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On a System of General Variational Inclusions

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Abstract In this paper, we introduce and consider some new systems of general variational inclusions involving five different operators. Using the resolvent operator technique, we show that the new systems of general variational inclusions are equivalent to the fixed point problems. This equivalent formulation is used to suggest and analyze some new iterative methods for this system of general variational inclusions. We also study the convergence analysis of the new iterative method under suitable conditions. Several special cases are also discussed. Results obtained in this paper can be viewed as a significant extension of the known results.
Keywords: Variational inclusions, iteration algorithms; System of general variational inclusions, convergence criteria, Hilbert spaces.

1. Introduction

System of variational inclusions can be viewed as natural and innovative generalizations of the system of variational inequalities. A wide class of problems, which arise in various branches of mathematical analysis, convex analysis, optimization, geometry, biology, elasticity, optimization, imaging processing, biomedical sciences, mathematical physics, etc. can be studied in the unified framework of system of variational inclusions. One can see an immense breadth of mathematics and its simplicity in the works of this research. These applications stipulated a significant interest to suggest and analyze iterative methods by using the resolvent/projection methods. It has been shown that variational

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inclusions contain variational inequalities and related optimization problems as special cases. In recent years, much attention has been given to consider the system of variational inclusions and variational inequalities, see [3,7, 27,28, 30-39] and the references therein. It is well known that the system of variational inclusions/inequalities can provide new insight regarding problems being studied and can stimulate new and innovative ideas for problem solving.

Inspired and motivated by research going on in this area, we introduce and consider a new system of variational inclusions involving five different nonlinear operators. This class of new systems includes the system of variational inclusions/inequalities involving four, three, two operators and quasi variational inclusions/inequalities as special cases. We establish the equivalence between the new system of general variational inclusions and the fixed point problem using the resolvent operator technique. This alternative equivalent formulation is used to suggest and analyze some iterative methods for solving this system of general variational inclusions. Several special cases of these iterative algorithms are also discussed. We also prove the convergence of the proposed iterative methods under some suitable conditions. Since the new system of general variational inclusions/inequalities includes the system of variational inclusions/ inequalities and related optimization problems as special cases, results proved in this paper continue to hold for these problems. Our result can be viewed as refinement and improvement of the previous results in this field. The interested readers are advised to explore this field further and discover some new and novel applications of these system of general variational inclusions/inequalities in various branches of pure and applied sciences. This field of study is not much studied and offers several opportunities for future research in this very active and interesting branch of discipline.

2. Preliminaries

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|.\|$ respectively Let K be a closed and convex set in H. Let $T_1, T_2, A, g, h : H \to H$ be nonlinear different operators and let $\varphi : H \longrightarrow R \cup \{+\infty\}$ be a continuous function.

We now consider the problem of finding $x^*, y^* \in H$ such that

$$\left.\begin{array}{ll} 0 \in \rho T_1(y^*) + \rho A(g(x^*)) - g(y^*) + g(x^*), & \rho > 0 \\ 0 \in \eta T_2(x^*) + \eta A(h(y^*)) + h(y^*) - h(x^*), & \eta > 0 \end{array}\right\}, \tag{1}$$

which is called the system of general variational inclusions involving five different operators().

We now discuss some special cases of the system of general variational inclusions (1).

I. If $T_1 = T_2 = T$, the problem (1) is equivalent to finding $x^*, y^*) \in H$ such that

$$\left.\begin{array}{ll} 0 \in \rho T(y^*) + \rho A(g(x^*)) - g(y^*) + g(x^*), & \rho > 0 \\ 0 \in \eta T(x^*) + \eta A(h(y^*)) + h(y^*) - h(x^*), & \eta > 0 \end{array}\right\}, \tag{2}$$

II. If $T_1 = T_2 = T$ and $g = h, \quad \rho = \eta, \quad x = x^* = y^*$, then problem (1) is equivalent to finding $x \in H$ such that

$$0 \in T(x) + A(g(x)), \tag{3}$$

which is known as the variational inclusion problem or finding the zero of the sum of two (more) monotone operators. It is well known that a wide class of linear and nonlinear problems can be studied via variational inclusion problems.

