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Complementarity Problems and General Equilibrium

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Abstract

A general equilibrium technique is used to show the existence of a solution for a class of complementarity or nonlinear complementarity problems which involves a function admitting a copositive extension.

Keywords Complementarity problem, general equilibrium, copositivity.

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1 Introduction

The† paper proposes a new method to study complementarity problems (CP) and nonlinear complementarity problems (NCP) and states existence results for a class of such problems. The CP(K, F) consists of finding a vector x in a given cone $K \subset R^n$ such that its image by a given function F belongs to the dual cone and is orthogonal to x. When $K = R^n_+$, the NCP(F) = CP(R^n_+, F) amounts to finding a pair $(x, F(x))$ of nonnegative and orthogonal vectors. Our existence argument is non algorithmic but relies on the search of a fixed point which can be implemented by means of an algorithm.

Research on the bimatrix game is at the origin of algorithmic methods [6, 7] which are now widely used to find solutions of linear and nonlinear complementarity problems. Many of them start from nonnegative variables and proceed to make them orthogonal [2]. One may imagine an alternative approach which starts from orthogonal variables and adjusts them in order that they become nonnegative. The problem is then similar to that of the existence of a general equilibrium in economics: the excess supply vector is always orthogonal to the price vector (Walras identity) and an equilibrium is reached when the excess supply is nonnegative. Since the existence result follows from the Gale-Nikaido-Debreu Lemma [3, 5, 8], the parallel suggests that it is possible to adapt it to complementarity problems. Section 3 proposes a way to implement the idea and states the basic result for the CP(K, F). Section 4 examines some applications and explains why more precise results are obtained for the NCP(F).

2 Notations

R_+ is the set of nonnegative scalars, R_{++} that of positive scalars. Notation $x \geq 0$ means that the column vector x is nonnegative, $x > 0$ that it is nonnegative and nonzero, and x^T denotes its transpose. $x^T y$ is the inner product of vectors x and y. Given a set $C \subset R^m$, a function $f : C \to R^m$ is copositive on C if $x^T f(x) \geq 0$ for any $x \in C$, and strictly copositive if the inequality is strict for any nonzero vector $x \in C$. The dual of a cone $C \subset R^m$ is the cone $C^* = \left\{x; \forall y \in C \;\; x^T y \geq 0\right\} \subset R^m$.

In this paper, K is a closed, convex and pointed cone in R^n, $F : K \to R^n$ is a given continuous function and we consider the CP(K, F)

$$x \in K, F(x) \in K^*, x^T F(x) = 0 \tag{1}$$

K admits a compact and convex basis S, i.e. S is the intersection of K with an adequate affine hyperplane and every nonzero vector x in K is written in a

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unique way as $x = \lambda x_0$, with $\lambda > 0$ and $x_0 \in S$. λ is denoted by $\|x\|$. With that definition, S is the subset of K made of vectors x with $\|x\| = 1$. Let Σ be the subset of R^{n+1} defined by $\Sigma = \{(x,t); x \in K, t \in [0,1], \|x\| + t = 1\}$, and Σ_+ be the subset of Σ with a positive last component t.

When K is the positive orthant of R^n, the CP(K,F) reduces to the NCP(F)

$$x \geq 0, F(x) \geq 0, x^T F(x) = 0 \tag{2}$$

If the chosen basis S is the unit simplex of R^n, $\|x\|$ is the norm $\|x\| = \sum_i |x_i|$ and Σ is the unit simplex of R^{n+1}.

