AMO - Advanced Modeling and Optimization, Volume 13, Number 2, 2011

Convergence of Iterative Sequences for Strict Pseudocontractions and Inverse-Strongly Monotone Mappings

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Abstract. In this work, the problem of finding a common element of the fixed point set of a strict pseudocontraction and the solution set of variational inequalities is investigated. As applications, the problem of approximating a common fixed point of three strict pseudocontractions is considered in a real Hilbert space.

Keywords: variational inequality; fixed point; nonexpansive mapping; strict pseudocontraction.

AMS Subject Classification: 47H05, 47H09, 47J25

1. Introduction and Preliminaries

Variational inequalities theory, which was introduced in early sixties, has witnessed an explosive growth in thoretical advances, algorithmic development and applications across all the discipline of pure and applied sciences. It is well known that the classical variational inequality is equivalent to a fixed point problem. This alternative equivalent formulation has played a major role in variational inequalities. In particular, the solution of the variational inequalities can be computed using iterative algorithms; see [2-5,7,9,13,15]. Indeed, many well known problems arising in various branches of science can be studied by using algorithms which are iterative in their nature. As an example, in computer tomography with limited data, each piece of information implies the existence of a convex set C_m in which the required solution lies. The problem of finding a point in the intersection $\cap_{m=1}^{N} C_m$ is then of crucial interest but cannot be usually solved directly. Therefore, an iterative algorithm must be used to approximate such a point.

In this paper, we consider the problem of finding a common element of the solution set of the classical variational inequality and in the fixed point set of strict pseudocontractions. The results presented in this paper improve the corresponding results announced in Iiduka and Takahashi [4] and Yao and Yao [15].

Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$. Let C be a nonempty, closed and convex subset of H and $A : C \to H$ a mapping.

Recall that the classical variational inequality is to find a point $u \in C$ such that

$$\langle Au, v - u \rangle \geq 0, \quad \forall v \in C. \tag{1.1}$$

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AMO - Advanced Modeling and Optimization. ISSN: 1841-4311

In this paper, we use $VI(C,A)$ to denote the solution set of the classical variational inequality (1.1).

Recall that the mapping A is said to be monotone if

$$\langle Ax - Ay, x - y\rangle \geq 0, \quad \forall x, y \in C.$$

A is said to be strongly monotone if there exists a positive constant α such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|x - y\|^2, \quad \forall x, y \in C.$$

In this case, A is also called an α-strongly monotone mapping. A is said to be inverse-strongly monotone if there exists a positive constant α such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2, \quad \forall x, y \in C.$$

In this case, A is also called an α-inverse-strongly monotone mapping.

Recall also that a set-valued mapping $R : H \to 2^H$ is called monotone if for all $x, y \in H$, $f \in Rx$ and $g \in Ry$ imply $\langle x - y, f - g\rangle \geq 0$. A monotone mapping $R : H \to 2^H$ is maximal if $G(R)$, the graph of R, is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping R is maximal if and only if for $(x, f) \in H \times H$, $\langle x - y, f - g\rangle \geq 0$ for every $(y, g) \in G(R)$ implies $f \in Rx$. Let A be a monotone mapping of C into H and let $N_C v$ be the normal cone to C at $v \in C$, i.e., $N_C v = \{w \in H : \langle v - u, w\rangle \geq 0,\ \forall u \in C\}$ and define

$$Rv = \begin{cases} Av + N_C v, & v \in C \\ \emptyset, & v \notin C. \end{cases}$$

Then R is maximal monotone and $0 \in Rv$ if and only if $v \in VI(C,A)$; see [11] for more details.

Let $T : C \to C$ be a mapping. We denote by $F(T)$ the fixed point set of T. Recall that T is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

T is said to be strictly pseudocontractive if there exits a constant $k \in [0,1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C.$$

In this case, T is also said to be a k-strict pseudocontraction. The class of strict pseudocontractions was introduced by Browder and Petryshyn in 1967; see [1] for more details.

Recently, many authors studied the problem of finding a common element of the solutions set of a variational inequality for an inverse-strongly monotone mapping and the fixed point set of a nonexpansive mapping; see, for example, [2-5,7-10,13,15] and the references therein. Iiduka and Takahashi [4] proved the following theorem.

