

A New Polynomial Interior-Point Algorithm for the Cartesian $P_*(\kappa)$
Second-Order Cone Linear Complementarity Problem ¹

G.Q. Wang^{a,b}, Y.J. Yue^a, B.J. He^a

^a College of Advanced Vocational Technology, Shanghai University of Engineering Science,
Shanghai 200437, P.R. China

^b Department of Mathematics, Shanghai Normal University, Shanghai 200234, P.R. China
e-mail: guoq.wang@hotmail.com

Abstract In this paper we present a new polynomial interior-point algorithm for the Cartesian $P_*(\kappa)$ second-order cone linear complementarity problem based on a finite kernel function. The symmetrization of the search directions used in this paper is based on the Nesterov and Todd scaling scheme. We derive the iteration bounds that match the currently best known iteration bounds for large- and small-update methods, namely, $O((1 + 2\kappa)\sqrt{N} \log N \log \frac{N}{\epsilon})$ and $O((1 + 2\kappa)\sqrt{N} \log \frac{N}{\epsilon})$, respectively, which are as good as the $P_*(\kappa)$ linear complementarity problem analogue. Moreover, this unifies the analysis for $P_*(\kappa)$ linear complementarity problem and second-order cone optimization.

Keywords Second-order cone linear complementarity problem, Large- and small-update methods, Interior-point algorithm, Kernel function, Iteration bound.

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1. INTRODUCTION

In this paper we consider the second-order cone linear complementarity problem (SOCLCP), which seeks vectors $x, s \in \mathbf{R}^n$ and $q \in \mathbf{R}^n$ such that

$$x \in \mathcal{K}, \quad s = \mathcal{A}(x) + q \in \mathcal{K}, \quad \text{and} \quad \langle x, s \rangle = 0,$$

where $\langle x, s \rangle = \mathbf{Tr}(x \circ s)$ denotes the Euclidean inner product, $\mathcal{A} : \mathcal{K} \rightarrow \mathcal{K}$ is a linear transformation, and $\mathcal{K} \subseteq \mathbf{R}^n$ is the Cartesian product of several second-order cones, i.e.,

$$\mathcal{K} = \mathcal{K}^1 \times \dots \times \mathcal{K}^N,$$

with

$$\mathcal{K}^j = \left\{ x^{(j)} = (x_1^{(j)}; x_{2:n_j}^{(j)}) \in \mathbf{R} \times \mathbf{R}^{n_j-1} : x_1^{(j)} \geq \|x_{2:n_j}^{(j)}\| \right\},$$

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for each $j \in J = \{1, \dots, N\}$, $x_{2:n_j}^{(j)} = (x_2^{(j)}; \dots; x_{n_j}^{(j)})$, and $n = \sum_{j=1}^N n_j$. This problem includes as a special case the well-known linear complementarity problem (LCP), corresponding to $n_j = 1$ for all j , i.e., \mathcal{K} is the nonnegative orthant \mathbf{R}_+^n , and the Karush-Kuhn-Tucker (KKT) optimality conditions for second-order cone optimization (SOCO) [1] can be written in the form of SOCLCP [10]. Additionally, many important practical problems in economics and engineering, such as facility location and Nash equilibrium, can be formulated as it. Many researchers have studied the second-order cone complementarity problem (SOCCP) and achieved plentiful and beautiful results [9, 10].

We call SOCLCP the Cartesian $P_*(\kappa)$ -SOCLCP if the linear transformation \mathcal{A} has the Cartesian $P_*(\kappa)$ -property, i.e., for any nonnegative real number κ , the linear transformation \mathcal{A} satisfies

$$(1) \quad (1 + 4\kappa) \sum_{\nu \in I_+(x)} \langle x^{(\nu)}, [\mathcal{A}(x)]^{(\nu)} \rangle + \sum_{\nu \in I_-(x)} \langle x^{(\nu)}, [\mathcal{A}(x)]^{(\nu)} \rangle \geq 0,$$

where

$$I_+(x) = \{\nu \in J : \langle x^{(\nu)}, [\mathcal{A}(x)]^{(\nu)} \rangle > 0\}, \text{ and } I_-(x) = \{\nu \in J : \langle x^{(\nu)}, [\mathcal{A}(x)]^{(\nu)} \rangle < 0\}$$

are two index sets. It should be pointed out that the Cartesian $P_*(\kappa)$ -property is a weaker property than the monotonicity unless $\kappa = 0$, and $0 \leq \kappa_1 \leq \kappa_2$ implies $P_*(\kappa_1) \subset P_*(\kappa_2)$. Moreover, the linear transformation \mathcal{A} with the Cartesian $P_*(\kappa)$ -property becomes the usual $P_*(\kappa)$ matrix when \mathcal{K} is specified to be \mathbf{R}_+^n , correspondingly, the Cartesian $P_*(\kappa)$ -SOCLCP reduces to $P_*(\kappa)$ -LCP [11]. The linear transformation \mathcal{A} has the Cartesian P_* -property if it has the Cartesian $P_*(\kappa)$ -property for some nonnegative κ , i.e.,

$$P_* = \bigcup_{\kappa \geq 0} P_*(\kappa).$$

We also recall that the linear transformation \mathcal{A} has

- (a) the Cartesian P -property, if for any $x \in \mathcal{K}$ and $x \neq 0$, there exists an index $\nu \in \{1, 2, \dots, N\}$, such that $\langle x^{(\nu)}, [\mathcal{A}(x)]^{(\nu)} \rangle > 0$;
- (b) the Cartesian P_0 -property, if for any $x \in \mathcal{K}$ and $x \neq 0$, there exists an index $\nu \in \{1, 2, \dots, N\}$ such that $x^{(\nu)} \neq 0$ and $\langle x^{(\nu)}, [\mathcal{A}(x)]^{(\nu)} \rangle \geq 0$.

It is clear that the Cartesian P_* class involves the Cartesian P class and turns out to be a special case in the Cartesian P_0 class. The concept of the Cartesian P_0 - and P -properties was first introduced by Chen and Qi [6] for a linear transformation between the space of symmetric matrices, and later extended by Pan and Chen [15] and Luo and Xiu [12] to the space of second-order cones and the general Euclidean Jordan algebra, respectively.

Recently, Bai et al. [2] presented a new efficient large-update primal-dual interior-point algorithm for linear optimization (LO) based on a finite kernel function as follows

$$(2) \quad \psi(t) = \frac{t^2 - 1}{2} + \frac{e^{\sigma(1-t)} - 1}{\sigma}, \quad \sigma \geq 1.$$

which is not a kernel function in the usual sense [4]. It has a finite value at the boundary of the feasible region, i.e.,

$$(3) \quad \lim_{t \rightarrow 0} \psi(t) = \frac{e^\sigma - 1}{\sigma} - \frac{1}{2} < \infty.$$

Despite this, the iteration bound of a large-update method based on this kernel function is shown to be $O(\sqrt{n} \log n \log \frac{n}{\varepsilon})$. Moreover, Bai et al. [3] proposed a class of polynomial interior-point algorithms for $P_*(\kappa)$ -LCP based on a parametric kernel function and obtained the currently best known iteration bounds for large- and small-update methods, namely, $O((1+2\kappa)\sqrt{n} \log n \log \frac{n}{\varepsilon})$ and $O((1+2\kappa)\sqrt{n} \log \frac{n}{\varepsilon})$, respectively.

The purpose of the paper is to extend the primal-dual interior-point algorithm for LO based on the finite kernel function in [2] to the Cartesian $P_*(\kappa)$ -SOCLCP. We adopt the basic analysis used in [2, 3] and revise them to be suited for the Cartesian $P_*(\kappa)$ -SOCLCP case. The symmetrization of the search directions used in this paper is based on the Nesterov and Todd (NT) scaling scheme. Finally, we derive the iteration bounds that match the currently best known iteration bounds for large- and small-update methods, namely, $O((1+2\kappa)\sqrt{N} \log N \log \frac{N}{\varepsilon})$ and $O((1+2\kappa)\sqrt{N} \log \frac{N}{\varepsilon})$, respectively. Moreover, our analysis is simple and straightforward to the $P_*(\kappa)$ -LCP analogue.