3 Main result

Lemma 1 *(GND Lemma). Let C be a compact and convex set in R^m and $s : p \in C \to s = s(p) \in R^m$ a continuous function satisfying the Walras inequality: $\forall p \in C \quad p^T s(p) \geq 0$. There exists $\overline{p} \in C$ such that $\overline{s} = s(\overline{p})$ belongs to the dual cone C^*.*

Proof. The image of the set C by s is included in a convex compact set C'. Let us consider the set-valued mapping φ from C' to C defined by $\varphi(x) = \left\{p; p \in C \quad p^T x = \min_{y \in C} y^T x\right\}$. The correspondence $s \times \varphi$ from $C \times C'$ into $C' \times C$ being upper semicontinuous with compact convex values, the existence of a fixed point $(\overline{p}, \overline{s})$ follows from the Kakutani theorem. Then $\overline{s} = s(\overline{p})$ and $\min_{y \in C} y^T \overline{s} = \varphi(\overline{s})^T \overline{s} = \overline{p}^T s(\overline{p}) \geq 0$. ■

In general equilibrium theory, the usual version of the Lemma assumes that C is the unit simplex of R^m, vector p is interpreted as a normalized price vector for m commodities, and $s(p)$ is the corresponding excess supply vector (or set-valued mapping, in a more general framework). By construction of $s(p)$, the Walras identity $p^T s(p) = 0$ holds. The conclusion of the Lemma is that $\overline{s}$ is nonnegative. In connection with equality $\overline{p}^T \overline{s} = 0$, that condition characterizes a general equilibrium. The excess supply function s is also homogenous of degree zero ('absence of monetary illusion'), but that important economic property is inessential for our purpose.

In order to find solutions to the CP(K,F), we introduce an additional scalar t to enforce the Walras identity. It is then possible to apply the Lemma for $m = n+1$, a strategy used in [1] for the study of prices of production. It is assumed that F obeys the basic assumption (H) below, which is made of two parts. Part (i) associates a function Φ with F and substitutes the search of a critical point of Φ for that of a solution to (1): a simple illustration of the condition is given by the functions defined on Σ_+ by $\Phi(x,t) = F(t^{-1}x) \in R^n$ and $E(x,t) = t^{-1}x \in K$. Part (ii) assumes that some function derived from Φ is bounded: that condition refers indirectly to the behavior of F at infinity.

(H) Assume the existence of continuous functions $\Phi : \Sigma_+ \to R^n$ and $e : \Sigma_+ \to R_{++}$ such that:

(i) $\left\{(x,t) \in \Sigma_+, \Phi(x,t) \in K^*, x^T\Phi(x,t) = 0\right\} \Rightarrow E(x,t) = e(x,t)x$ is a solution to the CP(K,F).

(ii) The function $t\Phi(x,t)$ is bounded on Σ_+.

For $x \in S$, let $l(x) \subset R^n$ be the set made of all the cluster points of $t_\varepsilon\Phi(x_\varepsilon,t_\varepsilon)$ when $(x_\varepsilon,t_\varepsilon) \in \Sigma_+$ tends to $(x,0) \in \Sigma$, and let L be the set

$$L = \left\{x{\in}S; \exists l \in l(x), l \in K^*, x^T l = 0\right\} \tag{3}$$

Theorem 1 *Let $F : K \to R^n$ be a continuous function satisfying assumption (H). If the following two conditions hold:*

$$\begin{aligned} -\infty &< \inf_{(x,t)\in\Sigma_+} x^T\Phi(x,t) && (4)\\ x_* &\in L \Rightarrow 0 < \liminf_{(x,t)\in\Sigma_+\to(x_*,0)} x^T\Phi(x,t) && (5)\end{aligned}$$

the CP(K,F) has a solution.

Proof. For $\varepsilon > 0$, let Σ_ε be the subset of Σ made of the vectors (x,t) with $t \geq \varepsilon > 0$. Since the continuous function defined on Σ_ε by $s(x,t) = (t\Phi(x,t), -x^T\Phi(x,t))$ satisfies the Walras identity $(x,t)^T s(x,t) = 0$, the GND Lemma asserts the existence of a point $(x_\varepsilon,t_\varepsilon) \in \Sigma_\varepsilon$ such that:

$$\forall (x,t) \in \Sigma_\varepsilon \quad x^T t_\varepsilon\Phi(x_\varepsilon,t_\varepsilon) - t x_\varepsilon^T\Phi(x_\varepsilon,t_\varepsilon) \geq 0 \tag{6}$$

