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A Smoothing Newton Method for Nonlinear Complementarity Problems *

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Abstract. In this paper, we propose a smoothing function to the nonlinear complementarity problem. The function possesses the Jacobian consistency property. Based on this function, we develop a smoothing Newton method for solving the nonlinear complementarity problem. Under appropriate conditions, we establish the global and superlinear convergence of the method. We also show that when applied to solve a linear complementarity problem, the method terminates at a solution after finitely many iterations.

Key words. nonlinear complementarity problem, smoothing function, smoothing Newton method

1 Introduction

Suppose $F : R^n \to R^n$ is continuously differentiable. We consider the nonlinear complementarity problem, NCP(F) for short, which is to find a vector $x \in R^n$ satisfying

$$x \geq 0, \quad F(x) \geq 0, \quad \text{and} \quad x^T F(x) = 0, \tag{1.1}$$

A popular way to solve the NCP(F) is to reformulate (1.1) to a nonsmooth equation

$$H(x) = 0 \tag{1.2}$$

via an NCP-function $\phi : R^2 \to R$:

$$\phi(a, b) = 0 \Leftrightarrow a \geq 0,\ b \geq 0,\ ab = 0.$$

One approach to solve the nonsmooth equation (1.2) is to use the so called smoothing methods. The feature of a smoothing method is to construct a smoothing approximation function $G : R^n \times R_{++} \to R^n$ of H such that for any $\epsilon > 0$ and $x \in R^n, G(\cdot, \epsilon)$ is continuously differentiable on R^n and satisfies

$$\|H(z) - G(z, \epsilon)\| \to 0, \quad \text{as} \quad \epsilon \downarrow 0,\ z \to x.$$

We then try to find a solution of (1.2) by solving approximately a sequence of smoothing equations

$$G_{\epsilon^k}(x^k) = 0, \tag{1.3}$$

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given positive sequence $\{\epsilon^k\}$. The merits of the smoothing method are global convergence and convenience in handling smooth functions instead of nonsmooth functions. However, the smooth equation (1.3), which needs to be solved at each step, is nonlinear in general.

The smoothing Newton method can be regarded as a variant of the smoothing method. At the kth step, the nonsmooth function H is approximated by a smooth function $G(\cdot, \epsilon)$, and the derivative of $G(\cdot, \epsilon)$ at x^k is used as the Newton iterative matrix. This kind of smoothing method has been extensively studied in the last two decades (see e.g., [3, 4, 5, 7, 11, 12, 13, 15, 17, 21, 22, 26, 31, 34, 35, 36] etc.).

The Jacobian consistency property introduce by Chen, Qi and Sun in [5] plays an important role for a smoothing Newton method to be globally and superlinearly convergent. It has been proved in [5] that many existing smoothing functions such as those in [3, 4, 7, 12, 22] have this property. Consequently, the related smoothing Newton methods are globally and superlinearly convergent. In particular, the method in [7] combined a finite termination strategy with Jacobian smoothing Newton method to get finitely convergent results for box constrained linear variational inequality problems. There are also smoothing functions that do not satisfy the Jacobian consistency property. In this case, a class of the so-called squared smoothing Newton methods were introduced in [26], which was then used in [19, 28, 37] for solving regularized reformulations of complementarity problems and variational inequality problems [10, 27, 30]. In general, in a smoothing Newton method, it is often assumed that the iteration matrices are nonsingular. In order to remove off the assumption, Qi and Sun [24] introduced an alternative step involving orthogonal projection operator. Recently there have been developed variant some one-step smoothing Newton methods for P_0-NCP, which needs only to solve one linear system of equations and perform one line search per iteration, see for [11, 31, 34, 35, 36] etc..