The paper is organized as follows. In Section 2, we review some relevant algebraic properties of the second-order cones and the properties of the finite kernel (barrier) function that are needed in the analysis of the algorithm. In Section 3, we first discuss the central path for the Cartesian $P_*(\kappa)$ -SOCLCP. Then we mainly derive the new search directions based on the finite kernel function for the Cartesian $P_*(\kappa)$ -SOCLCP. The generic polynomial interior-point algorithm for the Cartesian $P_*(\kappa)$ -SOCLCP is also presented. The analysis and complexity results of the algorithm are presented in Section 4 and 5, respectively. Finally, some conclusions and remarks follow in Section 6.

Some of the notations used throughout the paper are as follows. \mathbf{R}^n , \mathbf{R}_+^n and \mathbf{R}_{++}^n denote the set of vectors with n components, the set of nonnegative vectors and the set of positive vectors, respectively. $\| \cdot \|$ denotes the Frobenius norm for matrices, and the 2-norm for vectors. The Löwner partial order “ \succeq ” (or \succ) on second-order cones means that $x \succeq_{\mathcal{K}} s$ (or $x \succ_{\mathcal{K}} s$) if $x - s \in \mathcal{K}$ (or $x - s \in \mathcal{K}_+$), where \mathcal{K}_+ denotes the interior of \mathcal{K} . E_n denotes the $n \times n$ identity matrix. Finally, if $g(x) \geq 0$ is a real valued function of a real nonnegative variable, the notation $g(x) = O(x)$ means that $g(x) \leq \bar{c}x$ for some positive constant \bar{c} and $g(x) = \Theta(x)$ that $c_1x \leq g(x) \leq c_2x$ for two positive constants c_1 and c_2 .

2. PRELIMINARIES

2.1. Algebraic properties of the second-order cones. In this subsection we briefly recall some algebraic properties of the second-order cones that are needed in the analysis of the algorithm. Our presentation is mainly based on the references [5, 8, 16]. To ease discussion, we assume the second-order cone \mathcal{K} is defined with $N = 1$.

For any two vectors $x, s \in \mathbf{R}^n$, the bilinear operator \circ is defined by

$$(4) \quad x \circ s := (x^T s; x_1 s_2 + s_1 x_2; \cdots; x_1 s_n + s_1 x_n).$$

This operator is commutative. Moreover, (\mathbf{R}^n, \circ) is a Jordan algebra. Note that the map $s \mapsto x \circ s$ is linear. The matrix of this linear map, with respect to the standard basis, is denoted as $L(x)$, and one may easily verify that it is an arrow-shaped (symmetric) matrix

$$(5) \quad L(x) := \begin{pmatrix} x_1 & x_{2:n}^T \\ x_{2:n} & x_1 E_{n-1} \end{pmatrix}.$$

It follows that $L(x)s = x \circ s = s \circ x = L(s)x$. Moreover, $x \in \mathcal{K}$ (or \mathcal{K}_+) if and only if $L(x)$ is positive semidefinite (or positive definite) matrix.

Let $\lambda_{max}(x)$ and $\lambda_{min}(x)$ denote the maximal and minimal eigenvalues of $L(x)$, respectively, namely,

$$(6) \quad \lambda_{max}(x) := x_1 + \|x_{2:n}\|, \text{ and } \lambda_{min}(x) := x_1 - \|x_{2:n}\|.$$

It readily follows that

$$(7) \quad x \in \mathcal{K} \Leftrightarrow \lambda_{max}(x) \geq \lambda_{min}(x) \geq 0, \text{ and } x \in \mathcal{K}_+ \Leftrightarrow \lambda_{max}(x) \geq \lambda_{min}(x) > 0.$$

Furthermore, we can conclude that

$$(8) \quad |\lambda_{max}(x)| \leq \sqrt{2}\|x\|, \text{ and } |\lambda_{min}(x)| \leq \sqrt{2}\|x\|.$$

For any $x \in \mathbf{R}^n$, the trace of x associated with \mathcal{K} is defined by

$$(9) \quad \mathbf{Tr}(x) := \lambda_{max}(x) + \lambda_{min}(x) = 2x_1,$$

and the determinant of x associated with \mathcal{K} is given by

$$(10) \quad \det(x) := \lambda_{max}(x)\lambda_{min}(x) = x_1^2 - \|x_{2:n}\|^2.$$

The definition of the trace function, together with (4), implies that, for any $x, s \in \mathbf{R}^n$,

$$(11) \quad \langle x, s \rangle := \mathbf{Tr}(x \circ s) = 2x^T s.$$

It follows that

$$(12) \quad \mathbf{Tr}(x \circ x) = 2\|x\|^2.$$

Lemma 2.1 (Lemma 2.2 in [5]). *If $x, s \in \mathbf{R}^n$, then*

$$\lambda_{min}(x + s) \geq \lambda_{min}(x) - \sqrt{2}\|s\|.$$

Lemma 2.2 (Corollary 2.4 in [5]). *If $x, s \in \mathcal{K}$, then*

$$\lambda_{min}(x) \mathbf{Tr}(s) \leq \mathbf{Tr}(x \circ s) \leq \lambda_{max}(x) \mathbf{Tr}(s).$$

For any $x \in \mathbf{R}^n$, the so-called spectral decomposition of x is given by

$$(13) \quad x := \lambda_{max}(x)z_1 + \lambda_{min}(x)z_2,$$

where the Jordan frame $\{z_1, z_2\}$ is given by

$$(14) \quad z_1 := \frac{1}{2} \left(1; \frac{x_{2:n}}{\|x_{2:n}\|} \right), \text{ and } z_2 := \frac{1}{2} \left(1; \frac{-x_{2:n}}{\|x_{2:n}\|} \right),$$

which are the eigenvectors of $L(x)$ for the eigenvalues $\lambda_{max}(x)$ and $\lambda_{min}(x)$, respectively. Here by convention $\frac{x_{2:n}}{\|x_{2:n}\|} = 0$ if $x_{2:n} = 0$.

The importance of the spectral decomposition is that it enables us to extend the definition of any function $\psi : \mathbf{R} \rightarrow \mathbf{R}$ to a function that maps \mathbf{R}^n into \mathbf{R}^n . In particular this holds for the finite kernel function $\psi(t)$.

Let $\psi : \mathbf{R} \rightarrow \mathbf{R}$ and $x \in \mathbf{R}^n$ with the spectral decomposition as defined by (13). Then the vector valued function $\psi(x) : \mathbf{R}^n \rightarrow \mathbf{R}^n$ is defined as follows

$$(15) \quad \psi(x) := \psi(\lambda_{max}(x))z_1 + \psi(\lambda_{min}(x))z_2.$$

It follows that

$$(16) \quad \|\psi(x)\| = \frac{\sqrt{2}}{2} \sqrt{\psi^2(\lambda_{max}(x)) + \psi^2(\lambda_{min}(x))}$$

and

$$(17) \quad \mathbf{Tr}(\psi(x)) = \psi(\lambda_{max}(x)) + \psi(\lambda_{min}(x)).$$

If $\psi(t)$ is twice differentiable, like the finite kernel function $\psi(t)$, the derivative $\psi'(t)$ and $\psi''(t)$ exist for $t > 0$, and we also have the vector-valued functions $\psi'(x)$ and $\psi''(x)$, namely

$$(18) \quad \psi'(x) = \psi'(\lambda_{max}(x))z_1 + \psi'(\lambda_{min}(x))z_2,$$

and

$$(19) \quad \psi''(x) = \psi''(\lambda_{max}(x))z_1 + \psi''(\lambda_{min}(x))z_2.$$

For any $x \in \mathbf{R}^n$, we define

$$(20) \quad P(x) := 2L(x)^2 - L(x^2),$$

where $L(x)^2 = L(x)L(x)$ and $x^2 = x \circ x$. As a consequence we have

$$P(x) = \begin{bmatrix} \|x\|^2 & 2x_1x_{2:n}^T \\ 2x_1x_{2:n} & \det(x)E_{n-1} + 2x_1x_{2:n}^T \end{bmatrix}.$$

The map $P(x)$ is called the quadratic representation of \mathcal{K} , which is an essential concept in the theory of Jordan algebras.

Lemma 2.3 (NT-sacling, Lemma 3.2 in [8]). *Let $x, s \in \mathcal{K}_+$. Then there exists a unique $w \in \mathcal{K}_+$ such that*

$$x = P(w)s.$$

Moreover,

$$w = P(x^{\frac{1}{2}}) \left(P(x^{\frac{1}{2}})s \right)^{-\frac{1}{2}} \left[= P(s^{-\frac{1}{2}}) \left(P(s^{\frac{1}{2}})x \right)^{\frac{1}{2}} \right].$$

Note that $P(w)$ is an automorphism. The point w is called the scaling point of x and s (in this order). Hence there exists $\tilde{v} \in \mathcal{K}_+$ such that

$$(21) \quad \tilde{v} = P(w)^{-\frac{1}{2}}x = P(w)^{\frac{1}{2}}s,$$

which is the so-called NT-scaling of \mathbf{R}^n . In the following lemma we recall several properties of the NT-scaling scheme without its proof.