Theorem IT. *Let C be a closed convex subset of a real Hilbert space H. Let A be an α-inverse-strongly monotone mapping of C into H and let S be a nonexpansive mapping of C into itself such that $F(S) \cap VI(C,A) \neq \emptyset$. Suppose $x_1 = x \in C$ and $\{x_n\}$ is given by*

$$x_{n+1} = \alpha_n x + (1 - \alpha_n)SP_C(x_n - \lambda_n Ax_n)$$

for every $n = 1, 2, \ldots,$ *where* $\{\alpha_n\}$ *is a sequence in* $[0, 1)$ *and* $\{\lambda_n\}$ *is a sequence in* $[a, b]$*. If* $\{\alpha_n\}$ *and* $\{\lambda_n\}$ *satisfy that* $\{\lambda_n\} \in [a, b]$ *for some* a, b *with* $0 < a < b < 2\alpha$*,*

$$\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty, \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty \text{ and } \sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty,$$

then $\{x_n\}$ *converges strongly to* $P_{F(S)\cap VI(C,A)}x$*.*

Recently, Y. Yao and J.C. Yao [15] further considered the problem by introducing a general iterative method. To be more precise, they proved the following theorem.

Theorem YY. *Let* C *be a closed convex subset of a real Hilbert space* H*. Let* A *be an* α*-inverse-strongly monotone mapping of* C *into* H *and let* S *be a nonexpansive mapping of* C *into itself such that* $F(S) \cap \Omega \neq \emptyset$*, where* Ω *denotes the set of solutions of a variational inequality for the* α*-inverse-strongly monotone mapping. Suppose* $x_1 = u \in C$ *and* $\{x_n\}$*,* $\{y_n\}$ *are given by*

$$\begin{cases} x_1 = u \in C, \\ y_n = P_C(x_n - \lambda_n A x_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n S P_C(I - \lambda_n A) y_n, \quad n \geq 1, \end{cases}$$

where $\{\alpha_n\}$*,* $\{\beta_n\}$*,* $\{\gamma_n\}$ *are three sequences in* $[0, 1]$ *and* $\{\lambda\}$ *is a sequence in* $[0, 2a]$*. If* $\{\alpha_n\}$*,* $\{\beta_n\}$*,* $\{\gamma_n\}$ *and* $\{\lambda_n\}$ *satisfy that* $\lambda \in [a, b]$ *for some* a, b *with* $0 < a < b < 2a$ *and*

(1) $\alpha_n + \beta_n + \gamma_n = 1, \quad \forall n \geq 1;$

(2) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty;$

(3) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1;$

(4) $\lim_{n\to\infty}(\lambda_{n+1} - \lambda_n) = 0,$

then $\{x_n\}$ *converges strongly to* $P_{F(S)\cap\Omega}u$*.*

In this paper, motivated by the above results, we consider the class of strict pseudocontractions and the class of inverse-strongly monotone mappings. More precisely, we introduce a new iterative process to find a common element of the fixed point set of a strict pseudocontraction and the solution set of the variational inequality in a real Hilbert space. The results obtained in this paper improve and extend the recent ones announced in Iiduka and Takahashi [4], Yao and Yao [15] and many others.

In order to prove our main results, we need the following lemmas.

Lemma 1.1 (Suzuki [12]). *Let* $\{x_n\}$ *and* $\{y_n\}$ *be bounded sequences in a Banach space* E *and let* $\{\beta_n\}$ *be a sequence in* $[0, 1]$ *with* $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$*. Suppose* $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$ *for all integers* $n \geq 0$ *and*

$$\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$*.*

Lemma 1.2 (Xu [14]). *Assume that* $\{\alpha_n\}$ *is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \leq (1 - \gamma_n)\alpha_n + \delta_n,$$

where $\{\gamma_n\}$ *is a sequence in* $(0,1)$ *and* $\{\delta_n\}$ *is a sequence such that*

(1) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(2) $\limsup_{n\to\infty} \delta_n/\gamma_n \le 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} \alpha_n = 0$.