Let ε tend to zero. Assume first that the sequence $(x_\varepsilon,t_\varepsilon)$ admits a cluster point $(\overline{x},\overline{t})$ with $\overline{t} > 0$. Inequality (6) still holds at the limit, when $(x_\varepsilon,t_\varepsilon)$ is replaced by $(\overline{x},\overline{t})$ and $(x,t) \in \Sigma_\varepsilon$ by any point $(x,t) \in \Sigma$:

$$\forall (x,t) \in \Sigma \quad x^T\overline{t}\Phi(\overline{x},\overline{t}) - t\overline{x}^T\Phi(\overline{x},\overline{t}) \geq 0$$

For $t = 0$, this shows that $\Phi(\overline{x},\overline{t}) \in K^*$; for $(x = 0, t = 1)$, one obtains $\overline{x}^T\Phi(\overline{x},\overline{t}) \leq 0$, therefore $\overline{x}^T\Phi(\overline{x},\overline{t}) = 0$. According to assumption (H), $E(\overline{x},\overline{t})$ is a solution to the CP(K,F).

Assume on the contrary that $\lim t_\varepsilon = 0$. By assumption (H), the function $t\Phi(x,t)$ is bounded; by condition (4), the function $x^T\Phi(x,t)$ is lower bounded and, according to inequality (6) applied at point ($x = 0$, $t = 1$), negative or zero at point $(x_\varepsilon,t_\varepsilon)$. Therefore there exists a subsequence $(x_\varepsilon,t_\varepsilon)$ which tends to $(x_*,0){\in}\Sigma$ and such that $\lim t_\varepsilon\Phi(x_\varepsilon,t_\varepsilon) = l \in l(x_*)$ and $\lim x_\varepsilon^T\Phi(x_\varepsilon,t_\varepsilon) = \lambda \leq 0$. Going to the limit in relation (6), we obtain that

$$\forall (x,t) \in \Sigma_+ \quad x^T l + t(-\lambda) \geq 0$$

The inequality still holds at $t = 0$, therefore $l \in K^*$. Moreover, we have

$$x_*^T l = (\lim x_\varepsilon^T)(\lim t_\varepsilon\Phi(x_\varepsilon,t_\varepsilon)) = (\lim t_\varepsilon)(\lim x_\varepsilon^T\Phi(x_\varepsilon,t_\varepsilon)) = 0$$

To sum up, x_* belongs to L and the sequence $(x_\varepsilon, t_\varepsilon)$ violates condition (5). This case is therefore excluded. ■

Condition (4) imposes a significant restriction on function Φ: assume that $x^T l = -\alpha < 0$ for some $x \in S$ and some $l \in l(x)$. Then $x_t^T t\Phi(x_t, t) < -\alpha/2$ for a sequence $(x_t, t) \in \Sigma_+$ with $t > 0$ small enough, and condition (4) is violated. Therefore $x^T l \geq 0$ for any $x \in S$ and any $l \in l(x)$. In other words, the extension l of function $t\Phi$ to the frontier $t = 0$ is copositive on S:

$$x \in S \Rightarrow x^T l(x) \geq 0 \tag{7}$$

4 Applications

We have already given an example of a pair of functions Φ and E satisfying the condition (i) of assumption (H). A more general example is $\Phi(x, t) = d(x, t)F(e(x, t)x)$, where the real functions d and e are defined on Σ_+, continuous and positive. Thanks to the leeway provided by these parameters, condition (ii) is not that restrictive. Of course, the extension $l(x)$ of $t\Phi(x, t)$, the definition of L and conditions (4) and (5) depend on the choice of functions d and e.

In the following applications we stress the role of the cone K. In order to ease comparisons between results, the functions F we consider are all of the same type: F is the sum of two functions m and q such that m is copositive and has some type of homogeneity property (conditions (i) and (ii) of Corollary 1 below) and $q(x)$ becomes negligible at infinity with regard to $m(x)$ (condition (iii) of Corollary 1 covers the case $m(x) = 0$ and is a more precise expression of the idea). For that reason, and for an adequate choice of Φ and e, the limit set $l(x)$ in Corollaries 1 and 2 is reduced to the point $m(x)$ and the set L defined by (3) becomes L'

$$L' = \left\{x \in S; m(x) \in K^*, x^T m(x) = 0\right\} \tag{8}$$

Corollaries 1 and 3 deal with different cones, whereas Corollary 2 allows for a comparison with a classical result.