Lemma 2.4 (Proposition 3.2 in [16]). *Let $W = P(w)^{\frac{1}{2}}$ for some $w \in \mathcal{K}_+$. Then the following holds for any $x, s \in \mathbf{R}^n$*

- (i) $\mathbf{Tr}(Wx \circ W^{-1}s) = \mathbf{Tr}(x \circ s)$;
- (ii) $\det(Wx) = \det(w)\det(x)$, and $\det(W^{-1}s) = \det(w^{-1})\det(s)$;
- (iii) if w is the scaling point of x and s , then $\det(Wx \circ W^{-1}s) = \det(x)\det(s)$.

2.2. Back to the general case. In what follows we proceed by adapting the definitions and properties in this section to the general case where $N > 1$, when the cone underlying the given Cartesian $P_*(\kappa)$ -SOCLCP is the Cartesian product of N second-order cones \mathcal{K}^j . First we partition any vector $x \in \mathbf{R}^n$ according to the dimensions of the successive cones \mathcal{K}^j , so

$$(22) \quad x = \left(x^{(1)}; \dots; x^{(N)}\right), \text{ and } s = \left(s^{(1)}; \dots; s^{(N)}\right), \quad x^{(j)}, s^{(j)} \in \mathbf{R}^{n_j},$$

and we define the algebra (\mathbf{R}^n, \diamond) as a direct product of Jordan algebras

$$(23) \quad x \diamond s = \left(x^{(1)} \circ s^{(1)}; \dots; x^{(N)} \circ s^{(N)}\right).$$

Obviously, if $e^{(j)} \in \mathcal{K}^j$ is the unit element in the Jordan algebra for the j -th cone, then the vector

$$(24) \quad e = (e^{(1)}; \dots; e^{(N)})$$

is the unit element in (\mathbf{R}^n, \diamond) . Furthermore, we have

$$(25) \quad \mathbf{Tr}(x) = \sum_{j=1}^N \mathbf{Tr}(x^{(j)}), \quad \|x\| = \sqrt{\sum_{j=1}^N \|x^{(j)}\|^2}, \text{ and } \det(x) = \prod_{j=1}^N \det(x^{(j)}).$$

One can easily verify that $\mathbf{Tr}(e) = 2N$. It follows from (5) and (20) that

$$(26) \quad L(x) = \text{diag}(L(x^{(1)}), \dots, L(x^{(N)})),$$

and

$$(27) \quad P(x) = \text{diag}(P(x^{(1)}), \dots, P(x^{(N)})).$$

The NT-scaling scheme in this general case is now obtained as follows. Let $w^{(j)}$ be the scaling point $x^{(j)}, s^{(j)}$ in \mathcal{K}_+^j . Then

$$(28) \quad P(w^{(j)})^{-\frac{1}{2}}x^{(j)} = P(w^{(j)})^{\frac{1}{2}}s^{(j)}, \quad j \in J.$$

The scaling point of x and s in \mathcal{K} is then defined by

$$(29) \quad w = (w^{(1)}; \dots; w^{(N)}).$$

Since $P(w^{(j)})$ is symmetric and positive definite for each $j \in J$, the matrix

$$(30) \quad P(w) = \text{diag}(P(w^{(1)}), \dots, P(w^{(N)})),$$

is symmetric and positive definite as well and represents an automorphism of \mathcal{K} such that $P(w)s = x$. Therefore $P(w)$ can be used to rescale x and s to the same vector

$$(31) \quad v = (v^{(1)}; \dots; v^{(N)}).$$

As a consequence, we adapt the definitions of $\psi(v)$ as follows

$$(32) \quad \psi(v) = (\psi(v^{(1)}); \dots; \psi(v^{(N)})).$$

Since $L(v) = \text{diag}(L(v^{(1)}), \dots, L(v^{(N)}))$, we have

$$(33) \quad \lambda_{max}(v) = \lambda_{max}(L(v)) = \max\{\lambda_{max}(v^{(j)}) : j \in J\},$$

and

$$(34) \quad \lambda_{min}(v) = \lambda_{min}(L(v)) = \min\{\lambda_{min}(v^{(j)}) : j \in J\}.$$

2.3. Properties of the finite kernel (barrier) function. In this subsection we review some useful properties of the finite kernel function and the corresponding barrier function that are used in the analysis of the algorithm. The detailed can be found in [2]. For ease of reference, we give the first three derivatives of $\psi(t)$ with respect to t as follows

$$(35) \quad \psi'(t) = t - e^{\sigma(1-t)}, \quad \psi''(t) = 1 + \sigma e^{\sigma(1-t)}, \quad \psi'''(t) = -\sigma^2 e^{\sigma(1-t)}.$$

It is quite straightforward to verify

$$(36) \quad \psi(1) = \psi'(1) = 0, \quad \psi''(t) > 0, \quad t > 0, \quad \psi'''(t) < 0, \quad t > 0, \quad \lim_{t \rightarrow \infty} \psi(t) = +\infty.$$

Moreover, $\psi(t)$ is strictly convex and $\psi''(t)$ is monotonically decreasing in $t \in (0, +\infty)$.

Corresponding to the finite kernel function $\psi(t)$, we define the barrier function $\Psi(v)$ as follows

$$(37) \quad \Psi(x, s; \mu) := \Psi(v) := \text{Tr}(\psi(v)) = \sum_{j=1}^N \text{Tr}(\psi(v^{(j)})) = \sum_{j=1}^N \left(\psi(\lambda_{max}(v^{(j)})) + \psi(\lambda_{min}(v^{(j)})) \right).$$

Moreover, we can conclude that the derivative of the barrier function $\Psi(v)$ is exactly equal to the derivative of the vector valued function $\psi(v)$, i.e., $\nabla \Psi(v) = \psi'(v)$.

Theorem 2.5 (Proposition 2.9 in [16]). *$\Psi(v)$ is nonnegative and strictly convex with respect to $v \in \mathcal{K}_+$ and vanishes at its global minimal point $v = e = (1; 0; \dots; 0)$, i.e.,*

$$\Psi(v) = 0 \Leftrightarrow \psi(v) = 0 \Leftrightarrow \psi'(v) = 0 \Leftrightarrow v = e.$$

Lemma 2.6 (Lemma 2.4 in [2]). *If $t_1 \geq \frac{1}{\sigma}$ and $t_2 \geq \frac{1}{\sigma}$, then*

$$\psi(\sqrt{t_1 t_2}) \leq \frac{1}{2}(\psi(t_1) + \psi(t_2)).$$

As a consequence of Lemma 2.6, we have the following lemma, which is crucial for the analysis of the algorithm presented in Fig. 1.

Lemma 2.7 (Proposition 2.9 in [16]). *Let $\lambda_{\min}(x) \geq \frac{1}{\sigma}$, $\lambda_{\min}(s) \geq \frac{1}{\sigma}$. If $v \in \mathcal{K}_+$ satisfy*

$$\mathbf{det}(v \circ v) = \mathbf{det}(x) \mathbf{det}(s), \text{ and } \mathbf{Tr}(v \circ v) = \mathbf{Tr}(x \circ s),$$

then

$$\Psi(v) \leq \frac{1}{2}(\Psi(x) + \Psi(s)).$$

Note that $\psi(t)$ is exponentially convex, whenever $t \geq \frac{1}{\sigma}$. The following lemma makes clear that when v belongs to the level set $\{v : \Psi(v) \leq L\}$, for some given $L \geq 8$, the exponential convexity is guaranteed and it is proved that the value of σ is large enough.

Lemma 2.8 (Lemma 2.5 in [2]). *Let $L \geq 8$ and $\Psi(v) \leq L$. If $\sigma \geq 1 + 2 \log(1 + L)$, then $\lambda_{\min}(v) \geq \frac{3}{2\sigma}$.*

Lemma 2.9. *If $t \geq 1$, then*

$$\psi(t) \leq \frac{1 + \sigma}{2}(t - 1)^2.$$

Proof. By using Taylor's theorem and the fact that $\psi''(1) = 1 + \sigma$, the inequality is straightforward.

