class of inclusion problems, while generalizing Rockafellar's theorem (1976) on linear convergence using the proximal point algorithm in a real Hilbert space setting. Convergence analysis, based on the new model, turns out to be more general than that of the celebrated technique of Rockafellar that is limited to the Lipschitz continuity at 0 of the inverse of the set-valued mapping considered.

AMS (MOS) Mathematics Subject Classifications: 49J40, 47H10, 65B05

Keywords: Inclusion problems, Maximal monotone mapping, Relatively maximal $(m)-$relaxed monotone mapping, Generalized resolvent operator.

1. Introduction

We start with a real Hilbert space X that has the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$ on X. We consider the nonlinear variational inclusion problem: determine a solution to

$$0 \in M(x), \tag{1}$$

where $M : X \to 2^X$ is a set-valued mapping on X.

Rockafellar ([5], Theorem 2) investigated and studied general convergence of the proximal point algorithm in the context of solving (1), by showing for M maximal monotone, that the sequence $\{x^k\}$ generated for an initial point x^0 by the proximal point algorithm

$$x^{k+1} \approx P_k(x^k) \tag{2}$$

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AMO - Advanced Modeling and Optimization. ISSN: 1841-4311

converges strongly to a solution of (1), provided the approximation is made sufficiently accurate as the iteration proceeds, where $P_k = (I + c_k M)^{-1}$ is the resolvent operator for a sequence $\{c_k\}$ of positive real numbers, that is bounded away from zero. We observe from (2) that x^{k+1} is an approximate solution to inclusion problem

$$0 \in M(x) + c_k^{-1}(x - x^k). \tag{3}$$

Next, we recall Rockafellar's theorem ([5], Theorem 2) based on the approach of the Lipschitz continuity of M^{-1} instead of the strong monotonicity of M that turned out to be more application-oriented to convex programming.

Theorem 1.1. Let X be a real Hilbert space, and let $M : X \to 2^X$ be maximal monotone. For an arbitrarily chosen initial point x^0, let the sequence $\{x^k\}$ be generated by the proximal point algorithm

$$x^{k+1} \approx P_k(x^k) \tag{4}$$

such that

$$\|x^{k+1} - P_k(x^k)\| \le \epsilon_k,$$

where $P_k = (I + c_k M)^{-1}$, and the scalar sequences $\{\epsilon_k\}$ and $\{c_k\}$, respectively, satisfy $\Sigma_{k=0}^{\infty} \epsilon_k < \infty$ and $\{c_k\}$ is bounded away from zero.
We further suppose that sequence $\{x^k\}$ is generated by the proximal point algorithm

$$x^{k+1} \approx P_k(x^k) \tag{5}$$

such that

$$\|x^{k+1} - P_k(x^k)\| \le \delta_k \|x^{k+1} - x^k\|,$$

where scalar sequences $\{\delta_k\}$ and $\{c_k\}$, respectively, satisfy $\Sigma_{k=0}^{\infty} \delta_k < \infty$ and $c_k \uparrow c \le \infty$.
Also, assume that $\{x^k\}$ is bounded in the sense that there exists at least one solution to (1), and that M^{-1} is $(a)-$Lipschitz continuous at 0 for a>0. Let

$$\mu_k = \frac{a}{\sqrt{a^2 + c_k^2}} < 1.$$

Then the sequence $\{x^k\}$ converges strongly to x^*, a unique solution to (1) with

$$\|x^{k+1} - x^*\| \le \alpha_k \|x^k - x^*\| \, \forall \, k \ge k', \tag{6}$$

where

$$0 \le \alpha_k = \frac{\mu_k + \delta_k}{1 - \delta_k} < 1 \, \forall \, k \ge k', \tag{7}$$

and

$$\alpha_k \to 0 \;\; as \; c_k \to \infty. \tag{8}$$

As it seems that most the of variational problems, including minimization or maximization of functions, variational inequality problems, quasivariational inequality problems, and decision and management sciences can be unified into form (1), the general maximal monotonicity models have played a crucial role by providing a powerful framework to develop and use suitable proximal point algorithms in studying convex programming as well as variational inequalities. This algorithm turned out to be of more interest because of its role in certain computational methods based on duality, for instance the $Hestenes - Powell$ method of multipliers in nonlinear programming. For more details, we refer the reader [1-15].