In this paper, we will develop a smoothing Newton method based on the one in [7]. We use the nonsmooth equation reformulation (1.2) of (1.1) whose elements are given by

$$H_i(x) = \min\{x_i, F_i(x)\} = \frac{1}{2}(x_i + F_i(x) - |x_i(x) - F_i(x)|), \quad i = 1, \ldots, n. \tag{1.4}$$

Chen and Mangasarian [3] introduced a family of smoothing functions of "min" function, which unified the smoothing functions studied in [4, 8, 14, 29, 32, 33]. Chen, Qi and Sun [5] proved that the Chen-Mangasarian family have Jacobian consisitency property. Gabriel and Moré [12] extended Chen-Mangasarian's smoothing approach to box constrained variational inequalities. The smoothing Newton methods based on the class of smoothing functions by Chen-Mangasarian and Gabriel-Moré can be found in [2, 5, 7] etc..

Quite recently, Li, Wu and Zhang [16] proposed a new smoothing function that is totally different from the existing smoothing function. They first constructed a smoothing function to the generalized derivative of the absolute value function and then get a smoothing function to the absolute value function. This smoothing function enjoys some nice properties. The purpose of this paper is to propose a smoothing Newton method based on the smoothing function in [16]. Under appropriate conditions, we get the global and superlinear convergence of the method.

In the next section, we will review some interesting properties of the smoothing function proposed in [16]. We then derive a smoothing function to the nonsmooth equation reformulation to the nonlinear complementarity problem and propose a smoothing Newton method to solve the nonsmooth equation reformulation in Section 3. In Section 4, we show the global and superlinear/quadratic convergence of the method. We will also show in Section 4 that when applied to solve a linear complementarity problem, the method can terminate at the solution of the problem within finitely many iterations.

2 Properties of a Smoothing Approximation Function to the Absolute Value Function

In this section, we recall some good properties of a smoothing approximation function to the absolute value function proposed by Li, Wu and Zhang [16]. Let

$$\phi_\epsilon(t) = \frac{2}{\pi}\arctan\left(\frac{t}{\epsilon}\right). \tag{2.1}$$

It is clear that

$$\lim_{\epsilon\to 0^+}\phi_\epsilon(t) = \begin{cases} 1, & \text{if } t>0,\\ 0, & \text{if } t=0,\\ -1, & \text{if } t<0. \end{cases}$$

On the other hand, it is easy to see that the generalized derivative of the absolution function $|t|$ is given by

$$\text{sign }(t) \triangleq \begin{cases} 1, & \text{if } \ t>0,\\ [-1,1], & \text{if } \ t=0,\\ -1, & \text{if } \ t<0. \end{cases} \tag{2.2}$$

It is clear that the function ϕ_ϵ is an approximation to the generalized derivative of the absolute value function. More precisely, we have the following proposition.

Proposition 2.1 *For any given constant $\alpha>0$, there is a constant $M_\alpha>0$ independent of ϵ and x such that*

$$0\le \text{sign}(x)-\phi_\epsilon(x)\le M_\alpha\epsilon, \quad \forall x:\ x\ge\alpha, \quad \forall\epsilon>0, \tag{2.3}$$

and

$$0\le \phi_\epsilon(x)-\text{sign}(x)\le M_\alpha\epsilon, \quad \forall x:\ x\le-\alpha, \quad \forall\epsilon>0. \tag{2.4}$$

Proof We only prove (2.3). The inequalities in (2.4) can be proved in a similar way.

We first show that there are constant $\bar\epsilon>0$ and $\bar M_\alpha>0$ such that

$$0\le 1-\phi_\epsilon(x)\le \bar M_\alpha\epsilon, \quad \forall x:\ x\ge\alpha, \quad \forall\epsilon\in(0,\bar\epsilon]. \tag{2.5}$$

Indeed, we have by the monotonicity of $\phi_\epsilon(x)$

$$0\le 1-\phi_\epsilon(x)\le 1-\phi_\epsilon(\alpha) = 1-\frac{2}{\pi}\arctan\left(\frac{\alpha}{\epsilon}\right). \tag{2.6}$$

By an elementary deduction, we get

$$\lim_{\epsilon\to 0^+}\frac{1-\frac{2}{\pi}\arctan\left(\frac{\alpha}{\epsilon}\right)}{\epsilon} = \lim_{\epsilon\to 0^+}\frac{2}{\pi}\frac{\alpha/\epsilon^2}{1+(\alpha/\epsilon)^2} = \frac{2}{\pi}\alpha^{-1}.$$

Letting $\bar{M}_\alpha = \dfrac{4}{\pi}\alpha^{-1}$, the last inequality yields (2.5) with some constant $\bar{\epsilon}$.