Corollary 1 *Let $K \subset R^n$ be a pointed cone with a compact convex basis and let $F : K \to R^n$ be written $F = m + q$, where the continuous functions m and q are such that:*

(i) m is copositive;

(ii) For $x > 0$ we have $m(x) = g(x)m(\frac{x}{\|x\|})$, where the continuous function $g : R_+^n \to R_+$ is positive for $x > 0$ and $g(0) = 0$;

(iii) $\lim\limits_{\|x\| \to \infty} g(x)^{-1} q(x) = 0$;

(iv) $-\infty < \inf\limits_{\|x\|>1} g(x)^{-1} x^T q(x)$;

(v) $x_* \in L' \Rightarrow 0 < \liminf\limits_{\|x\| \to \infty, x/\|x\| \to x_*} g(x)^{-1} x^T q(x)$.

Then the CP(K, F) has a solution.

Proof. For $(x,t) \in \Sigma_+$ and $0 < t < 0.5$, let us set $d(x,t) = t^{-1}g(t^{-1}x)^{-1}$ and $e(x,t) = t^{-1}$. Both functions admit a continuous and positive extension to Σ_+.[‡] For $0 < t < 0.5$ we have $t\Phi(x,t) = g(t^{-1}x)^{-1}m(t^{-1}x) + g(t^{-1}x)^{-1}q(t^{-1}x) = m(\frac{x}{\|x\|}) + g(t^{-1}x)^{-1}q(t^{-1}x)$. When (x,t) tends to $(x_*,0)$, condition (iii) implies that $t\Phi(x,t)$ tends to $m(x_*)$. Since m is copositive, we have

$$\begin{aligned} x^T\Phi(x,t) &= t^{-1}x^T m(\frac{x}{\|x\|}) + t^{-1}g(t^{-1}x)^{-1}x^T q(t^{-1}x) \\ &\geq g(t^{-1}x)^{-1}(t^{-1}x)^T q(t^{-1}x) \end{aligned}$$

therefore assumptions (iv) and (v) of the Corollary ensure that conditions (4) and (5) of Theorem 1 are met. Hence, the existence result. ■

Condition (ii) of Corollary 1 holds if m is homogenous of degree k ($k > 0$), with $g(x) = \|x\|^k$; when function q is bounded, condition (iii) holds if $\lim_{x\to\infty} g(x) = +\infty$ and condition (iv) if $g(x)^{-1}\|x\|$ is bounded for $\|x\| > 1$. To illustrate Corollary 1 and show its connection with classical results, let us consider a very specific case:

Corollary 2 *Let K be a pointed cone with a compact convex basis and let $F(x) = m(x) + q$, where m is continuous, copositive on K and homogenous of degree one, and q is a given vector. If*

$$x \in L' \Rightarrow x^T q > 0 \tag{9}$$

the CP(K,F) has a solution.

Proof. In this case we have $g(x) = \|x\|$ and the result follows immediately from Corollary 1 and the compactness of L'. ■

Condition (9) can be compared with a well known result relative to the linear complementarity problem LCP(q,M): when $K = R^n_+$ and $m(x) = Mx$ with M copositive matrix, that result ensures the existence of a solution under the weaker condition that $x \in L'$ implies $x^T q \geq 0$ (Theorem 3.8.6 in [2]). With a nonlinear function m, however, that condition must be reinforced: let $n = 2$ and $K = R^2_+$, and consider the NCP(q,m) associated with $F(x) = m(x) + q$, with $q = (0,-1)^T$ and m defined by $m(0) = 0$ and $m(x_1,x_2) = \frac{x_2}{x_1+x_2}(-x_2,x_1)^T$ for $x > 0$. As $\{x \geq 0, m(x) \geq 0\}$ requires $x_2 = 0$, the condition $x \in L'$ does imply $x^T q \geq 0$. However, the NCP(q,m) has no solution since there is no $x \geq 0$ such that $m(x) + q \geq 0$.