In this communication, first we start examining the approximation solvability of inclusion problem (1) based on the notion of relatively maximal $(m)-$relaxed monotone mappings, and then derive some auxiliary results involving relatively maximal $(m)-$relaxed monotone and generalized cocoercive mappings. The notion of the relatively maximal $(m)-$relaxed monotonicity is based on $A-$maximal $(m)-$relaxed monotonicity introduced and studied in [9,10], but is more application-enhanced to convergence analysis. Our approach for the approximation solvability of (1) provides a simpler and compact proof, unlike the linear convergence method applied in ([5], Theorem 2). There exists a vast literature on proximal point algorithms and its applications to approximating solutions of inclusion problems of the form (1) in different space settings, especially in Hilbert and in Banach space settings.

2. Relative Maximal $(m)-$Relaxed Monotonicity

In this section, first we introduce the notion of the *relative maximal* $(m)-$*relaxed monotonicity*, and then we discuss some basic properties along with some auxiliary results for the problem on hand.

Let X be a real Hilbert space with the norm $\|\cdot\|$ and with the inner product $\langle\cdot,\cdot\rangle$.

Definition 2.1. Let X be a real Hilbert space, and let $M : X \to 2^X$ be a multivalued mapping and $A : X \to X$ be a single-valued mapping on X. The map M is said to be:

(i) Monotone if

$$\langle u^* - v^*, u - v\rangle \geq 0 \,\forall\, (u, u^*), (v, v^*) \in graph(M).$$

(ii) Strictly monotone if M is monotone and equality holds only if u=v.

(iii) $(r)-$ strongly monotone if there exists a positive constant r such that

$$\langle u^* - v^*, u - v\rangle \geq r\|u - v\|^2 \,\forall\, (u, u^*), (v, v^*) \in graph(M).$$

(iv) $(r)-$ expanding if there exists a positive constant r such that

$$\|u^* - v^*\| \geq r\|u - v\| \, \forall \, (u, u^*), (v, v^*) \in graph(M).$$

(v) $(m)-$ relaxed monotone if there exists a positive constant m such that

$$\langle u^* - v^*, u - v\rangle \geq -m\|u - v\|^2 \, \forall \, (u, u^*), (v, v^*) \in graph(M).$$

(vi) $(m)-$cocoercive if there exists a positive constant m such that

$$\langle u^* - v^*, u - v\rangle \geq m\|u^* - v^*\|^2 \, \forall \, (u, u^*), (v, v^*) \in graph(M).$$

(vii) Monotone with respect to A if

$$\langle u^* - v^*, A(u) - A(v)\rangle \geq 0 \, \forall \, (u, u^*), (v, v^*) \in graph(M).$$

(viii) Strictly monotone with respect to A if M is monotone with respect to A and equality holds only if u=v.

(ix) $(r)-$ strongly monotone with respect to A if there exists a positive constant r such that

$$\langle u^* - v^*, A(u) - A(v)\rangle \geq r\|u - v\|^2 \, \forall \, (u, u^*), (v, v^*) \in graph(M).$$

(x) $(c)-$cocoercive with respect to A if there exists a positive constant c such that

$$\langle u^* - v^*, A(u) - A(v)\rangle \geq c\|u^* - v^*\|^2 \, \forall \, (u, u^*), (v, v^*) \in graph(M).$$

(xi) Nonexpansive if

$$\|u^* - v^*\| \leq \|u - v\| \, \forall \, (u, u^*), (v, v^*) \in graph(M).$$

(xii) Cocoercive if

$$\langle u^* - v^*, u - v\rangle \geq \|u^* - v^*\|^2 \, \forall \, (u, u^*), (v, v^*) \in graph(M).$$