For $\epsilon > \bar{\epsilon}$, it is obvious that the inequality (2.3) holds with $M_\alpha = \bar{\epsilon}^{-1}$. □

The last proposition has shown that function ϕ_ϵ is an approximation to the generalized derivative of the absolute value function. Taking integral to function ϕ_ϵ, we can get the approximation to the absolute value function. Let

$$\psi_{1,\,\epsilon}(t) = \int \phi_\epsilon(t)dt = \frac{2}{\pi}\int \arctan(\frac{t}{\epsilon})dt = t\phi_\epsilon(t) - \frac{1}{\pi}\epsilon\ln(1+\frac{t^2}{\epsilon^2}), \tag{2.7}$$

$$\psi_{2,\,\epsilon}(t) = \frac{2}{\pi}t\arctan(\frac{t}{\epsilon}) = t\phi_\epsilon(t). \tag{2.8}$$

Function $\psi_{1,\,\epsilon}$ is obtained by integrating function ϕ_ϵ while function $\psi_{2,\,\epsilon}$ is obtained by removing the term $-\frac{1}{\pi}\epsilon\ln(1+\frac{t^2}{\epsilon^2})$ in $\psi_{1,\,\epsilon}$. It is clear that for any $\epsilon > 0$, $\psi_{1,\,\epsilon}(t)$ and $\psi_{2,\,\epsilon}(t)$ are continuously differentiable.

In what follows, we derive the relations between functions $\psi_{1,\,\epsilon}(t)$ and $\psi_{2,\,\epsilon}(t)$. Let constants α and β satisfy $\beta > \alpha > 0$. Then we have for any $t \in [\alpha,\,\beta] \cup [-\beta,\,-\alpha]$ and any $\epsilon > 0$

$$0 < \psi_{2,\,\epsilon}(t) - \psi_{1,\,\epsilon}(t) = \frac{1}{\pi}\epsilon\ln(1+\frac{t^2}{\epsilon^2}) \le \frac{1}{\pi}\ln(1+\frac{\beta^2}{\epsilon^2})\epsilon. \tag{2.9}$$

We also have

$$|\psi'_{2,\,\epsilon}(t) - \psi'_{1,\,\epsilon}(t)| = \frac{2\epsilon}{\pi}\frac{|t|}{1+t^2/\epsilon^2} \le \frac{2\beta}{\pi}\epsilon. \tag{2.10}$$

The following proposition shows that both functions $\psi_{1,\,\epsilon}(t)$ and $\psi_{2,\,\epsilon}(t)$ are smoothing approximation functions to $|t|$.

Proposition 2.2 *Let functions $\psi_{1,\epsilon}$ and $\psi_{2,\epsilon}$ be defined by (2.7) and (2.8) respectively. Then the following statements hold.*

(i) For any $\epsilon > 0$, it holds that

$$\psi_{1,\,\epsilon}(t) \le |t|, \quad \forall t.$$

The inequality holds strictly for all $t \neq 0$.

(ii) For given constants $\beta \ge \alpha > 0$, there is a constant $M_{\alpha\beta} > 0$ such that

$$0 < |t| - \psi_{1,\,\epsilon}(t) \le M_{\alpha\beta}\epsilon, \quad \forall t \in [\alpha, \beta] \cup [-\beta, -\alpha].$$

(iii) For given constants $\beta \ge \alpha > 0$, there is a constant $T_{\alpha\beta} > 0$ such that

$$0 < |t| - \psi_{2,\,\epsilon}(t) \le T_{\alpha\beta}\epsilon, \quad \forall t \in [\alpha, \beta] \cup [-\beta, -\alpha]$$

hold.

(iv) Denote $h(t) = |t|$. Then the following relations hold for any t,

$$\lim_{\epsilon\downarrow 0}\mathrm{dist}(\psi'_{1,\,\epsilon}(t), \partial h(t)) = 0,$$

$$\lim_{\epsilon\downarrow 0}\mathrm{dist}(\psi'_{2,\,\epsilon}(t), \partial h(t)) = 0,$$

where $\partial h(t) = \mathrm{sign}(t)$ is the generalized derivative of $h(t) = |t|$ given by (2.2) and $\mathrm{dist}(v, S)$ denotes the distance from the point v to the set S.