Lemma 1.3 (Zhou [16]). *Let* C *be a nonempty closed convex subset of a real Hilbert space* H *and* $T : C \to C$ *a* λ*-strict pseudo-contraction with a fixed point. Define* $S : C \to C$ *by* $Sx = \alpha x + (1-\alpha)Tx$ *for each* $x \in C$*. Then, as* $\alpha \in [\lambda, 1)$*,* S *is nonexpansive such that* $F(S) = F(T)$*.*

Lemma 1.4 (Marino and Xu [6]). *Let* C *be a nonempty closed and convex subset of a Hilbert space* H *and* $T : C \to C$ *be a* k*-strict pseudocontraction. Then*

(1) T *is* $\frac{1+k}{1-k}$*-Lipschitz continuous;*

(2) *the mapping* $I - T$ *is demiclosed at zero; that is, if* $\{x_n\}$ *is a sequence in* C *such that* $x_n \rightharpoonup x$ *and* $(I-T)x_n \to 0$*, then* $(I-T)x = 0$.

2. Main results

Theorem 2.1. *Let* H *be a real Hilbert space,* C *a nonempty closed convex subset of* H*. Let* $A : C \to H$ *be an* α*-inverse-strongly monotone mapping,* $B : C \to H$ *a* β*-inverse-strongly monotone mapping and* $T : C \to C$ *a* k*-strict pseudocontraction. Assume that* $\mathcal{F} = F(T) \cap VI(C,A) \cap VI(C,B) \neq \emptyset$*. Let* $\{x_n\}$ *be a sequence generated by the following algorithm:*

$$\begin{cases} x_0 \in C, \\ z_n = \eta_n x_n + (1-\eta_n) P_C(x_n - \rho_n A x_n), \\ y_n = \delta_n P_C(z_n - \lambda_n B z_n) + (1-\delta_n) T P_C(z_n - \lambda_n B z_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n y_n, \quad n \ge 0, \end{cases} \tag{Υ}$$

where u *is fixed element in* C*,* $\{\alpha_n\}$*,* $\{\beta_n\}$*,* $\{\gamma_n\}$*,* $\{\delta_n\}$ *and* $\{\eta_n\}$ *are sequences in* $(0,1)$ *and* $\{\lambda_n\}$ *and* $\{\rho_n\}$ *are positive sequences. Assume that the above control sequences satisfy the following restrictions:*

(a) $\alpha_n + \beta_n + \gamma_n = 1$, $\forall n \ge 0$;

(b) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(c) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$;

(d) $\lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = \lim_{n\to\infty} |\delta_{n+1} - \delta_n| = \lim_{n\to\infty} |\rho_{n+1} - \rho_n| = 0$;

(e) $\{\lambda_n\}, \{\rho_n\} \in [a,b]$ *for some* a, b *with* $0 < a < b < \min\{2\alpha, 2\beta\}$*;*

(f) $\lim_{n\to\infty} \eta_n = 0$ *and* $k \le \delta_n \le e < 1$*, where* $e \in [k,1)$ *is some constant.*

Then the sequence $\{x_n\}$ converges strongly to $\bar{x} \in \mathcal{F}$, where $\bar{x} = P_{\mathcal{F}}u$.

Proof. Put $A_n = P_C(I - \rho_n A)$ and $B_n = P_C(I - \lambda_n B)$ for each $n \geq 0$. Next, we show that A_n and B_n are nonexpansive for each $n \geq 0$. Indeed, for any $x, y \in C$, we obtain from the condition (e) that

$$\begin{aligned}\|A_nx - A_ny\|^2 &\leq \|(I - \rho_n A)x - (I - \rho_n A)y\|^2\\ &= \|(x - y) - \rho_n(Ax - Ay)\|^2\\ &= \|x - y\|^2 - 2\rho_n\langle Ax - Ay, x - y\rangle + \rho_n^2\|Ax - Ay\|^2\\ &\leq \|x - y\|^2 - \rho_n(2\alpha - \rho_n)\|Ax - Ay\|^2\\ &\leq \|x - y\|^2,\end{aligned}$$

which implies the mapping A_n is nonexpansive for each $n \geq 0$. Similarly, we can show that B_n is also nonexpansive, for each $n \geq 0$.