(xiii) Cocoercive with respect to A if

$$\langle u^* - v^*, A(u) - A(v)\rangle \geq \|u^* - v^*\|^2 \, \forall \, (u, u^*), (v, v^*) \in graph(M).$$

Definition 2.2. Let X be a real Hilbert space. Let $A : X \to X$ be a single-valued mapping. The map $M : X \to 2^X$ is said to be relatively maximal $(m)-$relaxed monotone $(m > 0)$ if

(i) M is $(m)-$relaxed monotone with respect A, that is,

$$\langle u^* - v^*, A(u) - A(v)\rangle \geq -m\|u - v\|^2 \, \forall \, (u, u^*), (v, v^*) \in graph(M),$$

(ii) $R(I + \rho M) = X$ for $\rho > 0$.

Proposition 2.1. Let X be a real Hilbert space. Let $A : X \to X$ be an $(r)-$strongly monotone mapping, and let $M : X \to 2^X$ be a relatively maximal $(m)-$relaxed monotone mapping. Then the operator $(I+\rho M)^{-1}$ is single-valued for $r - \rho m > 0$.

Proof. The proof follows from the definition of the resolvent operator. For any $u \in X$, let $x, y \in (I + \rho M)^{-1}(u)$. It follows from this that $-x + u \in \rho M(x)$ and $-y + u \in \rho M(y)$. Since M is relatively maximal $(m)-$relaxed monotone, and A is $(r)-$strongly monotone, we have

$$\frac{1}{\rho}\langle -x + u - (-y + u), A(x) - A(y)\rangle \geq -m\|x - y\|^2.$$

This is equivalent to

$$-\rho m\|x - y\|^2 \leq \langle -(x - y), A(x) - A(y)\rangle \leq -r\|x - y\|^2,$$

and this is equivalent to

$$(r - \rho m)\|x - y\|^2 \leq \langle -(x - y), A(x) - A(y)\rangle \leq 0.$$

This implies that x=y for $r - \rho m > 0$, and hence $(I + \rho M)^{-1}$ is single-valued.

Definition 2.3. Let X be a real Hilbert space. Let $A : X \to X$ be an $(r)-$strongly monotone mapping, and let $M : X \to 2^X$ be a relatively maximal $(m)-$relaxed monotone mapping. Then the generalized resolvent operator $R^M_{\rho,m,A} : X \to X$ is defined by

$$R^M_{\rho,m,A}(u) = (I + \rho M)^{-1}(u)\, for\, r - \rho m > 0.$$

Definition 2.4. Let X be a real Hilbert space. A map $M : X \to 2^X$ is said to be maximal monotone if

(i) M is monotone, that is,

$$\langle u^* - v^*, u - v\rangle \geq 0\, \forall\, (u, u^*), (v, v^*) \in graph(M),$$

(ii) $R(I + \rho M) = X$ for $\rho > 0$.

Definition 2.5. Let X be a real Hilbert space. Let $M : X \to 2^X$ be a maximal monotone mapping. Then the resolvent operator $J^M_\rho : X \to X$ is defined by

$$J^M_\rho(u) = (I + \rho M)^{-1}(u).$$

Next, we include an example on the notion of relative $(m)-$relaxed monotonicity. Note that all relatively monotone mappings are relatively $(m)-$ relaxed monotone for $(m > 0)$.

Example 2.1. Let $X = (-\infty, +\infty)$, $A(x) = -\frac{1}{2}x$ and $M(x) = -x$ for all $x \in X$. Then M is relatively monotone but not monotone, while M is relatively $(m)-$relaxed monotone for $m > 0$.

3. Linear Convergence Analysis

This section deals with a generalization to Rockafellar's theorem ([5], Theorem 2) based on the new approach under the framework of relatively maximal $(m)-$relaxed monotonicity, while solving (1).