Proof (i) Denote $\eta_\epsilon(t) = |t| - \psi_{1,\epsilon}(t)$. Then we have $\eta_\epsilon(0) = 0$. It is easy to get $\eta'_\epsilon(t) > 0$, $\forall t > 0$; and $\eta'_\epsilon(t) < 0$, $\forall t < 0$. This means that $\eta_\epsilon(t)$ attains its minimum at $t = 0$, and the conclusion in statement 1 holds.

(ii) For any $t > 0$, we have

$$0 < \eta_\epsilon(t) = t(1 - \phi_\epsilon(t)) + \frac{1}{\pi}\epsilon \ln(1 + \frac{t^2}{\epsilon^2}).$$

It follows from Proposition 2.1 that there is a constant $M_{\alpha\beta} > 0$ such that $0 < \eta_\epsilon(t) \le M_{\alpha\beta}\epsilon$ holds for any $t \in [\alpha, \beta]$. Since $\eta_\epsilon(t) = \eta_\epsilon(-t)$ for any $t \in R$, we can also get $0 < \eta_\epsilon(t) \le M_{\alpha\beta}\epsilon$ for any $t \in [-\beta, -\alpha]$.

Conclusion (iii) can be obtained by conclusion (i) and inequality (2.9) while conclusion (iv) can be obtained by Proposition 2.1 and (2.10). □

3 A Smoothing Newton Method for NCP(F)

The concept of semismooth function was introduced by Mifflin [18] and extended by Qi and Sun [23]. The following definition is due to Qi and Sun [23].

Definition 3.1 *We say that function $G : R^n \to R^n$ is semismooth at a point $x \in R^n$ if it is locally Lipschitzian at x and*

$$\lim_{V \in \partial F(x+th'), h' \to h, t \downarrow 0} Vh'$$

exists for any $h \in R^n$, where $\partial F(x)$ is the generalized Jacobian of F at x. It is said to be strongly semismooth at $x \in R^n$ if for any $d \to 0$ and any $H \in \partial G(x + d)$,

$$Hd - G'(x; d) = O(\|d\|^2),$$

where $G'(x; d)$ denotes the directional derivative of G at x along direction d.

In Section 2, we introduced two smooth functions $\psi_{1,\epsilon}(t)$ and $\psi_{2,\epsilon}(t)$. As shown in Proposition 2.2, both functions are approximations to the absolute value function $|t|$. The purpose of this section is to develop a smoothing Newton method for solving a nonsmooth equation reformulation to the NCP(F). To this end, we first construct the smoothing function $G(x, \epsilon)$ of $H(x)$ defined by (1.2) whose elements are given by (1.4) as follows. For given $\epsilon > 0$, we let $G(x, \epsilon) : R^n \to R^n$ with elements $G_i(x, \epsilon)$, $i = 1, 2, \ldots, n$ defined by

$$\begin{aligned} G_i(x, \epsilon) &= \frac{1}{2}(x_i + F_i(x) - \psi_\epsilon(x_i - F_i(x))) \\ &= \frac{1}{2}\Big[x_i + F_i(x) - (x_i - F_i(x)) \arctan\Big(\frac{x_i - F_i(x)}{\epsilon}\Big) + \frac{1}{\pi}\epsilon \ln\Big(1 + \frac{(x_i - F_i(x))^2}{\epsilon^2}\Big)\Big], \end{aligned} \tag{3.1}$$

where ψ_ϵ is the abbreviation of $\psi_{1,\epsilon}$. By simple calculation, we get

$$G'_x(x, \epsilon) = \frac{1}{2}(I + F'(x) - D_1(I - F'(x))),$$

where $I \in R^{n \times n}$ stands for the identity matrix and

$$D_1 = \frac{2}{\pi}\text{diag}\ (\arctan\Big(\frac{x_1 - F_1(x)}{\epsilon}\Big), \cdots, \arctan\Big(\frac{x_n - F_n(x)}{\epsilon}\Big)).$$