Lemma 2.10. *If $\sigma \geq 2$ and $t \geq 1$, then*

$$t \leq 1 + \sqrt{t\psi(t)}.$$

Proof. Defining $f(t) := t\psi(t) - (t - 1)^2$, we have $f(1) = 0$ and $f'(t) = \psi(t) + t\psi'(t) - 2(t - 1)$. Hence $f'(1) = 0$ and $f''(t) = 2\psi'(t) + t\psi''(t) - 2 = 2(t - 1) + (\sigma t - 2)e^{\sigma(1-t)} + t > 0$. The last inequality holds since $\sigma \geq 2$ and $t \geq 1$. This implies the lemma.

Lemma 2.11. *If $t \geq 1$, then*

$$t\psi'(t) \geq \psi(t).$$

Proof. Defining $f(t) := t\psi'(t) - \psi(t)$, we have $f(1) = 0$ and

$$f'(t) = t\psi''(t) \geq 0.$$

This implies the lemma.

Lemma 2.12. *Let $\varrho : [0, \infty) \rightarrow [1, \infty)$ be the inverse function of $\psi(t)$ for $t \geq 1$. If $\sigma \geq 1$, then*

$$(38) \quad \sqrt{1 + 2s} \leq \varrho(s) \leq \left(2s + \frac{2 + \sigma}{\sigma}\right)^{\frac{1}{2}}.$$

If $\sigma \geq 2$, then

$$(39) \quad \varrho(s) \leq 1 + \sqrt{s} \left(2s + \frac{2 + \sigma}{\sigma}\right)^{\frac{1}{4}}.$$

Proof. Let $\varrho(s) = t \geq 1$. Then $s = \psi(t)$. Hence, using $t \geq 1$,

$$\frac{t^2 - 1}{2} - \frac{1}{\sigma} \leq s = \frac{t^2 - 1}{2} + \frac{e^{\sigma(1-t)} - 1}{\sigma} \leq \frac{t^2 - 1}{2}.$$

The left inequality gives

$$(40) \quad \varrho(s) = t \leq \left(1 + 2 \left(s + \frac{1}{\sigma}\right)\right)^{\frac{1}{2}} = \left(2s + \frac{2 + \sigma}{\sigma}\right)^{\frac{1}{2}},$$

and the right inequality

$$\varrho(s) = t \geq (1 + 2s)^{\frac{1}{2}}.$$

Now we turn to the case that $\sigma \geq 2$. By Lemma 2.10 we have

$$t \leq 1 + \sqrt{t\psi(t)} = 1 + \sqrt{ts}.$$

Substituting the upper bound for t given by (40) we obtain (39). This proves the lemma.

For the analysis of the algorithm, we define the norm-based proximity $\delta(v)$ as follows

$$(41) \quad \delta(v) := \frac{1}{\sqrt{2}} \|\nabla \Psi(v)\| = \frac{1}{2} \sqrt{\sum_{j=1}^N ((\psi'(\lambda_{\max}(v^{(j)})))^2 + (\psi'(\lambda_{\min}(v^{(j)})))^2)}.$$

One can easily verify that $\delta(v) \geq 0$, and $\delta(v) = 0$ if and only if $\Psi(v) = 0$.

It is clear that $\delta(v)$ and $\Psi(v)$ depend only on the eigenvalues $\lambda_{\max}(v^{(j)})$ and $\lambda_{\min}(v^{(j)})$ of the vectors $v^{(j)}$ from (37) and (41), for $j \in J$. This observation makes it possible to apply Theorem 4.9 in [4], with z being the vector in \mathbf{R}^{2N} consisting of all the eigenvalues of the second-order cone v . The theorem below immediately follows.

Theorem 2.13. *If $v \in \mathcal{K}_+$, then*

$$\delta(v) \geq \frac{1}{2} \psi'(\varrho(\Psi(v))).$$

Corollary 2.14. *If $v \in \mathcal{K}_+$ and $\Psi(v) \geq \tau \geq 1$, then*

$$\delta(v) \geq \frac{1}{6} \sqrt{\Psi(v)}.$$

Proof. Since $\Psi(v) \geq 1$ and $\sigma \geq 1$, from (38), we have

$$\varrho(\Psi(v)) \leq \left(2\Psi(v) + \frac{2 + \sigma}{\sigma}\right)^{\frac{1}{2}} \leq (5\Psi(v))^{\frac{1}{2}} \leq 3(\Psi(v))^{\frac{1}{2}}.$$

By applying Theorem 2.13 and Lemma 2.11, we have

$$\delta(v) \geq \frac{1}{2} \psi'(\varrho(\Psi(v))) \geq \frac{\psi(\varrho(\Psi(v)))}{2\varrho(\Psi(v))} = \frac{\Psi(v)}{2\varrho(\Psi(v))} \geq \frac{\Psi(v)}{6(\Psi(v))^{\frac{1}{2}}} = \frac{1}{6} \sqrt{\Psi(v)}.$$

This proves the corollary.

In the analysis of the algorithm, which will be presented below, we need to consider the derivatives with respect to a real parameter t of the functions $\psi'(x(t))$ and $\Psi(x(t))$, where

$$x(t) = (x^{(1)}(t); \dots; x^{(N)}(t))$$

with

$$x^{(j)}(t) = (x_1^{(j)}(t); \cdots; x_{n_j}^{(j)}(t)), \quad j \in J.$$

The usual concepts of continuity, differentiability and integrability can be naturally extended to the vectors of functions, by interpreting them entry-wise.

The following lemma provides measure the first-order directional derivative of a general function $\Psi(x(t))$ and bound its second-order derivative with respect to t .

Lemma 2.15 (Lemma 2.10 in [16]). *Suppose that $x(t)$ is a mapping from \mathbf{R} into \mathbf{R}^n . If $x(t)$ is twice differentiable with respect to t for all $t \in (l_t, u_t)$ and $\psi(t)$ is also twice continuously differentiable function in a suitable domain that contains all the eigenvalues $\lambda_{max}(x(t))$ and $\lambda_{min}(x(t))$ of $x(t)$, then*

$$\frac{d}{dt} \mathbf{Tr}(\psi(x(t))) = \mathbf{Tr}(\psi'(x(t)) \diamond x'(t)), \quad \forall t \in (l_t, u_t),$$

and

$$\frac{d^2}{dt^2} \mathbf{Tr}(\psi(x(t))) \leq \omega \mathbf{Tr}(x(t) \diamond x'(t)) + \mathbf{Tr}(\psi''(x(t)) \diamond x''(t)),$$

where

$$\omega = \max \left\{ |\psi''(\lambda_{max}(x(t)))|, |\psi''(\lambda_{min}(x(t)))|, \frac{|\psi'(\lambda_{max}(x(t))) - \psi'(\lambda_{min}(x(t)))|}{2\|x_{2:n}(t)\|} \right\}.$$

3. POLYNOMIAL INTERIOR-POINT ALGORITHM FOR THE CARTESIAN $P_*(\kappa)$ -SOCLCP

3.1. The central path for the Cartesian $P_*(\kappa)$ -SOCLCP. Until now majority of well known polynomial interior-point methods (IPMs) used the so-called central path as a guideline to the optimal set, and some variant of the Newton's method to follow the central path approximately. Kojima et al. [11] first proved the existence and uniqueness of the central path for any $P_*(\kappa)$ -LCP and unified the theory of $P_*(\kappa)$ -LCP from the view point of IPMs. Analogously to the $P_*(\kappa)$ -LCP case, the concept of the central path can also be extended to the Cartesian $P_*(\kappa)$ -SOCLCP. In addition, the existence and uniqueness of the central path associated with the Cartesian $P_*(\kappa)$ -SOCLCP are precisely the special case of the Cartesian $P_*(\kappa)$ -SCLCP established by Luo and Xiu [12]. For more details we refer to [11, 12].