III. We note that if $A(.) = \partial\varphi(.)$, the subdifferential of a proper, convex and lower-semicontinuous function, then the system of variational inclusions (1) is equivalent to finding $x^*, y^* \in H$ such that

$$\left.\begin{array}{ll} 0 \in \rho T_1(y^*) + \rho\partial\varphi(g(x^*)) - g(y^*) + g(x^*), & \rho > 0 \\ 0 \in \eta T_2(x^*) + \eta\partial\varphi(h(y^*)) + h(y^*) - h(x^*), & \eta > 0 \end{array}\right\}, \tag{4}$$

or equivalently the problem of finding $x^*, y^* \in H$ such that

$$\left.\begin{array}{lll} \langle \rho T_1(y^*) + g(x^*) - g(y^*), x - g(x^*)\rangle \geq \rho\varphi(g(x^*)) - \rho\varphi(x), & \forall x \in H, & \rho > 0 \\ \langle \eta T_2(x^*) + h(y^*) - h(x^*), x - h(y^*)\rangle \geq \eta\varphi(h(y^*)) - \eta\varphi(x), & \forall x \in H, & \eta > 0 \end{array}\right\}, \tag{5}$$

which is called the system of mixed general variational inequalities involving four different nonlinear operators and appears to be a new one.

IV. If $T_1 = T_2 = T$, then problem (5) reduces to the following system of mixed general variational inequalities of finding $x^*, y^* \in H$ such that

$$\left.\begin{array}{lll} \langle \rho T(y^*) + g(x^*) - g(y^*), x - g(x^*)\rangle \geq \rho\varphi(g(x^*)) - \rho\varphi(x), & \forall x \in H, & \rho > 0 \\ \langle \eta T(x^*) + h(y^*) - h(x^*), x - h(y^*)\rangle \geq \eta\varphi(h(y^*)) - \eta\varphi(x), & \forall x \in H, & \eta > 0 \end{array}\right\}. \tag{6}$$

V. If $T_1 = T_1 = T$ is univariate operator and $\quad g = h, \quad \rho = \eta, \quad x = x^* = y^*$, then problem (5) is equivalent to finding $x \in H$ such that

$$\langle Tx, y - g(x)\rangle \geq \varphi(g(x)) - \varphi(y), \quad \forall y \in H, \tag{7}$$

which is known as the mixed general variational inequality or variational inequality of the second type. For the applications and numerical methods for solving the mixed variational inequalities, see [34].

VI. If φ is an indicator function of a closed convex set K in H, then problem (5) is equivalent to finding $x^*, y^* \in H : g(x^*), h(y^*) \in K$ such that

$$\left.\begin{array}{lll} \langle \rho T_1(y^*) + g(x^*) - g(y^*), x - g(x^*)\rangle \geq 0, & \forall x \in K, & \rho > 0 \\ \langle \eta T_2(x^*) + h(y^*) - h(x^*), x - h(y^*)\rangle \geq 0, & \forall x \in K, & \eta > 0 \end{array}\right\}, \tag{8}$$

is called the system of general variational inequalities.

VII. If $T_1 = T_2 = T$, then the problem (8) is equivalent to the following system of variational inequalities of finding $x^*, y^* \in H : g(x^*), h(y^*) \in K$ such that

$$\left.\begin{array}{l} \langle \rho T(y^*) + g(x^*) - g(y^*), x - g(x^*)\rangle \geq 0, \quad \forall x \in K, \quad \rho > 0 \\ \langle \eta T(x^*) + h(y^*) - h(x^*), x - h(y^*)\rangle \geq 0, \quad \forall x \in K, \quad \eta > 0 \end{array}\right\}, \tag{9}$$

which can be viewed as generalization of the system considered in [3,7].

VIII. If $\varphi(.)$ is the indicator function of a closed convex set K, then problem (7) is equivalent to finding $x^* \in H : g(x^*) \in K$ such that

$$Tx^*, x - g(x^*) \geq 0, \quad \forall x \in K, \tag{10}$$

which is known as the general variational inequality introduced and studied by Noor [11] in 1988. It has been shown [26] that the minimum of a differentiable nonconvex function on the nonconvex set can be characterized by the general variational inequality (10). In particular, for $g = h$, one can obtain the various classes of variational inclusions and variational inequalities studied by Noor and Noor [35]. This shows that the system of general variational inclusions involving five different operators (1) is more general and includes several classes of variational inclusions/inequalities and related optimization problems as special cases. For the recent applications, numerical methods and formulations of variational inequalities and variational inclusions, see [1-40] and the references therein.