Next, we show that the sequence $\{x_n\}$ is bounded. Letting $p \in \mathcal{F}$, we see that

$$p = Tp = A_np = B_np, \quad \forall n \geq 0.$$

Note that

$$\begin{aligned}\|z_n - p\| &= \|\eta_n(x_n - p) + (1 - \eta_n)(A_nx_n - p)\|\\ &\leq \eta_n\|x_n - p\| + (1 - \eta_n)\|x_n - p\|\\ &= \|x_n - p\|.\end{aligned} \tag{2.1}$$

Put $U_n = \delta_n I + (1 - \delta_n)T$ for each $n \geq 0$. From Lemma 1.3, we see that U_n is nonexpansive and $F(T) = F(U_n)$ for each $n \geq 0$. It follows that

$$\|y_n - p\| = \|U_nB_nz_n - p\| \leq \|B_nz_n - p\| \leq \|z_n - p\|,$$

which combines with (2.1) gives that

$$\|y_n - p\| \leq \|x_n - p\|. \tag{2.2}$$

In view of (2.1) and (2.2), we arrive at

$$\begin{aligned}\|x_{n+1} - p\| &= \|\alpha_n(u - p) + \beta_n(x_n - p) + \gamma_n(y_n - p)\|\\ &\leq \alpha_n\|u - p\| + \beta_n\|x_n - p\| + \gamma_n\|y_n - p\|\\ &\leq \alpha_n\|u - p\| + (1 - \alpha_n)\|x_n - p\|\\ &\leq \max\{\|x_n - p\|, \|u - p\|\}.\end{aligned}$$

By simple inductions, we have

$$\|x_n - p\| \leq \max\{\|x_0 - p\|, \|u - p\|\},$$

which gives that the sequence $\{x_n\}$ is bounded. Note that

$$\begin{aligned}&\|A_nx_n - A_{n+1}x_{n+1}\|\\ &\leq \|(I - \rho_n A)x_n - (I - \rho_{n+1}A)x_{n+1}\|\\ &= \|(x_n - \rho_n Ax_n) - (x_{n+1} - \rho_n Ax_{n+1}) + (\rho_{n+1} - \rho_n)Ax_{n+1}\|\\ &\leq \|x_n - x_{n+1}\| + |\rho_{n+1} - \rho_n|\|Ax_{n+1}\|\end{aligned} \tag{2.3}$$

and

$$\begin{aligned} &\|B_n z_n - B_{n+1} z_{n+1}\| \\ &\le \|(I-\lambda_n B)z_n - (I-\lambda_{n+1}B)z_{n+1}\| \\ &= \|(z_n - \lambda_n B z_n) - (z_{n+1} - \lambda_n B z_{n+1}) + (\lambda_{n+1} - \lambda_n) B z_{n+1}\| \\ &\le \|z_n - z_{n+1}\| + |\lambda_{n+1} - \lambda_n| \|B z_{n+1}\|. \end{aligned} \tag{2.4}$$

From the iterative process (Υ), we see that

$$\begin{aligned} z_{n+1} - z_n = {}& \eta_{n+1}(x_{n+1} - x_n) + (1-\eta_{n+1})(A_{n+1}x_{n+1} - A_n x_n) \\ & + (\eta_{n+1} - \eta_n)(x_n - A_n x_n). \end{aligned}$$

It follows from (2.3) that

$$\begin{aligned} \|z_{n+1} - z_n\| \le {}& \eta_{n+1}\|x_{n+1} - x_n\| + (1-\eta_{n+1})\|A_{n+1}x_{n+1} - A_n x_n\| \\ & + |\eta_{n+1} - \eta_n| \|x_n - A_n x_n\| \\ \le {}& \|x_{n+1} - x_n\| + M_1(|\rho_{n+1} - \rho_n| + |\eta_{n+1} - \eta_n|), \end{aligned} \tag{2.5}$$

where M_1 is an appropriate constant such that

$$M_1 = \max\{\sup_{n\ge 0}\{\|Ax_n\|\}, \sup_{n\ge 0}\{\|x_n - A_n x_n\|\}\}.$$

Note that

$$\|U_{n+1}B_n z_n - U_n B_n z_n\| \le |\delta_{n+1} - \delta_n| \|B_n z_n - TB_n z_n\|,$$