Taking limits in the expression of $G'_x(x,\epsilon)$ as $\epsilon \to 0^+$, we get

$$G^0(x) = \lim_{\epsilon \downarrow 0} G'_x(x,\epsilon) = \frac{1}{2}(I + F'(x) - D_2(I - F'(x))),$$

where

$$D_2 = \text{diag}\ (\text{sgn}\ (x_1 - F_1(x)), \cdots, \text{sgn}\ (x_n - F_n(x))),$$

Here without confliction, we abuse the notation sgn(t) to stand the one dimensional real-valued function whose value is 1 if $t > 0$, -1 if $t < 0$, and 0 if $t = 0$. It is a little different from the function defined by (2.2). It is not difficult to see from Proposition 2.2 that the function $G(x,\epsilon)$ satisfies the Jacobian consistency property.

The following lemma gives some regularity conditions. It can be proved by the use of Theorems 4.2 and 4.3 in [12].

Lemma 3.2 *If F is a P_0 function, then derivative of $G(x,\epsilon)$ on x, denoted by $G'_x(x,\epsilon)$, is nonsingular for any $x \in R^n$ and $\epsilon > 0$. If F is a uniform P-function, then for any $x \in R^n$, matrix $G^0(x)$ is nonsingular.*

Now, we are going to develop the smoothing Newton method. The method is very similar to the one in [7]. The major difference between our method and the method in [7] lies in the use of the smoothing functions. The smoothing function in [7] is constructed based on the Gabriel-Moré smoothing function while our smoothing function is obtained on the basis of the smoothing function (3.1).

Define two functions as follows.

$$\theta_k(x) = \frac{1}{2}\|G(x,\epsilon^k)\|^2,$$

and

$$\theta(x) = \frac{1}{2}\|H(x)\|^2.$$

The steps of our smoothing Newton method are stated below.

Algorithm 1

Step 0 Given $\rho, \alpha, \eta \in (0,1)$, $\sigma \in (0, \frac{1}{2}(1-\alpha))$. Given a starting point $x^0 \in R^n$. Let $\beta_0 = \|H(x^0)\|$. Choose $\mu > 0$ and $\epsilon^0 = \frac{\alpha}{2\mu}\beta_0$ satisfying $\|H(x^0) - G(x^0,\epsilon^0)\| \leq \mu\epsilon^0$. Set $k := 0$.

Step 1 Solve the system of linear equations

$$H(x^k) + G^0(x^k)d = 0 \tag{3.2}$$

to get $\hat{d}^k$. If $\|H(x^k + \hat{d}^k)\| \leq \eta\beta_k$, then we let $x^{k+1} = x^k + \hat{d}^k$ and go to Step 3.1. Otherwise perform Step 2.

Step 2 Solve the system of linear equations

$$H(x^k) + G'_x(x^k,\epsilon^k)d = 0 \tag{3.3}$$

to get d^k.

Let m_k be the smallest nonnegative integer m such that

$$\theta_k(x^k + \rho^m d^k) - \theta_k(x^k) \leq -2\sigma\rho^m\theta(x^k).$$

Set $t_k = \rho^{m_k}$ and $x^{k+1} = x^k + t_k d^k$.

Step 3.1 If $\|H(x^{k+1})\| = 0$, terminate.

Step 3.2 If $\|H(x^{k+1})\| > 0$ and

$$\|H(x^{k+1})\| \le \max\{\eta\beta_k, \alpha^{-1}\|H(x^{k+1}) - G(x^{k+1}, \epsilon^k)\|\}, \tag{3.4}$$

we let

$$\beta_{k+1} = \|H(x^{k+1})\|$$

and choose an ϵ^{k+1} satisfying

$$0 < \epsilon^{k+1} = \min\{\frac{\alpha}{2\mu}\beta_{k+1}, \frac{\epsilon^k}{2}\}. \tag{3.5}$$

Step 3.3 Otherwise, we set $\beta_{k+1} := \beta_k$ and $\epsilon^{k+1} := \epsilon^k$.