Throughout the paper, we assume that the Cartesian $P_*(\kappa)$ -SOCLCP satisfies the interior-point condition (IPC), i.e., there exists $(x^0 \succ_{\kappa} 0, s^0 \succ_{\kappa} 0)$ such that $s^0 = \mathcal{A}(x^0) + q$. For this and other properties of the Cartesian $P_*(\kappa)$ -SOCLCP, we refer to [12]. Under the IPC holds, by relaxing the complementarity slackness $x \diamond s = 0$, we obtain

$$(42) \quad \begin{pmatrix} \mathcal{A}(x) - s \\ x \diamond s \end{pmatrix} = \begin{pmatrix} -q \\ \mu e \end{pmatrix}, \quad x, s \succ_{\kappa} 0,$$

where $\mu > 0$ is a parameter. The parameterized system (42) has a unique solution, for each $\mu > 0$ (cf. Section 3 in [12]). This solution is denoted as $(x(\mu), s(\mu))$ and we call $(x(\mu), s(\mu))$ the μ -center of the Cartesian $P_*(\kappa)$ -SOCLCP. The set of μ -centers (with μ running through all positive real numbers) gives a homotopy path, which is called *the central path* of the Cartesian $P_*(\kappa)$ -SOCLCP. If $\mu \rightarrow 0$, then the limit

of the central path exists and since the limit points satisfy the complementarity condition $x \circ s = 0$, the limit yields an optimal solution for the Cartesian $P_*(\kappa)$ -SOCLCP.

3.2. The new search directions for the Cartesian $P_*(\kappa)$ -SOCLCP. IPMs follow the central path approximately and find an approximate solution of the Cartesian $P_*(\kappa)$ -SOCLCP by letting μ go to zero. For any strictly feasible $x \succ_{\mathcal{K}} 0$ and $s \succ_{\mathcal{K}} 0$, we want to find displacements Δx and Δs such that

$$(43) \quad \begin{pmatrix} \mathcal{A}(x + \Delta x) - (s + \Delta s) \\ (x + \Delta x) \diamond (s + \Delta s) \end{pmatrix} = \begin{pmatrix} -q \\ \mu e \end{pmatrix}.$$

Neglecting the term $\Delta x \diamond \Delta s$ in the left-hand side expression of the second equation, we obtain the following system

$$(44) \quad \begin{pmatrix} \mathcal{A}(\Delta x) - \Delta s \\ s \diamond \Delta x + x \diamond \Delta s \end{pmatrix} = \begin{pmatrix} 0 \\ \mu e - x \diamond s \end{pmatrix}.$$

Due to the fact that x and s do not operator commute in general, i.e., $L(x)L(s) \neq L(s)L(x)$, this system doesn't always have a unique solution. It is well known that this difficulty can be solved by applying a scaling scheme. This goes as follows.

Lemma 3.1 (Lemma 28 in [18]). *Let $u \in \mathcal{K}_+$. Then*

$$x \diamond s = \mu e \Leftrightarrow P(u)x \diamond P(u)^{-1}s = \mu e.$$

Now we replace the second equation of the system (43) by

$$(45) \quad P(u)(x + \Delta x) \diamond P(u)^{-1}(s + \Delta s) = \mu e.$$

Applying Newton's method again, and neglecting the term $P(u)\Delta x \diamond P(u)^{-1}\Delta s$, we get

$$(46) \quad \begin{pmatrix} \mathcal{A}(\Delta x) - \Delta s \\ P(u)^{-1}(s) \diamond P(u)\Delta x + P(u)(x) \diamond P(u)^{-1}\Delta s \end{pmatrix} = \begin{pmatrix} 0 \\ \mu e - P(u)(x) \diamond P(u)^{-1}(s) \end{pmatrix}.$$

In this paper we consider the so-called NT scaling scheme [13, 14]. Let $u = w^{-\frac{1}{2}}$, where

$$w = P(x)^{\frac{1}{2}} \left(P(x^{\frac{1}{2}})s \right)^{-\frac{1}{2}} \left[= P(s^{-\frac{1}{2}}) \left(P(s^{\frac{1}{2}})x \right)^{\frac{1}{2}} \right].$$

Furthermore, we define

$$(47) \quad v := \frac{P(w)^{-\frac{1}{2}}x}{\sqrt{\mu}} \left[= \frac{P(w)^{\frac{1}{2}}s}{\sqrt{\mu}} \right],$$

and

$$(48) \quad \bar{\mathcal{A}} := P(w)^{\frac{1}{2}}\mathcal{A}P(w)^{\frac{1}{2}}, \quad d_x := \frac{P(w)^{-\frac{1}{2}}\Delta x}{\sqrt{\mu}}, \quad d_s := \frac{P(w)^{\frac{1}{2}}\Delta s}{\sqrt{\mu}}.$$

It should be mentioned that the transformation $\bar{\mathcal{A}}$ also has the Cartesian $P_*(\kappa)$ -property if the linear transformation \mathcal{A} has the Cartesian $P_*(\kappa)$ -property (cf. Proposition 3.4 in [12]). From (47) and (48), after some elementary reductions, we obtain the scaled Newton system as follows

$$(49) \quad \begin{pmatrix} \bar{\mathcal{A}}(d_x) - d_s \\ d_x + d_s \end{pmatrix} = \begin{pmatrix} 0 \\ v^{-1} - v \end{pmatrix}.$$

Since the linear transformation $\bar{\mathcal{A}}$ has the Cartesian $P_*(\kappa)$ -property, the system (49) has a unique solution [12]. So far we have described the scheme that defines the classical NT search direction for the Cartesian $P_*(\kappa)$ -SOCLCP. The approach in this paper differs only in one detail: we replace the right-hand side of the second equation in (49) by $-\nabla\Psi(v)$. Thus we will use the following system to define our new search direction

$$(50) \quad \begin{pmatrix} \bar{\mathcal{A}}(d_x) - d_s \\ d_x + d_s \end{pmatrix} = \begin{pmatrix} 0 \\ -\nabla\Psi(v) \end{pmatrix}.$$

Since (50) has the same matrix of coefficients as (49), (50) also has a unique solution.² The new search directions d_x and d_s are obtained by solving (50) so that Δx and Δs are computed via (48). If $(x, s) \neq (x(\mu), s(\mu))$ then $(\Delta x, \Delta s)$ is nonzero. By taking a default step size along the search directions, we get the new iteration point as follows

$$(51) \quad x_+ := x + \alpha\Delta x, \text{ and } s_+ := s + \alpha\Delta s.$$

Furthermore, we have, by Theorem 2.5,

$$(52) \quad x \diamond s = \mu e \Leftrightarrow v = e \Leftrightarrow \psi'(v) = 0 \Leftrightarrow \psi(v) = 0 \Leftrightarrow \Psi(v) = 0.$$

Hence, the value of $\Psi(v)$ can be considered as a measure for the distance between the given iterate (x, s) and the μ -center $(x(\mu), s(\mu))$.

3.3. The generic interior-point algorithm for the Cartesian $P_*(\kappa)$ -SOCLCP.

It is clear from the above description that the closeness of (x, s) to $(x(\mu), s(\mu))$ is measured by the value of $\Psi(v)$, with $\tau > 0$ as a threshold value. If $\Psi(v) \leq \tau$ then we start a new *outer iteration* by performing a μ -update, otherwise we enter an *inner iteration* by computing the search directions at the current iterates with respect to the current value of μ and apply (51) to get the new iterates. If necessary, we repeat the procedure until we find the iterates that are in the neighborhood of $(x(\mu), s(\mu))$. Then μ is again reduced by the factor $1 - \theta$ with $0 < \theta < 1$ and we apply Newton method targeting at the new μ -centers, and so on. This process is repeated until μ is small enough, say until $N\mu < \varepsilon$, at this stage we have found an ε -approximate solution of the Cartesian $P_*(\kappa)$ -SOCLCP. The parameters τ, θ and

²It may be worth mentioning that if we use the kernel function of the classical logarithmic barrier function, i.e., $\psi(t) = \frac{1}{2}(t^2 - 1) - \log t$, then $\psi'(t) = t - t^{-1}$, whence $-\nabla\Psi(v) = v^{-1} - v$, and hence system (50) then coincides with the classical system (49).

the step size α should be chosen in such a way that the algorithm is ‘optimized’ in the sense that the number of iterations required by the algorithm is as small as possible.

The generic polynomial interior-point algorithm for the Cartesian $P_*(\kappa)$ -SOCLCP is now presented in Figure 1.