3. Iterative algorithms

In this Section, we suggest some explicit iterative algorithms for solving the system of general variational inclusion (1). First of all, we establish the equivalence between the system of variational inclusions and fixed point problems. For this purpose, we recall the following well known result.

Definition 3.1[2]. For any maximal monotone operator T, the resolvent operator associated with T, for any $\rho > 0$, is defined as

$$J_T(u) = (I + \rho T)^{-1}(u), \qquad \forall u \in H.$$

It is well known that an operator T is maximal monotone if and only if its resolvent operator J_T is defined everywhere. It is single-valued and nonexpansive, that is,

$$\|J_A u - J_A v\| \leq \|u - v\|, \quad \forall u, v \in H.$$

If $\varphi(.)$ is a proper, convex and lower-semicontinuous function, then its subdifferential $\partial\varphi(.)$ is a maximal monotone operator. In this case, we can define the resolvent operator

$$J_\varphi(u) = (I + \rho\partial\varphi)^{-1}(u), \quad \forall u \in H$$

associated with the subdifferential $\partial\varphi(.)$. The resolvent operator J_φ has the following useful characterization.

Lemma 3.1[2]. For a given $z \in H$, $u \in H$ satisfies the inequality

$$\langle u - z, v - u\rangle + \rho\varphi(v) - \rho\varphi(u) \geq 0, \quad \forall v \in H$$

if and only if $u = J_\varphi(z)$, where $J_\varphi = (I + \rho\partial\varphi)^{-1}$ is the resolvent operator. It is well known the resolvent operator J_φ is nonexpansive.

We now show that the system of variational inclusions (1) is equivalent to the fixed-point problem and this is the motivation of our next result.

Lemma 3.2. If the operator A is maximal monotone, then $(x^*, y^*) \in H$ is a solution of the system of general variational inclusions (1) if and only if $x^*, y^* \in H$ satisfies

$$g(x^*) = J_A[g(y^*) - \rho T_1(y^*)], \tag{11}$$

$$h(y^*) = J_A[h(x^*) - \eta T_2(x^*)]. \tag{12}$$

Proof. Let $(x^*, y^*) \in H$ be a solution of (1). Then

$$\left.\begin{array}{l} g(y^*) - \rho T_1(y^*) \in (I + \rho A)(g(x^*)) \\ h(x^*) - \eta T_2(x^*) \in (I + \eta A)(h(y^*)) \end{array}\right\}$$

which is equivalent to

$$g(x^*) = J_A[g(y^*) - \rho T_1(y^*)],$$

$$h(y^*) = J_A[h(x^*) - \eta T_2(x^*)],$$

the required result. □

This equivalent formulation is used to suggest and analyze a number of iterative methods for solving the system of general variational inclusions (1). To do so, one rewrite the equations (11) and (12) in the following form.

$$x^* = x^* - g(x^*) + J_A[g(y^*) - \rho T_1(y^*)], \tag{13}$$

$$y^* = y^* - h(y^*) + J_A[h(x^*) - \eta T_2(x^*)]. \tag{14}$$

This alternative equivalence formulation enables us to suggest the following explicit iterative method for solving the system of general variational inclusions (1).

Algorithm 3.1. For a given initial value $y_0 \in K$ compute the sequence $\{x_n\}$ and $\{y_n\}$ by

$$\begin{aligned} x_{n+1} &= (1 - a_n)x_n + a_n(x_{n+1} - g(x_{n+1})) + a_n J_A[g(y_n) - \rho T_1(y_n)], \\ y_{n+1} &= y_{n+1} - h(y_{n+1}) + J_A[h(x_{n+1}) - \eta T_2(x_{n+1})], \end{aligned}$$

where $a_n \in [0,1]$, for all $n \geq 0$, satisfies some suitable conditions.

For $a_n = 0$, the Algorithm 3.1 reduces to the Algorithm 3.2.

Algorithm 3.2. For a given initial value $y_0 \in K$ compute the sequence $\{x_n\}$ and $\{y_n\}$ by

$$x_{n+1} = x_{n+1} - g(x_{n+1}) + J_A[g(y_n) - \rho T_1(y_n)], \quad (15)$$

$$y_{n+1} = y_{n+1} - h(y_{n+1}) + J_A[h(x_{n+1}) - \eta T_2(x_{n+1})]. \quad (16)$$

For suitable and appropriate choice of the operators T_1, T_2, A, g, h and spaces, one can obtain a wide class of iterative methods for solving different classes of variational inclusions and related optimization problems. This shows that Algorithm 3.1 is quite flexible and general and includes various known and new algorithms for solving variational inequalities and related optimization problems as special cases.