When $K = R^n_+$, the specific structure of the cone R^n_+ allows for a more precise application of Theorem 1 and more results relative to the NCP(F). Let us first state a property which can be viewed as a nonlinear version of a Theorem of the Alternative:

[‡] The construction avoids considering the quantity $g(t^{-1}x)^{-1}$ for $(x = 0, t = 1)$.

Lemma 2 *Let C be a compact metric space and consider m continuous functions $f_i : C \to R^n$. Exactly one of the following two properties holds:*

- either there is some $x \in C$ with $f_i(x) \leq 0$ for $i = 1, ..., m$,

- or there exist m continuous and positive functions $\delta_i : C \to R_{++}$ such that $\sum_i \delta_i f_i$ is positive on C.

Proof. It is clear that the second condition excludes the first. Conversely, assume that the first condition does not hold. Given $\varepsilon > 0$, we define the set

$$C_{i\varepsilon} = \{x; f_i(x) \leq \varepsilon\}$$

Without loss of generality, $C_{i\varepsilon}$ is non void, otherwise function f_i would be greater than ε and the second condition would hold by choosing $\delta_i(x) = 1$ for that function and $\delta_j(x)$ positive and small enough for the others. Since $C_{i\varepsilon}$ is compact and $\underset{\varepsilon}{\cap} C_{i\varepsilon} = \{x; f_i(x) \leq 0\}$, the assumption that $\underset{\varepsilon,i}{\cap} C_{i\varepsilon}$ is void implies the existence of some $\varepsilon > 0$ such that $\underset{i}{\cap} C_{i\varepsilon}$ is void. Let us choose such a value of ε. For any $x \in C$, we consider the distance $d(x, C_{i\varepsilon})$ of x to $C_{i\varepsilon}$. Since the continuous function $\max_i d(x, C_{i\varepsilon})$ does not reach value 0 on C, it admits a positive minimum α: we thus have found values $\varepsilon > 0$ and α such that, for any $x \in C$, we have $d(x, C_{i\varepsilon}) \geq \alpha$ for some $C_{i\varepsilon}$. For a given scalar $\delta > 0$, let the function f_i^δ be defined on C for $i = 1, ..., m$ by

$$\begin{array}{rcll} f_i^\delta(x) & = & f_i(x) & \text{if } x \in C_{i\varepsilon} \\ f_i^\delta(x) & = & \varepsilon + \delta d(x, C_{i\varepsilon}) & \text{if } x \notin C_{i\varepsilon} \end{array}$$

Function f_i^δ is continuous (its value is ε on the boundary of $C_{i\varepsilon}$) and, for any $x \in C$, we have $\sum_i f_i^\delta(x) \geq \varepsilon + \delta\alpha + (m-1)\min_i f_i$. Let us choose a great enough value of δ, such that the right-hand side of that expression is positive. Let the real function δ_i be defined on C for $i = 1, ..., m$ by

$$\begin{array}{rcll} \delta_i(x) & = & 1 & \text{if } x \in C_{i\varepsilon} \\ \delta_i(x) & = & f_i^\delta(x)/f_i(x) & \text{if } x \notin C_{i\varepsilon} \end{array}$$

Function δ_i is positive and continuous (its value is 1 on the boundary of $C_{i\varepsilon}$). As $\delta_i f_i = f_i^\delta$, the function $\sum_i \delta_i f_i = \sum_i f_i^\delta$ is positive on C. ■

The proof of the following statement makes use of the fact that, when the cone K is the positive orthant, condition (i) of assumption (H) is met by the function $\Phi(x,t) = D(x,t)F(t^{-1}x)$ where $D(x,t)$ is a diagonal matrix with positive diagonal entries.