which combines with (2.4) gives that

$$\begin{aligned} \|y_{n+1} - y_n\| &\le \|U_{n+1}B_{n+1}z_{n+1} - U_{n+1}B_n z_n\| + \|U_{n+1}B_n z_n - U_n B_n z_n\| \\ &\le \|z_n - z_{n+1}\| + M_2(|\lambda_{n+1} - \lambda_n| + |\delta_{n+1} - \delta_n|), \end{aligned} \tag{2.6}$$

where M_2 is an appropriate constant such that

$$M_2 = \max\{\sup_{n\ge 0}\{\|Bz_n\|\}, \sup_{n\ge 0}\{\|B_n z_n - TB_n z_n\|\}\}.$$

Combining (2.5) with (2.6), we arrive at

$$\|y_{n+1} - y_n\| \le \|x_{n+1} - x_n\| + M_3(|\rho_{n+1} - \rho_n| + |\eta_{n+1} - \eta_n| + |\lambda_{n+1} - \lambda_n| + |\delta_{n+1} - \delta_n|), \tag{2.7}$$

where M_3 is an appropriate constant such that

$$M_3 = \max\{M_1, M_2\}.$$

Put

$$l_n = \frac{x_{n+1} - \beta_n x_n}{1-\beta_n}, \quad \forall n \ge 0.$$

That is, $x_{n+1} = (1-\beta_n)l_n + \beta_n x_n$. Now, we compute $\|l_{n+1} - l_n\|$. From

$$\begin{aligned} l_{n+1} - l_n &= \frac{\alpha_{n+1}u + (1-\alpha_{n+1}-\beta_{n+1})y_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n u + (1-\alpha_n-\beta_n)y_n}{1-\beta_n} \\ &= \frac{\alpha_{n+1}}{1-\beta_{n+1}}(u - y_{n+1}) - \frac{\alpha_n}{1-\beta_n}(u - y_n) + y_{n+1} - y_n, \end{aligned}$$

we obtain that

$$\|l_{n+1} - l_n\| \le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|u - y_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|u - y_n\| + \|y_{n+1} - y_n\|,$$

which combines with (2.7) yields that

$$\begin{aligned}
&\|l_{n+1} - l_n\| - \|x_{n+1} - x_n\| \\
&\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|u - y_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|u - y_n\| \\
&\quad + M_3(|\rho_{n+1} - \rho_n| + |\eta_{n+1} - \eta_n| + |\lambda_{n+1} - \lambda_n| + |\delta_{n+1} - \delta_n|).
\end{aligned}$$

In view of the conditions (b), (c), (d) and (f), we obtain that

$$\limsup_{n\to\infty}(\|l_{n+1} - l_n\| - \|x_{n+1} - x_n\|) \le 0.$$

From Lemma 1.1, we see that $\lim_{n\to\infty}\|l_n - x_n\| = 0$. It follows that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = \lim_{n\to\infty}(1-\beta_n)\|l_n - x_n\| = 0. \tag{2.8}$$

Note that

$$x_{n+1} - x_n = \alpha_n(u - x_n) + \gamma_n(y_n - x_n),$$

which combines with the condition (c) gives that

$$\lim_{n\to\infty}\|y_n - x_n\| = 0. \tag{2.9}$$

On the other hand, we have

$$\begin{aligned}
\|A_n x_n - p\|^2 &\le \|(x_n - p) - \rho_n(Ax_n - Ap)\|^2 \\
&= \|x_n - p\|^2 - 2\rho_n\langle Ax_n - Ap, x_n - p\rangle + \rho_n^2\|Ax_n - Ap\|^2 \\
&\le \|x_n - p\|^2 - \rho_n(2\alpha - \rho_n)\|Ax_n - Ap\|^2.
\end{aligned} \tag{2.10}$$

Similarly, we can obtain that

$$\|B_n z_n - p\|^2 \le \|x_n - p\|^2 - \lambda_n(2\beta - \lambda_n)\|Bz_n - Bp\|^2. \tag{2.11}$$