Definition 3.2. A mapping $T : H \rightarrow H$ is called r-strongly monotone, if there exists a constant $r > 0$, such that

$$\langle Tx - Ty, x - y\rangle \geq r||x - y||^2, \quad \forall x, y \in H.$$

Definition 3.3. A mapping $T : H \rightarrow H$ is called μ-Lipschitzian, if there exists a constant $\mu > 0$, such that

$$||Tx - Ty|| \leq \mu||x - y||, \quad \forall x, y \in H.$$

4. Main Result

In this Section, we consider the convergence criteria of Algorithm 3.2 under some suitable conditions and this is the main motivation of this paper.

Theorem 4.1. Let x^*, y^* be the solution of (1). If $T_1 : H \rightarrow H$ is r_1-strongly monotone and μ_1-Lipschitzian and $T_2 : H \rightarrow H$ is r_2-strongly monotone and μ_3-Lipschitzian. Let g be a r_3-strongly and μ_3-Lipschitzian. Let the operator h be r_4-strongly monotone and μ_4-Lipschitzian. If

$$\left|\rho - \frac{r_1}{\mu_1^2}\right| < \frac{\sqrt{(r_1^2 - \mu_1^2 k(2-k)}}{\mu_1^2}, \quad r_1 > \mu_1\sqrt{k(2-k)}, \quad k < 1, \quad (17)$$

$$\left|\eta - \frac{r_2}{\mu_2^2}\right| < \frac{\sqrt{(r_2^2 - \mu_2^2 k_1(2-k_1)}}{\mu_2^2}, \quad r_2 > \mu_2\sqrt{k_1(2-k_1)}, \quad k_1 < 1, \quad (18)$$

where

$$k = 2\sqrt{1 - 2r_3 + \mu_3^2} \quad (19)$$

$$k_1 = 2\sqrt{1 - 2r_4 + \mu_4^2}, \quad (20)$$

then, for a given initial value $y_0 \in H$, x_n and y_n obtained from Algorithm 3.2 converge strongly to x^* and y^*, respectively.

Proof. To prove the result, we need first evaluate $||x_{n+1} - x^*||$ for all $n \geq 0$. From (13), (15), and the nonexpansive property of the resolvent operator J_A, we have

$$\begin{aligned} & ||x_{n+1} - x^*|| \\ = \ & ||x_{n+1} - g(x_{n+1}) + J_A[g(y_n) - \rho T_1(y_n)] - (x^* - g(x^*)) - J_A[g(y^*) - \rho T_1(y^*)]|| \\ \leq \ & ||x_{n+1} - x^* - (g(x_{n+1}) - g(x^*))|| + ||J_A[g(y_n) - \rho T_1(y_n)] - J_A[g(y^*) - \rho T_1(y^*)]|| \\ \leq \ & ||x_{n+1} - x^* - (g(x_{n+1}) - g(x^*))|| + ||[g(y_n) - \rho T_1(y_n)] - [g(y^*) - \rho T_1(y^*)]|| \\ = \ & ||x_{n+1} - x^* - (g(x_{n+1}) - g(x^*))|| + ||y_n - y^* - \rho[T_1(y_n) - T_1(y^*)]|| \\ & + \|y_n - y^* - (g(y_n) - g(y^*))\|. \end{aligned} \tag{21}$$

From the r_1-strongly monotonicity and μ_1-Lipschitzian of T_1, we have

$$\begin{aligned} & \|y_n - y^* - \rho[T_1(y_n) - T_1(y^*)]\|^2 \\ = \ & \|y_n - y^*\|^2 - 2\rho\langle T_1(y_n) - T_1(y^*), y_n - y^*\rangle + \rho^2\|T_1(y_n) - T_1(y^*)\|^2 \\ = \ & [1 - 2\rho r_1 + \rho^2\mu_1^2]\|y_n - y^*\|^2. \end{aligned} \tag{22}$$