Corollary 3 *Let $F : R_+^n \to R^n$ admit a decomposition $F = m + q$ where the continuous functions m and q are such that:*

(i) $\forall x \in S \;\; \exists i \;\; x_i m_i(x) > 0$,

(ii) For $x > 0$ we have $m_i(x) = g_i(x)m_i(\frac{x}{\|x\|})$, where the continuous function $g_i : R^n_+ \to R_+$ is positive for $x > 0$ and $g_i(0) = 0$,

(iii) $\lim\limits_{\|x\|\to\infty} g_i(x)^{-1}q_i(x) = 0$.

Then the NCP(F) has a solution.

Proof. According to condition (i) and Lemma 2 applied to the compact set S and the n functions $x_i m_i(x)$, there exist n continuous functions δ_i such that $\sum\limits_i x_i\delta_i(x)m_i(x) > 0$ for any $x \in S$. Thus the function $\Delta_0 m : S \to R^n$, where Δ_0 is the matrix with entries $\delta_i(x)$ on the diagonal, is strictly copositive. Let us write $\delta_i(x, 0)$ for $\delta_i(x)$. $\delta_i(x, 0)$ admits a continuous and positive extension $\delta_i(x, t)$ to Σ_{low}, i.e. to the subset of Σ corresponding to $0 \leq t \leq 0.5$. For $(x, t) \in \Sigma_{low}$, let us choose $d_i(x, t) = \delta_i(x, t)[tg_i(t^{-1}x)]^{-1} > 0$ and $e(x, t) = t^{-1}$. Then

$$\begin{aligned} td_i(x,t)m_i(t^{-1}x) &= \delta_i(x,t)[g_i(t^{-1}x)]^{-1}[g_i(t^{-1}x)m_i(\frac{t^{-1}x}{\|t^{-1}x\|})] \\ &= \delta_i(x,t)m_i(\frac{x}{\|x\|}) \\ td_i(x,t)q_i(t^{-1}x) &= \delta_i(x,t)[g_i(t^{-1}x)]^{-1}q_i(t^{-1}x) \end{aligned}$$

Let $D(x, t)$ be the diagonal matrix with the positive diagonal entries $d_i(x, t)$. $D(x, t)$, which is defined for $(x, t) \in \Sigma_{low}$, can be extended by continuity to the whole simplex Σ, as a matrix with the same characteristics. From the above calculations and hypothesis (iii) of the Corollary, it turns out that $t\Phi(x, t) = tD(x, t)F(t^{-1}x)$ tends to $\Delta_0 m(x_*)$ when $(x, t) \in \Sigma_+$ tends to $(x_*, 0)$. As $\Delta_0 m$ is strictly copositive, the set L is void and condition (5) holds. Moreover, since the continuous function $tx^T\Phi(x, t)$ admits the positive extension $x^T\Delta_0 m(x)$ for $t = 0$, it is positive on a range $0 \leq t \leq \varepsilon$, therefore the function $x^T\Phi(x, t)$ admits a positive lower bound on Σ_ε and a finite lower bound on Σ_+: condition (4) holds and Theorem 1 applies. ■

Condition (i) implies that function m_i is positive on the ray generated by the ith unit vector and condition (ii) that $m_i(x)$ holds the same sign along a ray (the origin excepted). These properties do not prevent $m_i(x)$ from being negative on some rays: for instance, Corollary 3 applies to $n = 2$ and $F(x) = (x_1 - x_2 + q_1(x), 2x_2^2 - x_1^2 + q_2(x))$, where $q_1(x)/\|x\|$ and $q_2(x)/\|x\|^2$ tend to zero when $\|x\|$ tends to infinity.

This example points at a significant difference between Corollary 1, which deals with a general cone, and Corollary 3, which assumes $K = R^n_+$: condition (ii) in Corollary 1 holds for a homogenous function, whereas its more general counterpart in Corollary 3 allows for a componentwise homogenous function. Returning to the initial condition (H), it turns out that a difference between the CPs and the NCPs stems from the existence, in the second case, of a large family of transformations $D : R^n \to R^n$ (the class of diagonal matrices with positive and continuous diagonal entries) such that the NCP(F) and the NCP(DF) have the same solutions. This suggests that the identification of a class of transformations with similar properties for a given cone K may be a clue to the existence of a solution to the CP(K, F).

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