It follows from (2.10) that

$$\begin{aligned}
\|z_n - p\|^2 &= \|\eta_n(x_n - p) + (1-\eta_n)(A_n x_n - p)\|^2 \\
&\le \eta_n\|x_n - p\|^2 + (1-\eta_n)\|A_n x_n - p\|^2 \\
&\le \|x_n - p\|^2 - (1-\eta_n)\rho_n(2\alpha - \rho_n)\|Ax_n - Ap\|^2.
\end{aligned} \tag{2.12}$$

It follows from (2.11) that

$$\begin{aligned}
\|y_n - p\|^2 &= \|U_n B_n z_n - p\|^2 \\
&\le \|B_n z_n - p\|^2 \\
&\le \|x_n - p\|^2 - \lambda_n(2\beta - \lambda_n)\|Bz_n - Bp\|^2.
\end{aligned} \tag{2.13}$$

Note that

$$\begin{aligned}\|x_{n+1}-p\|^2 &= \|\alpha_n(u-p)+\beta_n(x_n-p)+\gamma_n(y_n-p)\|^2\\ &\le \alpha_n\|u-p\|^2+\beta_n\|x_n-p\|^2+\gamma_n\|y_n-p\|^2\\ &\le \alpha_n\|u-p\|^2+\|x_n-p\|^2-\gamma_n\lambda_n(2\beta-\lambda_n)\|Bz_n-Bp\|^2,\end{aligned}$$

which yields that

$$\begin{aligned}\gamma_n\lambda_n(2\beta-\lambda_n)\|Bz_n-Bp\|^2 &\le \alpha_n\|u-p\|^2+\|x_n-p\|^2-\|x_{n+1}-p\|^2\\ &\le \alpha_n\|u-p\|^2+(\|x_n-p\|+\|x_{n+1}-p\|)\|x_n-x_{n+1}\|.\end{aligned}$$

From (2.8) and the conditions (C2), (C3), (C5) and (C6), we obtain that

$$\lim_{n\to\infty}\|Bz_n-Bp\|=0. \tag{2.14}$$

On the other hand, we see from (2.12) that

$$\begin{aligned}\|x_{n+1}-p\|^2 &= \|\alpha_n(u-p)+\beta_n(x_n-p)+\gamma_n(y_n-p)\|^2\\ &\le \alpha_n\|u-p\|^2+\beta_n\|x_n-p\|^2+\gamma_n\|y_n-p\|^2\\ &\le \alpha_n\|u-p\|^2+\beta_n\|x_n-p\|^2+\gamma_n\|z_n-p\|^2\\ &\le \alpha_n\|u-p\|^2+\|x_n-p\|^2-\gamma_n(1-\eta_n)\rho_n(2\alpha-\rho_n)\|Ax_n-Ap\|^2,\end{aligned}$$

which yields that

$$\begin{aligned}&\gamma_n(1-\eta_n)\rho_n(2\alpha-\rho_n)\|Ax_n-Ap\|\\ &\le \alpha_n\|u-p\|^2+\|x_n-p\|^2-\|x_{n+1}-p\|^2\\ &\le \alpha_n\|u-p\|^2+(\|x_n-p\|+\|x_{n+1}-p\|)\|x_n-x_{n+1}\|.\end{aligned}$$

From (2.8) and the conditions (b), (c), (e) and (f), we obtain that

$$\lim_{n\to\infty}\|Ax_n-Ap\|=0. \tag{2.15}$$

Observe that

$$\begin{aligned}&\|A_nx_n-p\|^2\\ &= \|P_C(I-\rho_nA)x_n-P_C(I-\rho_nA)p\|^2\\ &\le \langle (I-\rho_nA)x_n-(I-\rho_nA)p, A_nx_n-p\rangle\\ &= \frac{1}{2}\{\|(I-\rho_nA)x_n-(I-\rho_nA)p\|^2+\|A_nx_n-p\|^2\\ &\quad -\|(I-\rho_nA)x_n-(I-\rho_nA)p-(A_nx_n-p)\|^2\}\\ &\le \frac{1}{2}\{\|x_n-p\|^2+\|A_nx_n-p\|^2-\|x_n-A_nx_n-\rho_n(Ax_n-Ap)\|^2\}\\ &= \frac{1}{2}\{\|x_n-p\|^2+\|A_nx_n-p\|^2-\|x_n-A_nx_n\|^2-\rho_n^2\|Ax_n-Ap\|^2\\ &\quad +2\rho_n\langle x_n-A_nx_n, Ax_n-Ap\rangle\},\end{aligned}$$

which yields that

$$\|A_nx_n-p\|^2 \le \|x_n-p\|^2-\|x_n-A_nx_n\|^2+2\rho_n\|x_n-A_nx_n\|\|Ax_n-Ap\|. \tag{2.16}$$