Theorem 3.1. Let X be a real Hilbert space, let $M : X \to 2^X$ be relatively maximal $(m)-$relaxed monotone, and $A : X \to X$ be $(r)-$strongly monotone. Then the following statements are mutually equivalent:

(i) An element $u \in X$ is a solution to (1).

(ii) For an $u \in X$, we have

$$u = R^M_{\rho,m,A}(u),$$

where

$$R^M_{\rho,m,A}(u) = (I + \rho M)^{-1}(u) \; for \; r - \rho m > 0.$$

Proof. It follows from the definition of resolvent operator corresponding to M.

Theorem 3.2. Let X be a real Hilbert space, let $A : X \to X$ be $(r)-$ strongly monotone, and let $M : X \to 2^X$ be relatively maximal $(m)-$relaxed monotone. Furthermore, suppose that $(I - A)\text{o}R^M_{\rho_k,m,A}$ is $(\lambda)-$cocoercive.
(i) For an arbitrarily chosen initial point x^0, suppose that the sequence $\{x^k\}$ is generated by the proximal point algorithm

$$x^{k+1} \approx R^M_{\rho_k,m,A}(x^k) \tag{9}$$

such that

$$\|x^{k+1} - R^M_{\rho_k,m,A}(x^k)\| \leq \epsilon_k, \tag{10}$$

where $\Sigma^{\infty}_{k=0}\epsilon_k < \infty$, $R^M_{\rho_k} = (I + \rho_k M)^{-1}$, $(r - m\rho_k) > 1$, and the scalar sequence $\{\rho_k\}$ satisfies $\rho_k \uparrow \rho \leq \infty$. Suppose that the sequence $\{x^k\}$ is bounded in the sense that there exists at least one solution to (1).

(ii) In addition to assumptions in (i), we further suppose that, for an arbitrarily chosen initial point x^0, the sequence $\{x^k\}$ is generated by the proximal point algorithm

$$x^{k+1} \approx R^M_{\rho_k,m,A}(x^k) \tag{11}$$

such that

$$\|x^{k+1} - R^M_{\rho_k,m,A}(x^k)\| \leq \delta_k \|x^{k+1} - x^k\|, \tag{12}$$

where $\delta_k \to 0$, $R^M_{\rho_k,m,A} = (I + \rho_k M)^{-1}$, and the scalar sequences $\{\delta_k\}$ and $\{\rho_k\}$, respectively, satisfy $\Sigma_{k=0}^{\infty}\delta_k < \infty$, and $\rho_k \uparrow \rho \leq \infty$.
Then the following implications hold:

(i) The sequence $\{x^k\}$ converges strongly to a solution of (1).

(ii) Rate of convergence

$$0 \leq \lim_{k\to\infty} \frac{\delta_k + ((\gamma - m\rho_k)r)^{-1}}{1 - \delta_k} < 1,$$

where $\frac{1}{r - m\rho_k)} < 1$.

Proof. Suppose that x^* is a zero of M. We start with the proof for

$$\|R^M_{\rho_k,m,A}(x^k) - R^M_{\rho_k,m,A}(x^*)\| \leq \frac{1}{(r - m\rho_k)}\|x^k - x^*\|,$$

crucial to linear convergence.
Based on the definition of the generalized resolvent operator $R^M_{\rho_k,m,A}$ and other hypotheses, we have

$$\begin{aligned} & \langle x^k - x^* - (R^M_{\rho_k,m,A}(x^k) - R^M_{\rho_k,m,A}(x^*)), A(R^M_{\rho_k,m,A}(x^k)) - A(R^M_{\rho_k,m,A}(x^*))\rangle \\ \geq \; & -m\rho_k\|R^M_{\rho_k,m,A}(x^k) - R^M_{\rho_k,m,A}(x^*)\|^2 \end{aligned}$$