Interior-Point Algorithm for the Cartesian $P_*(\kappa)$ -SOCLCP

Input:

A threshold parameter $\tau \geq 1$;
 an accuracy parameter $\varepsilon > 0$;
 a fixed barrier update parameter θ , $0 < \theta < 1$;
 a strictly feasible (x^0, s^0) and $\mu^0 = (x^0)^T s^0 / N$ such that $\Psi(x^0, s^0; \mu^0) \leq \tau$.

begin

$x := x^0$; $s := s^0$; $\mu := \mu^0$;

while $N\mu \geq \varepsilon$ **do**

begin

$\mu := (1 - \theta)\mu$;

while $\Psi(x, s; \mu) > \tau$ **do**

begin

solve system (50) and use (48) to obtain $(\Delta x, \Delta s)$;

choose a suitable step size α ;

update $x = x + \alpha\Delta x$, $s = s + \alpha\Delta s$;

end

end

end

FIGURE 1. Algorithm

4. ANALYSIS OF THE ALGORITHM

4.1. Growth behavior of the barrier function. Note that during the course of the algorithm the largest values of $\Psi(v)$ occur just after the update of μ . So next we derive an estimate for the effect of a μ -update on the value of $\Psi(v)$.

It follows from (37) and (15) with (31) that

$$\Psi(\beta v) = \sum_{j=1}^N \left(\psi(\beta \lambda_{max}(v^{(j)})) + \psi(\beta \lambda_{min}(v^{(j)})) \right).$$

As in the previous theorem, the variables are essentially only the eigenvalues $\lambda_{max}(v^{(j)})$ and $\lambda_{min}(v^{(j)})$ of the vectors $v^{(j)}$, for $j \in J$. Applying Theorem 3.2 in [4], with z being the vector in \mathbf{R}^{2N} consisting of all the eigenvalues of the second-order cone v , the theorem below immediately follows.

Theorem 4.1. *If $v \in \mathcal{K}_+$ and $\beta \geq 1$, then*

$$\Psi(\beta v) \leq 2N\psi \left(\beta \varrho \left(\frac{\Psi(v)}{2N} \right) \right).$$

Corollary 4.2. *Let $0 \leq \theta < 1$ and $v_+ = \frac{v}{\sqrt{1-\theta}}$. If $\Psi(v) \leq \tau$, then*

$$\Psi(v_+) \leq 2N\psi \left(\frac{\varrho(\frac{\tau}{2N})}{\sqrt{1-\theta}} \right).$$

Proof. With $\beta = \frac{1}{\sqrt{1-\theta}} \geq 1$ and $\Psi(v) \leq \tau$, the corollary follows immediately from Theorem 4.1.

As we will show in the next section, after each μ -update the subsequent inner iterations will give rise to decreasing values of $\Psi(v)$. Hence, due to Corollary 4.2, we conclude that

$$(53) \quad L := 2N\psi \left(\frac{\varrho(\frac{\tau}{2N})}{\sqrt{1-\theta}} \right)$$

is an upper bound for the value that $\Psi(v)$ attains during the course of the algorithm.

4.2. Decrease of the barrier function and choice of the default step. In each inner iteration after a feasible step, we get a new primal-dual pair

$$x_+ := x + \alpha \Delta x, \text{ and } s_+ := s + \alpha \Delta s$$

and we can write

$$x_+ = \sqrt{\mu} P(w)^{\frac{1}{2}} (v + d_x) \text{ and } s_+ = \sqrt{\mu} P(w)^{-\frac{1}{2}} (v + d_s).$$

Define

$$v_+ := \frac{1}{\sqrt{\mu}} P(w_+)^{-\frac{1}{2}} x_+ = \frac{1}{\sqrt{\mu}} P(w_+)^{\frac{1}{2}} s_+.$$

We have

$$v_+ = P(w_+)^{-\frac{1}{2}} P(w)^{\frac{1}{2}} (v + \alpha d_x) = P(w_+)^{\frac{1}{2}} P(w)^{-\frac{1}{2}} (v + \alpha d_s),$$

where $w_+ := P(x_+)^{\frac{1}{2}} (P(x_+)^{\frac{1}{2}} s_+)^{-\frac{1}{2}} = P(s_+)^{-\frac{1}{2}} (P(s_+)^{\frac{1}{2}} x_+)^{\frac{1}{2}}$. Now, we consider the decrease in $\Psi(v)$ as a function of α and define

$$f(\alpha) := \Psi(v_+) - \Psi(v).$$

Our aim is to find an upper bound for $f(\alpha)$ by using the exponential convexity of $\psi(t)$, and according to Lemma 2.6. In order to do this we assume for the moment that

$$(54) \quad \lambda_{\min}(v^{(j)} + \alpha d_x^{(j)}) \geq \frac{1}{\sigma}, \text{ and } \lambda_{\min}(v^{(j)} + \alpha d_s^{(j)}) \geq \frac{1}{\sigma}, \quad j \in J.$$

Since v_+ is the scaled vector resulting from the NT-scaling, we can conclude, by Lemma 2.4 and (48), that

$$\mathbf{Tr}((v_+)^2) = \frac{1}{\mu} \mathbf{Tr}((x + \alpha \Delta x) \diamond (s + \alpha \Delta s)) = \mathbf{Tr}((v + \alpha d_x) \diamond (v + \alpha d_s)),$$

$$\det((v_+)^2) = \frac{1}{\mu^2} \det((x + \alpha \Delta x) \diamond (s + \alpha \Delta s)) = \det(v + \alpha d_x) \det(v + \alpha d_s).$$

Theorem 2.7 implies that

$$\Psi(v_+) \leq \frac{1}{2}(\Psi(v + \alpha d_x) + \Psi(v + \alpha d_s)).$$

We have

$$f(\alpha) \leq f_1(\alpha) := \frac{1}{2}(\Psi(v + \alpha d_x) + \Psi(v + \alpha d_s)) - \Psi(v).$$

It is obvious that $f(0) = f_1(0) = 0$. From Lemma 2.15, we have

$$(55) \quad f'_1(\alpha) = \frac{1}{2}(\mathbf{Tr}(\psi'(v + \alpha d_x) \diamond d_x) + \mathbf{Tr}(\psi'(v + \alpha d_s) \diamond d_s)),$$

and

$$(56) \quad f''_1(\alpha) = \frac{1}{2} \frac{d^2}{d\alpha^2} \mathbf{Tr}(\psi(v + \alpha d_x) + \psi(v + \alpha d_s)) \leq \frac{1}{2}(\omega_1 \|d_x\|^2 + \omega_2 \|d_s\|^2),$$

where

$$(57) \quad \omega_1 = \max \left\{ |\psi''(\lambda_{max}(v + \alpha d_x))|, |\psi''(\lambda_{min}(v + \alpha d_x))|, \frac{|\psi'(\lambda_{max}(v + \alpha d_x)) - \psi'(\lambda_{min}(v + \alpha d_x))|}{2\|(v + \alpha d_x)_{2:n}\|} \right\},$$

$$(58) \quad \omega_2 = \max \left\{ |\psi''(\lambda_{max}(v + \alpha d_s))|, |\psi''(\lambda_{min}(v + \alpha d_s))|, \frac{|\psi'(\lambda_{max}(v + \alpha d_s)) - \psi'(\lambda_{min}(v + \alpha d_s))|}{2\|(v + \alpha d_s)_{2:n}\|} \right\}.$$

It follows from (55) and (50) that

$$(59) \quad f'_1(0) = \frac{1}{2} \mathbf{Tr}(\nabla \Psi(v) \diamond (d_x + d_s)) = -\frac{1}{2} \mathbf{Tr}(\nabla \Psi(v) \diamond \nabla \Psi(v)) = -\|\nabla \Psi(v)\|^2 = -2\delta(v)^2 < 0,$$

which means that $f'_1(\alpha)$ is monotonically decreasing in a neighborhood of $\alpha = 0$.

Since the linear transformation \mathcal{A} has the Cartesian $P_*(\kappa)$ -property and $\mathcal{A}(\Delta x) = \Delta s$ from (44), we obtain

$$(60) \quad (1 + 4\kappa) \sum_{\nu \in J_+} \langle \Delta x^{(\nu)}, \Delta s^{(\nu)} \rangle + \sum_{\nu \in J_-} \langle \Delta x^{(\nu)}, \Delta s^{(\nu)} \rangle \geq 0,$$

where $J_+ = \{1 \leq \nu \leq N : \langle \Delta x^{(\nu)}, \Delta s^{(\nu)} \rangle > 0\}$ and $J_- = \{1 \leq \nu \leq N : \langle \Delta x^{(\nu)}, \Delta s^{(\nu)} \rangle < 0\}$ are two index sets.