Similarly, we can prove that

$$\|B_n z_n - p\|^2 \leq \|x_n - p\|^2 - \|z_n - B_n z_n\|^2 + 2\lambda_n \|z_n - B_n z_n\| \|Bz_n - Bp\|. \tag{2.17}$$

Note that

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|\alpha_n(u-p) + \beta_n(x_n - p) + \gamma_n(y_n - p)\|^2 \\ &\leq \alpha_n \|u-p\|^2 + \beta_n \|x_n - p\|^2 + \gamma_n \|B_n z_n - p\|^2, \end{aligned} \tag{2.18}$$

which combines with (2.17) gives that

$$\begin{aligned} &\gamma_n \|z_n - B_n z_n\|^2 \\ &\leq \alpha_n \|u-p\|^2 + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + 2\gamma_n \lambda_n \|z_n - B_n z_n\| \|Bz_n - Bp\| \\ &\leq \alpha_n \|u-p\|^2 + (\|x_n - p\|^2 + \|x_{n+1} - p\|^2) \|x_n - x_{n+1}\| \\ &\quad + 2\gamma_n \lambda_n \|z_n - B_n z_n\| \|Bz_n - Bp\|, \end{aligned}$$

Thanks to (2.14) and (2.8), we obtain from the conditions (b) and (c) that

$$\lim_{n\to\infty} \|z_n - B_n z_n\| = 0. \tag{2.19}$$

On the other hand, it follows from (2.16) that

$$\begin{aligned} &\|x_{n+1} - p\|^2 \\ &\leq \alpha_n \|u-p\|^2 + \beta_n \|x_n - p\|^2 + \gamma_n \|z_n - p\|^2 \\ &\leq \alpha_n \|u-p\|^2 + \beta_n \|x_n - p\|^2 + \gamma_n \|\eta_n(x_n - p) + (1-\eta_n)(A_n x_n - p)\|^2 \\ &\leq \alpha_n \|u-p\|^2 + \beta_n \|x_n - p\|^2 + \gamma_n \big(\eta_n \|x_n - p\|^2 + (1-\eta_n)\|A_n x_n - p\|^2\big) \\ &\leq \alpha_n \|u-p\|^2 + \|x_n - p\|^2 - \gamma_n(1-\eta_n)\|x_n - A_n x_n\|^2 \\ &\quad + 2\rho_n \|x_n - A_n x_n\| \|Ax_n - Ap\|. \end{aligned}$$

It follows that

$$\begin{aligned} &\gamma_n(1-\eta_n)\|x_n - A_n x_n\|^2 \\ &\leq \alpha_n \|u-p\|^2 + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + 2\rho_n \|x_n - A_n x_n\| \|Ax_n - Ap\| \\ &\leq \alpha_n \|u-p\|^2 + (\|x_n - p\| + \|x_{n+1} - p\|)\|x_n - x_{n+1}\| + 2\rho_n \|x_n - A_n x_n\| \|Ax_n - Ap\|. \end{aligned}$$

In view of (2.8) and (2.15), we see from conditions (b), (c) and (f) that

$$\lim_{n\to\infty} \|x_n - A_n x_n\| = 0. \tag{2.20}$$

Next, we show that

$$\limsup_{n\to\infty} \langle u - \bar{x}, x_n - \bar{x} \rangle \leq 0.$$

To show it, we choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle u - \bar{x}, x_n - \bar{x} \rangle = \lim_{i\to\infty} \langle u - \bar{x}, x_{n_i} - \bar{x} \rangle.$$