or

$$\begin{aligned} & \langle x^k - x^*, A(R^M_{\rho_k,m,A}(x^k)) - A(R^M_{\rho_k,m,A}(x^*))\rangle \\ \geq \; & \langle R^M_{\rho_k,m,A}(x^k) - R^M_{\rho_k,m,A}(x^*), A(R^M_{\rho_k,m,A}(x^k)) - A(R^M_{\rho_k,m,A}(x^*))\rangle \\ - \; & m\rho_k\|R^M_{\rho_k,m,A}(x^k) - R^M_{\rho_k,m,A}(x^*)\|^2 \\ \geq \; & (r - m\rho_k)\|R^M_{\rho_k,m,A}(x^k) - R^M_{\rho_k,m,A}(x^*)\|^2. \end{aligned}$$

Thus, we have (for $r - \rho_k m > 0$)

$$\langle x^k - x^*, A(R^M_{\rho_k,m,A}(x^k)) - A(R^M_{\rho_k,m,A}(x^*))\rangle \geq (r - m\rho_k)\|R^M_{\rho_k,m,A}(x^k) - R^M_{\rho_k,m,A}(x^*)\|^2.$$

It follows that

$$
\begin{aligned}
& \langle x^k - x^*, R^M_{\rho_k,m,A}(x^k) - R^M_{\rho_k,m,A}(x^*)\rangle \\
\geq\ & (r - m\rho_k)\|R^M_{\rho_k,m,A}(x^k) - R^M_{\rho_k,m,A}(x^*)\|^2 \\
+\ & \lambda\|(I-A)(R^M_{\rho_k,m,A}(x^k)) - (I-A)(R^M_{\rho_k,m,A}(x^k))\|^2 \\
\geq\ & (r - m\rho_k)\|R^M_{\rho_k,m,A}(x^k) - R^M_{\rho_k,m,A}(x^*)\|^2 \, for\, r - \rho_k m > 0,\ \lambda > 0.
\end{aligned}
$$

Next, we estimate

$$
\begin{aligned}
& \|x^{k+1} - x^*\| \leq \|R^M_{\rho_k,m,A}(x^k) - x^*\| + \epsilon_k \\
=\ & \|R^M_{\rho_k,m,A}(x^k) - R^M_{\rho_k,m,A}(x^*)\| + \epsilon_k \\
\leq\ & \frac{1}{(r - m\rho_k)}\|x^k - x^*\| + \epsilon_k.
\end{aligned}
$$

Since $(r - m\rho_k) > 1$, combining for all k, we have

$$
\begin{aligned}
& \|x^{k+1} - x^*\| \leq \|x^0 - x^*\| + \sum_{i=0}^{k} \epsilon_i \ \forall\ i \\
\leq\ & \|x^0 - x^*\| + \sum_{i=0}^{\infty} \epsilon_i. \qquad (13)
\end{aligned}
$$

Hence, $\{x^k\}$ is bounded.

Next we turn our attention to the linear convergence. Since

$$
\begin{aligned}
& \|x^{k+1} - x^*\| \leq \|x^{k+1} - R^M_{\rho_k,m,A}(x^k)\| \\
+\ & \|R^M_{\rho_k,m,A}(x^k) - R^M_{\rho_k,m,A}(x^*)\|, \qquad (14)
\end{aligned}
$$

and

$$
\begin{aligned}
& \|x^{k+1} - R^M_{\rho_k,m,A}(x^k)\| \leq \delta_k\|x^{k+1} - x^k\| \\
\leq\ & \delta_k[\|x^{k+1} - x^*\| + \|x^k - x^*\|], \qquad (15)
\end{aligned}
$$

we get

$$
\begin{aligned}
& \|x^{k+1} - x^*\| \leq \|x^{k+1} - R^M_{\rho_k,m,A}(x^k)\| \\
+\ & \|R^M_{\rho_k,m,A}(x^k) - R^M_{\rho_k,m,A}(x^*)\| \\
\leq\ & \delta_k[\|x^{k+1} - x^*\| + \|x^k - x^*\|] \\
+\ & \frac{1}{(\gamma - m\rho_k)r}\|x^k - x^*\| \, for\, k \geq k',
\end{aligned}
$$

where $\frac{1}{(r-m\rho_k} < 1$.