It follows from (48) that

$$\langle d_x, d_s \rangle = \frac{\langle \Delta x, \Delta s \rangle}{\mu}.$$

Thus we can rewrite (60) as

$$(61) \quad (1 + 4\kappa) \sum_{\nu \in J_+} \langle d_x^{(\nu)}, d_s^{(\nu)} \rangle + \sum_{\nu \in J_-} \langle d_x^{(\nu)}, d_s^{(\nu)} \rangle \geq 0.$$

In order to facilitate discussion, we denote

$$(62) \quad \delta := \delta(v), \quad \delta_+ := \sum_{\nu \in J_+} \langle d_x^{(\nu)}, d_s^{(\nu)} \rangle, \quad \text{and} \quad \delta_- := - \sum_{\nu \in J_-} \langle d_x^{(\nu)}, d_s^{(\nu)} \rangle.$$

Lemma 4.3. *One has*

$$\|d_x\| \leq \sqrt{2(1 + 2\kappa)}\delta, \quad \text{and} \quad \|d_s\| \leq \sqrt{2(1 + 2\kappa)}\delta.$$

Proof. By (62), we have

$$\delta_+ = \sum_{\nu \in J_+} \langle d_x^{(\nu)}, d_s^{(\nu)} \rangle \leq \frac{1}{4} \sum_{\nu \in J_+} \left\| d_x^{(\nu)} + d_s^{(\nu)} \right\|^2 \leq \frac{1}{4} \sum_{\nu \in J} \left\| d_x^{(\nu)} + d_s^{(\nu)} \right\|^2 = \frac{1}{4} \|d_x + d_s\|^2 = \frac{1}{2} \delta^2.$$

It follows from (61) that

$$(1 + 4\kappa)\delta_+ - \delta_- \geq 0.$$

Then

$$\delta_- \leq (1 + 4\kappa)\delta_+ \leq \frac{1 + 4\kappa}{2} \delta^2.$$

Hence, we have

$$2\delta^2 = \|d_x + d_s\|^2 = \|d_x\|^2 + \|d_s\|^2 + 2(\delta_+ - \delta_-) \geq \|d_x\|^2 + \|d_s\|^2 - \frac{8k}{1 + 4k} \delta_-.$$

Thus

$$(63) \quad \|d_x\|^2 + \|d_s\|^2 \leq 2\delta^2 + \frac{8k}{1 + 4k} \delta_- \leq 2(1 + 2\kappa)\delta^2.$$

This implies the lemma.

Lemma 4.4. *One has*

$$f_1''(\alpha) \leq 2(1 + 2\kappa)\delta^2 \psi''(\lambda_{\min}(v) - 2\alpha\sqrt{1 + 2\kappa}\delta).$$

Proof. From Lemma 2.1 and Lemma 4.3, we obtain

$$\lambda_{\min}(v + \alpha d_x) \geq \lambda_{\min}(v) - \sqrt{2}\alpha \|d_x\| \geq \lambda_{\min}(v) - 2\alpha\sqrt{1 + 2\kappa}\delta,$$

$$\lambda_{\min}(v + \alpha d_s) \geq \lambda_{\min}(v) - \sqrt{2}\alpha \|d_s\| \geq \lambda_{\min}(v) - 2\alpha\sqrt{1 + 2\kappa}\delta.$$

Combining the choice of ω_1 as given by (57) and the mean value theorem, we can conclude that there exists a constant $\zeta_* \in [\lambda_{\min}(v + \alpha d_x), \lambda_{\max}(v + \alpha d_x)]$ satisfying

$$\omega_1 = |\psi''(\zeta_*)|.$$

Since $\psi''(t)$ is nonnegative and monotonically decreasing in $t \in (0, +\infty)$, we have

$$\omega_1 = \psi''(\zeta_*) \leq \psi''(\lambda_{\min}(v + \alpha d_x)) \leq \psi''(\lambda_{\min}(v) - 2\alpha\sqrt{1 + 2\kappa}\delta).$$

Similarly, we have

$$\omega_2 \leq \psi''(\lambda_{\min}(v) - 2\alpha\sqrt{1 + 2\kappa}\delta).$$

From the above consequences and (56), we have

$$f_1''(\alpha) \leq \frac{1}{2} \psi''(\lambda_{\min}(v) - 2\alpha\sqrt{1 + 2\kappa}\delta) (\|d_x\|^2 + \|d_s\|^2) \leq 2(1 + 2\kappa)\delta^2 \psi''(\lambda_{\min}(v) - 2\alpha\sqrt{1 + 2\kappa}\delta),$$

which completes the proof of the lemma.

The following strategy for choosing the step size is almost a “word-by-word” extension of LO [2] and $P_*(\kappa)$ -LCP [3]. We can easily verify that the following three lemmas are valid for the Cartesian $P_*(\kappa)$ -SOCLCP by just applying the corresponding lemmas in [3] to the vector x in \mathbf{R}^{2N} consisting of all the eigenvalues of the second-order cone x . Therefore, for the proofs of the following lemmas we refer to [2, 3].

Lemma 4.5 (Lemma 5.6 in [3]). *If the step size α satisfies*

$$(64) \quad -\psi'(\lambda_{\min}(v) - 2\alpha\sqrt{1+2\kappa}\delta) + \psi'(\lambda_{\min}(v)) \leq \frac{2\delta}{\sqrt{1+2\kappa}},$$

then $f_1(\alpha) \leq 0$.

Lemma 4.6 (Lemma 5.7 in [3]). *Let $\rho(s) : [0, \infty) \rightarrow (0, 1]$ be the inverse function of $-\frac{1}{2}\psi'(t)$ for $t \leq 1$. The largest possible value of the step size of α satisfying (64) is given by*

$$(65) \quad \bar{\alpha} = \frac{1}{2\sqrt{1+2\kappa}\delta} \left(\rho(\delta) - \rho \left(\left(1 + \frac{1}{\sqrt{1+2\kappa}} \right) \delta \right) \right).$$

Lemma 4.7 (Lemma 5.8 in [3]). *One has*

$$\bar{\alpha} \geq \frac{1}{(1+2\kappa)\psi'' \left(\rho \left(\left(1 + \frac{1}{\sqrt{1+2\kappa}} \right) \delta \right) \right)}.$$

From Lemma 4.7 and the definition of ρ , we have

$$(66) \quad \bar{\alpha} \geq \frac{1}{(1+2\kappa)(1+\sigma e^{\sigma(1-t)})}, \quad t \in \left[\frac{1}{\sigma}, 1 \right] \text{ where it is such that } e^{\sigma(1-t)} - t = 2 \left(1 + \frac{1}{\sqrt{1+2\kappa}} \right) \delta.$$

From the second equation of (66), we get

$$e^{\sigma(1-t)} = t + 2 \left(1 + \frac{1}{\sqrt{1+2\kappa}} \right) \delta \leq 1 + 2 \left(1 + \frac{1}{\sqrt{1+2\kappa}} \right) \delta \leq 1 + 4\delta.$$

It follows from Corollary 2.14 and $\Psi(v) \geq 1$ that

$$\delta \geq \frac{1}{6} \sqrt{\Psi(v)} \geq \frac{1}{6}.$$

Substituting this in the first equation of (66), we have

$$\bar{\alpha} \geq \frac{1}{\sigma(1+2\kappa)(1+e^{\sigma(1-t)})} \geq \frac{1}{2\sigma(1+2\kappa)(1+2\delta)} \geq \frac{1}{16\sigma\delta(1+2\kappa)}.$$

In what follows we use the notation

$$(67) \quad \tilde{\alpha} = \frac{1}{16\sigma\delta(1+2\kappa)}.$$

And we will use $\tilde{\alpha}$ as the default step size. It is obvious that $\bar{\alpha} \geq \tilde{\alpha}$.

Now, to validate the above analysis we need to show that $\tilde{\alpha}$ satisfies (2.8). In fact, from Lemmas 2.1, 2.8 and 4.3, we have

$$\lambda_{\min}(v^{(j)} + \tilde{\alpha}d_x^{(j)}) \geq \lambda_{\min}(v^{(j)}) - \sqrt{2}\tilde{\alpha}\|d_x^{(j)}\| \geq \frac{3}{2\sigma} - \frac{1}{8\sigma\sqrt{1+2\kappa}} \geq \frac{11}{8\sigma} \geq \frac{1}{\sigma}, \quad j \in J,$$

and

$$\lambda_{\min}(v^{(j)} + \tilde{\alpha}d_s^{(j)}) \geq \lambda_{\min}(v^{(j)}) - \sqrt{2}\tilde{\alpha}\|d_s^{(j)}\| \geq \frac{3}{2\sigma} - \frac{1}{8\sigma\sqrt{1+2\kappa}} \geq \frac{11}{8\sigma} \geq \frac{1}{\sigma}, \quad j \in J.$$

Lemma 4.8 (Lemma 12 in [16]). *Let $h(t)$ be a twice differentiable convex function with $h(0) = 0$, $h'(0) < 0$ and let $h(t)$ attain its (global) minimum at $t^* > 0$. If $h''(t)$ is increasing for $t \in [0, t^*]$, then*

$$h(t) \leq \frac{th'(0)}{2}, \quad 0 \leq t \leq t^*.$$

From Lemma 4.8, we have the following lemma, which gives an upper bound for the decreasing value of the barrier function $\Psi(v)$ in each inner iteration.