In a similar way, using the r_3-strongly monotonicity and μ_3-Lipschitz continuity of the operator g, we have

$$\|y_n - y^* - (g(y_n) - g(y^*))\| \leq k\|y_n - y^*\|, \tag{23}$$

where k is defined by (19). Set

$$\theta_1 = \frac{\{\frac{k}{2} + [1 - 2\rho r_1 + \rho^2\mu_1^2]^{1/2}\}}{1 - \frac{k}{2}}$$

It is clear from the condition (17) that $0 \leq \theta_1 < 1$. Hence from (23),(21) and (22), it follows that

$$\|x_{n+1} - x^*\| \leq \theta_1\|y_n - y^*\|. \tag{24}$$

Similarly, from the r_2-strongly monotonicity and μ_2-Lipschitzian of T_2, we obtain

$$\begin{aligned} & \|x_{n+1} - x^* - \eta[T_2(x_{n+1}) - T_2(x^*)]\|^2 \\ = \ & \|x_{n+1} - x^*\|^2 - 2\eta\langle T_2(x_{n+1}) - T_2(x^*), x_{n+1} - x^*\rangle \\ & + \eta^2\|T_2(x_{n+1}) - T_2(x^*)\|^2 \\ = \ & [1 + 2\eta\gamma_2\mu_2^2 - 2\eta r_2 + \eta^2\mu_2^2]\|x_{n+1} - x^*\|^2. \end{aligned} \tag{25}$$

Also, using the r_4-strongly monotonicity and μ_4-Lipschitz continuity of the operator h, we have

$$\|y_n - y^* - (h(y_n) - h(y^*))\| \leq k_1\|y_n - y^*\|, \tag{26}$$

where k_1 is defined by (20).

Hence from (14), (16), (23), (25) and (26), we have

$$\begin{aligned}\|y_{n+1} - y^*\| &= \|y_{n+1} - y^* - (g(y_{n+1}) - g(y^*))\| \\ &\quad + J_A[h(x_{n+1}) - \eta T_2(x_{n+1})] - J_A[h(x^*) - \eta T_2(x^*)]|| \\ &\leq \|y_{n+1} - y^* - (g(y_{n+1}) - g(y^*))\| + \|x_{n+1} - x^* - \eta(T_2(x_{n+1})] - T_2(x^*))|| \\ &\quad + \|x_{n+1} - x^* - (h(x_{n+1}) - h(x^*)),\end{aligned}$$

which implies that

$$\|y_{n+1} - y^*\| \leq \theta_2 \|x_{n+1} - x^*\|, \tag{27}$$

where

$$\theta_2 = \frac{[\frac{k_1}{2} + [1 - 2\rho r_1 + \rho^2 \mu_1^2]^{1/2}}{1 - \frac{k_1}{2}}.$$

From (18), it follows that $\theta_2 < 1$.

From (24) and (27), we obtain that

$$\|x_{n+1} - x^*\| \leq \theta_1 \theta_2 \|x_n - x^*\|.$$

Since $\theta_1\theta_2 < 1$, it follows that $\lim_{n\to\infty}\{\|x_n - x^*\|\} = 0$. Hence the result $\lim_{n\to\infty}\{\|y_n - y^*\|\} = 0$ is from (27). This completes the proof. □

Remark 3.1. We would like to point out that the problem of the existence of a unique solution of the system of general variational inclusions is an open problem, which needs further research. Algorithm 3.2 and Algorithm 3.1 are explicit type iterative methods for solving the system of general variational inclusions (1).. In order to implement Algorithm 3.2, one only needs the initial value of y_0. This value is enough to find the approximate x_1, from (15). Then, one can find the approximate solution y_1 using (16). In this way, one find the approximate solution (x_n, y_n) of the system (1). It is an interested problem to find the novel and important applications of the new system of general variational inclusions (1) and its variant forms. The interested reader is advised to explore its various applications of this new system in different branches of the mathematical and engineering sciences.

Conclusion. In this paper, we have introduced a new system of general variational inclusions involving five different operators. Using the resolvent operator technique, we have established the equivalence between the system of general variational inclusions and the fixed point problems. We have used this alternative equivalent formulation to suggest and analyze a number of iterative resolvent methods for solving this new system. Convergence analysis is also considered. Several special cases of these results are also discussed. Some new classes of system of general variational inclusions have been studied by Noor and Noor [34] and Noor et al. [36] recently. It is interested to compare the idea and technique of this paper with other techniques. These problems can be very useful in practice. Much more work is needed in all these areas to develop a sound basis for applications of the system of general variational inclusions in engineering and physical sciences.

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