It follows that

$$\|x^{k+1} - x^*\| \leq \frac{((r-m\rho_k)^{-1} + \delta_k}{1-\delta_k}\|x^k - x^*\| \, for \ k \ \geq k'. \quad (16)$$

It appears that (16) holds since $\frac{1}{(r-m\rho_k)} < 1$ (seems to hold) and $\delta_k \to 0$.

Hence, the sequence $\{x^k\}$ converges strongly to x^*.

Finally, to examine the uniqueness of the solution to (1), assume that x^* is a zero of M. Then using $\|x^k - x^*\| \leq \|x^0 - x^*\| + \sum_{k=0}^{\infty} \epsilon_k \, \forall \, k$, we have

$$a^* = \lim_{k\to\infty} \ inf \, \|x^k - x^*\|$$

is nonnegative and finite, and as a result, $\|x^k - x^*\| \to \ a^*$. Consider x_1^* and x_2^* be two limit points of $\{x^k\}$. Then we have

$$\|x^k - x_1^*\| = a_1, \ \|x^k - x_2^*\| = a_2$$

and both exist and are finite. If we express

$$\|x^k - x_2^*\|^2 = \|x^k - x_1^*\|^2 + 2\langle x^k - x_1^*, x_1^* - x_2^*\rangle + \|x_1^* - x_2^*\|^2,$$

then it follows that

$$\lim_{k\to\infty} \langle x^k - x_1^*, x_1^* - x_2^*\rangle = \frac{1}{2}[a_2^2 - a_1^2 - \|x_1^* - x_2^*\|^2].$$

Since x_1^* is a limit point of $\{x^k\}$, the left hand side limit must tend to zero. Therefore,

$$a_1^2 = a_2^2 - \|x_1^* - x_2^*\|^2.$$

Similarly, we obtain

$$a_2^2 = a_1^2 - \|x_1^* - x_2^*\|^2.$$

This results $x_1^* = x_2^*$. □

Remark 3.1. When $M : X \to 2^{X^*}$ equals ∂f, the subdifferential of f, where $f : X \to (-\infty, +\infty]$ is a functional on a Hilbert space X, it can be applied for minimizing f. The function f is proper if $f \not\equiv +\infty$ and is convex if

$$f((1-\lambda)x + \lambda y) \leq (1-\lambda)f(x) + \lambda f(y),$$

where $x, y \in X$ and $0 < \lambda < 1$. Furthermore, the function f is lower semicontinuous on X if the set

$$\{x : x \in X, f(x) \leq \lambda \, \forall \, \lambda \in R\}$$

is closed in X.

The subdifferential $\partial f : X \to 2^{X^*}$ of f at x is defined by

$$\partial f(x) = \{x^* \in X^* : f(y) - f(x) \geq \langle y - x, x^*\rangle \, \forall \, y \in X\}.$$

In an earlier work [7], Rockafellar has shown that if f is a lower semicontinuous proper convex functional on X, then $\partial f : X \to 2^{X^*}$ is maximal monotone on X, where X is any real Banach space. Several other special cases can be derived.

Suppose that $A : X \to X$ be $(r)-$strongly monotone and $(\gamma)-$cocoercive, and let $f : X \to R$ be $(\tau)-$locally Lipschitz (for $\tau \geq 0$) such that $\partial f : X \to 2^{X^*}$ is $(m)-$relaxed monotone with respect to A, that is,

$$\langle u^* - v^*, A(u) - A(v)\rangle \geq -m\|u - v\|^2 \,\forall\, u, v \in X, \tag{17}$$

where $u^* \in \partial f(u)$, and $v^* \in \partial f(v)$.

Then ∂f is relatively maximal $(m)-$relaxed monotone.

Acknowledgment

The author is greatly indebted to the referees for their valuable comments and suggestions leading to the revised version of the submission.

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