Lemma 4.9 (Lemma 5.10 in [3]). *If the step size α is such that $\alpha \leq \bar{\alpha}$, then*

$$f(\alpha) \leq -\alpha\delta^2.$$

Theorem 4.10. *With $\tilde{\alpha}$ being the default step size as given by (67), one has*

$$f(\tilde{\alpha}) \leq -\frac{\sqrt{\Psi(v)}}{96(1+2\kappa)(1+\sigma)}.$$

Proof. It follows from Lemma 4.9 and (67) that

$$f(\tilde{\alpha}) \leq -\tilde{\alpha}\delta^2 = -\frac{\delta}{16\sigma(1+2\kappa)}.$$

It is obvious that the last expression above is monotonically decreasing in δ . Thus, by Corollary 2.14, after some elementary reductions, we obtain

$$f(\tilde{\alpha}) \leq -\frac{\sqrt{\Psi(v)}}{96\sigma(1+2\kappa)}.$$

This proves the theorem.

5. COMPLEXITY OF THE ALGORITHM

5.1. Iteration bound for large-update method. For the complexity of the algorithm, we need to count how many inner iterations are required to return to the situation where $\Psi(v) \leq \tau$. We use the value of $\Psi(v)$ after the μ -update by Ψ_0 , the subsequent values in the same outer iteration are denoted as Ψ_k , $k = 1, 2, \dots, K$, where K denotes the total number of inner iterations in the outer iteration.

According to the decrease of $f(\tilde{\alpha})$, we obtain

$$(68) \quad \Psi_{k+1} \leq \Psi_k - \beta(\Psi_k)^{1-\gamma}, \quad k = 0, 1, \dots, K,$$

where $\beta = \frac{1}{96\sigma(1+2\kappa)}$, and $\gamma = \frac{1}{2}$.

Lemma 5.1. (Lemma 14 in [16]) *Suppose t_0, t_1, \dots, t_K be a sequence of positive numbers such that*

$$t_{k+1} \leq t_k - \beta t_k^{1-\gamma}, \quad k = 0, 1, \dots, K-1,$$

where $\beta > 0$ and $0 < \gamma \leq 1$. Then $K \leq \left\lceil \frac{t_0^\gamma}{\beta\gamma} \right\rceil$.

The following theorem gives an upper bound for the number of inner iterations produced by the algorithm presented in Figure 1.

Theorem 5.2. *One has*

$$K \leq 192\sigma(1+2\kappa)\sqrt{\Psi_0}.$$

Proof. The theorem follows immediately from Lemma 5.1 and (68).

The number of outer iterations is bounded above by $\frac{1}{\theta} \log \frac{N}{\varepsilon}$ (cf. Lemma II.17 in [17]). By multiplying the number of outer iterations and the number of inner iterations we get an upper bound for the total number of iterations, namely,

$$\frac{192\sigma(1+2\kappa)}{\theta} \sqrt{\frac{N}{1-\theta} \left(\frac{\tau}{N} + \frac{2+\sigma}{\sigma} \right)} \log \frac{N}{\varepsilon}.$$

Due to (53), (38), and $\psi(t) \leq \frac{t^2-1}{2}$ when $t \geq 1$, we have

$$\Psi_0 \leq L = 2N\psi \left(\frac{\varrho\left(\frac{\tau}{2N}\right)}{\sqrt{1-\theta}} \right) \leq 2N\psi \left(\frac{\sqrt{\frac{\tau}{N} + \frac{2+\sigma}{\sigma}}}{\sqrt{1-\theta}} \right) \leq \frac{N}{1-\theta} \left(\frac{\tau}{N} + \frac{2+\sigma}{\sigma} \right).$$

From the above expression with $\theta = \Theta(1)$ and $\tau = O(N)$, and also applying Lemma 2.8, one can conclude that $\sigma = O(\log N)$.

After some elementary reductions, we have the following theorem, which yields the iteration bound for large-update method.

Theorem 5.3. *For large-update method, one takes $\theta = \Theta(1)$ and $\tau = O(N)$, then the algorithm requires at most*

$$O \left((1+2\kappa)\sqrt{N} \log N \log \frac{N}{\varepsilon} \right).$$

iterations. The output gives an ε -approximate solution of the Cartesian $P_(\kappa)$ -SOCLCP.*

5.2. Iteration bound for small-update method. It is not hard to show that if the above analysis is used for small-update method the iteration bound would not be as good as it can be for these types of methods. For the analysis of the iteration bound of small-update method, we need to estimate the upper bound of Ψ_0 more accurately. It should be noted that the following analysis only holds for $\sigma \geq 2$.

From (53), (39), Lemma 2.9 and $1 - \sqrt{1-\theta} = \frac{\theta}{1+\sqrt{1-\theta}} \leq \theta$, we have

$$\begin{aligned} \Psi_0 \leq L &= 2N\psi \left(\frac{\varrho\left(\frac{\tau}{2N}\right)}{\sqrt{1-\theta}} \right) \leq 2N\psi \left(\frac{1 + \sqrt{\frac{\tau}{2N} \left(\frac{\tau}{N} + \frac{2+\sigma}{\sigma} \right)^{\frac{1}{4}}}}{\sqrt{1-\theta}} \right) \\ &\leq N(1+\sigma) \left(\frac{1 + \sqrt{\frac{\tau}{2N} \left(\frac{\tau}{N} + \frac{2+\sigma}{\sigma} \right)^{\frac{1}{4}}}}{\sqrt{1-\theta}} - 1 \right)^2 \leq \frac{1+\sigma}{1-\theta} \left(\theta\sqrt{N} + \sqrt{\frac{\tau}{2} \left(\frac{\tau}{N} + \frac{2+\sigma}{\sigma} \right)^{\frac{1}{4}}} \right)^2. \end{aligned}$$

From the above expression with $\theta = \Theta\left(\frac{1}{\sqrt{N}}\right)$ and $\tau = O(1)$, and also applying Lemma 2.8, we obtain $\sigma = O(1)$. Using Theorem 5.2, we can conclude that the total number of iterations is bounded above by

$$\frac{192\sigma\sqrt{1+\sigma}(1+2\kappa)}{\theta\sqrt{1-\theta}} \left(\theta\sqrt{N} + \sqrt{\frac{\tau}{2} \left(\frac{\tau}{N} + \frac{2+\sigma}{\sigma} \right)^{\frac{1}{4}}} \right) \log \frac{N}{\varepsilon},$$

which gives the iteration bound for small-update method.

Theorem 5.4. *For small-update method, namely, $\theta = \Theta\left(\frac{1}{\sqrt{N}}\right)$ and $\tau = O(1)$, then the algorithm requires at most*

$$O\left((1 + 2\kappa)\sqrt{N} \log \frac{N}{\varepsilon}\right).$$

iterations. The output gives an ε -approximate solution of the Cartesian $P_(\kappa)$ -SOCLCP.*

6. CONCLUSIONS AND REMARKS

In this paper we have generalized a new primal-dual interior-point algorithm for LO based on the finite kernel function to the Cartesian $P_*(\kappa)$ -SOCLCP. The currently best known iteration bounds for large- and small-update methods are obtained, namely, $O((1 + 2\kappa)\sqrt{N} \log N \log \frac{N}{\varepsilon})$ and $O((1 + 2\kappa)\sqrt{N} \log \frac{N}{\varepsilon})$, respectively. Moreover, the resulting analysis is simple and straightforward to the $P_*(\kappa)$ -LCP analogue.

Some interesting topics for further research remain. Firstly, the search directions used in this paper are based on the NT-scaling scheme. It may be possible to design similar algorithms using other scaling schemes and still obtain polynomial-time iteration bounds. Secondly, the extensions to the $P_*(\kappa)$ linear complementarity problem over symmetric cones deserve to be investigated. Finally, the numerical test is an interesting work for investigating the behavior of the algorithm so as to be compared with other existing approaches.

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