Step 3.4 Let $k := k+1$. Go to Step 1.

Remark It follows from Lemma 3.2 that when F is a uniform P-function, matrix $G^0(x)$ is nonsingular. It is not difficult to show from Lemma 3.1 in [5] and Lemma 3.2 above that if F is a P_0 function, the line search step is well defined. Consequently, Algorithm 1 is well defined.

The remainder of this paper is to show the global and superlinear convergence of Algorithm 1. We will also show that the method will terminate within finitely many iterations when applied to solve the linear complementarity problem LCP(M, q)

$$F(x) = Mx + q \ge 0, \quad x \ge 0, \quad F(x)^T x = 0.$$

To show the global and superlinear convergence of Algorithm 1, we need the following assumption.

Assumption 1. The level set

$$\Omega = \{x \in R^n : \theta(x) \le (1+\alpha)^2\theta(x^0)\}$$

is bounded.

Assumption 2. For any $\epsilon \in R_{++} = \{\epsilon : \epsilon > 0, \epsilon \in R\}$ and every $x \in \Omega$, matrix $G'_x(x, \epsilon)$ is nonsingular.

The conditions that guarantee Assumption 1 to hold can be found in [2, 7, 27].

In a way similar to the proof of Theorem 3.1 of [7], we can establish the following global convergence theorem of Algorithm 1.

Theorem 3.3 *Suppose that Assumptions 1 and 2 hold. Then Algorithm 1 is well defined and the generated sequence* $\{x^k\}$ *remains in* Ω *and satisfies*

$$\lim_{k\to\infty} H(x^k) = 0.$$

In particular, every accumulation point of $\{x^k\}$ *is a solution to the nonlinear complementarity problem (1.1).*

The following lemma shows that when applied to solve the linear complementarity problem LCP(M, q), if an iterate is sufficiently close to a solution of the problem, then Algorithm 1 finds the solution in one step. Define

$$\gamma(x) = \min_{1\le i\le n}\{|x - Mx - q|_i : (x - Mx - q)_i \ne 0\}.$$

Lemma 3.4 *Let x^* be a solution of the LCP(M, q). Let*

$$S = \begin{cases} \{x \in R^n : \|x - x^*\|_\infty \le \gamma(x^*)/\|I - M\|_\infty\}, & \text{if} \quad M \ne I, \\ R^n, & \text{otherwise.} \end{cases} \tag{3.6}$$

Then for any $x \in S$, it holds that

$$H(x) + G^0(x)(x^* - x) = 0. \tag{3.7}$$

Proof If $M = I$, then we have $G^0(x) = I$ for any $x \in R^n$. Moreover, $x^* = \max(0, -q)$ is a solution of $H(x) = 0$. In this case, we have for any $x \in R^n$

$$H(x) + G^0(x)(x^* - x) = x - \max(0, -q) + x^* - x = 0.$$

Consider the case where $M \ne I$ and $x \in S$, we have

$$|x - Mx - q - x^* + Mx^* + q|_i \le \|I - M\|_\infty \|x - x^*\|_\infty \le \gamma(x^*). \tag{3.8}$$

For a fixed $i \in \{1, 2, \ldots, n\}$, we consider two cases.

Case 1. $(x^* - Mx^* - q)_i \ne 0$.

By the definition of $\gamma(x)$, we get from (3.8)

$$(x^* - Mx^* - q)_i < 0 \Rightarrow (x - Mx - q)_i < 0 \tag{3.9}$$

and

$$(x^* - Mx^* - q)_i > 0 \Rightarrow (x - Mx - q)_i > 0. \tag{3.10}$$

If $(x^* - Mx^* - q)_i > 0$, then $(Mx^* + q)_i = 0$. By (3.10), we get $G_i^0(x) = M_i$, where for an $n \times n$ matrix, A_i stand for its $i-$th row, and

$$H_i(x) + G_i^0(x)(x^* - x) = x_i - (x - Mx - q)_i + M_i(x^* - x) = (Mx^* + q)_i = 0. \tag{3.11}$$

If $(x^* - Mx^* - q)_i < 0$, then $x_i^* = 0$. It follows from (3.9) that $G_i^0(x) = I_i$ and

$$H_i(x) + G_i^0(x)(x^* - x) = x_i + I_i(x^* - x) = x_i^* = 0. \tag{3.12}$$

Case 2. $(x^* - Mx^* - q)_i = 0$.

If $(x^* - Mx^* - q)_i = 0$, then $x_i^* = 0$ and $(Mx^* + q)_i = 0$. If $(x - Mx - q)_i > 0$ or $(x - Mx - q)_i < 0$, following the same argument as the proofs of (3.11) and (3.12), it is not difficult to show $H_i(x) + G_i^0(x)(x^* - x) = 0$. If $(x - Mx - q)_i = 0$, then $G^0(x) = \frac{1}{2}(I + M)$ and

$$H_i(x) + G_i^0(x)(x^* - x) = x_i + \frac{1}{2}(I + M)_i(x^* - x) = \frac{1}{2}(Mx^* + q)_i = 0.$$

The proof is complete. □

The following theorem shows the superlinear/quadratic convergence of Algorithm 1.

Theorem 3.5 *Let $\{x^k\}$ be generated by Algorithm 1. Suppose that Assumptions 1 and 2 hold. Suppose further that for an accumulation point x^* of $\{x^k\}$, there are an open ball $\mathcal{S} \triangleq S(x^*, r^*) = \{x : \|x - x^*\| < r^*\}$ and a positive number τ such that for any $x \in \mathcal{S}$, $G^0(x)$ is nonsingular and $\|G^0(x)^{-1}\| \le \tau$. Then the sequence $\{x^k\}$ converges to x^* superlinearly. Moreover, if F has a locally Lipschitz continuous derivative around x^*, then the convergence rate of $\{x^k\}$ is quadratic. In addition, if F is an affine function, the convergence is finite.*

Proof Let x^* be an accumulation point of $\{x^k\}$. Then Theorem 3.3 implies that x^* satisfies $H(x^*) = 0$. Moreover, it follows from Lemma 2.4 in [1] that x^* is the unique solution of $H(x) = 0$ in S. Let $\{x^k\}_{K_*}$ be a subsequence of $\{x^k\}$ converging to x^*, i.e.,

$$\lim_{k\to\infty, k\in K_*} x^k = x^*.$$

Since for any $x \in R^n, G^0(x) \in \partial_C H(x)$ by Lemma 2.2 in [1], we obtain

$$\begin{aligned} \|x^k + \hat{d}^k - x^*\| &= \|x^k - G^0(x^k)^{-1}H(x^k) - x^*\| \\ &= \|G^0(x^k)^{-1}(G^0(x^k)(x^k - x^*) - H(x^k) + H(x^*))\| \\ &\le \tau\|H(x^k) - H(x^*) - G^0(x^k)(x^k - x^*)\| \\ &= o(\|x^k - x^*\|). \end{aligned} \tag{3.13}$$

Similar to the proof of Theorem 3.1 in [20], we can derive

$$\|H(x^k + \hat{d}^k)\| = o(\|H(x^k)\|).$$

This implies that there is $k_* \in K_*$ such that the following inequality holds for all $k \ge k_*$

$$\|H(x^k + \hat{d}^k)\| \le \eta\|H(x^k)\|.$$

By Step 1 of Algorithm 1, we have $x^{k+1} = x^k + \hat{d}^k$. This shows that $\{x^k\}$ converges to x^* superlinearly. If F has a locally Lipschitz continuous derivative around x^*, then H is strongly semismooth at x^*. Since G satisfies the Jacobian consistency property and $G^0(x) \in \partial_C H(x)$ for any $x \in R^n$, we have

$$\|H(x^k) - H(x^*) - G^0(x^k)(x^k - x^*)\| = O(\|x^k - x^*\|^2).$$

In a way similar to the proof of (3.13), it is not difficult to show the quadratic convergence of $\{x^k\}$.

The finite convergence of $\{x^k\}$ in the linear case follows from Lemma 3.4 and that $k \in K_*$ for all $k \ge k_*$. $\square$ **Acknowledgement.** The authors would like to thank the reviewers in the revised paper for their helpful comments.

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