As $\{x_{n_i}\}$ is bounded, we have that there is a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ converges weakly to q. We may assume, without loss of generality, that $x_{n_i} \rightharpoonup q$. From (2.20), we see that

$$A_{n_i}x_{n_i} \rightharpoonup q \quad \text{as } n \to \infty. \tag{2.21}$$

Next, we show that $q \in VI(C, A)$. Put

$$Rv = \begin{cases} Av + N_C v, & v \in C \\ \emptyset, & v \notin C. \end{cases}$$

Then R is maximal monotone. Let $(v, w) \in G(R)$, where $G(R)$ is the graph of R. Since $w - Av \in N_C v$ and $A_n x_n \in C$, we have

$$\langle v - A_n x_n, w - Av \rangle \geq 0.$$

On the other hand, we see from $A_n x_n = P_C(I - \rho_n A)x_n$ that

$$\langle v - A_n x_n, A_n x_n - (I - \rho_n A)x_n \rangle \geq 0$$

and hence

$$\langle v - A_n x_n, \frac{A_n x_n - x_n}{\rho_n} + Ax_n \rangle \geq 0.$$

It follows that

$$\begin{aligned}
\langle v - A_{n_i}x_{n_i}, w \rangle &\geq \langle v - A_{n_i}x_{n_i}, Av \rangle \\
&\geq \langle v - A_{n_i}x_{n_i}, Av \rangle - \langle v - A_{n_i}x_{n_i}, \frac{A_{n_i}x_{n_i} - x_{n_i}}{\rho_{n_i}} + Ax_{n_i} \rangle \\
&\geq \langle v - A_{n_i}x_{n_i}, Av - \frac{A_{n_i}x_{n_i} - x_{n_i}}{\rho_{n_i}} - Ax_{n_i} \rangle \\
&= \langle v - A_{n_i}x_{n_i}, Av - AA_{n_i}x_{n_i} \rangle + \langle v - A_{n_i}x_i, AA_{n_i}x_{n_i} - Ax_{n_i} \rangle \\
&\quad - \langle v - A_{n_i}x_{n_i}, \frac{A_{n_i}x_{n_i} - x_{n_i}}{\rho_{n_i}} \rangle \\
&\geq \langle v - A_{n_i}x_{n_i}, AA_{n_i}x_{n_i} - Ax_{n_i} \rangle - \langle v - A_{n_i}x_{n_i}, \frac{A_{n_i}x_{n_i} - x_{n_i}}{\rho_{n_i}} \rangle.
\end{aligned}$$

In view of (2.20), we see that $\langle v - q, w \rangle \geq 0$. We have $q \in R^{-1}0$ and hence $q \in VI(C, A)$.

On the other hand, we see from the condition (f) that $z_n - A_n x_n \to 0$ as $n \to \infty$. It follows from (2.20) that

$$\lim_{n\to\infty} \|z_n - x_n\| = 0, \tag{2.22}$$

which combines with (2.19) gives that

$$\lim_{n\to\infty} \|B_n z_n - x_n\| = 0. \tag{2.23}$$

It follows that

$$B_{n_i}z_{n_i} \rightharpoonup q \quad \text{as } n \to \infty.$$

In a similar way, we can obtain that $q \in VI(C, B)$.

Next, we show that $q \in F(T)$. It follows from Lemma 1.4 that

$$\begin{aligned}
&\|x_n - Tx_n\| \\
&\le \|x_n - y_n\| + \|y_n - Tx_n\| \\
&\le \|x_n - y_n\| + \delta_n\|B_n z_n - Tx_n\| + (1-\delta_n)\|TB_n z_n - Tx_n\| \\
&\le \|x_n - y_n\| + \delta_n\|B_n z_n - x_n\| + \delta_n\|x_n - Tx_n\| + (1-\delta_n)\frac{1+k}{1-k}\|B_n z_n - x_n\| \\
&\le \|x_n - y_n\| + \frac{2}{1-k}\|B_n z_n - x_n\| + \delta_n\|x_n - Tx_n\|,
\end{aligned}$$

which implies that

$$(1-\delta_n)\|x_n - Tx_n\| \le \|x_n - y_n\| + \frac{2}{1-k}\|B_n z_n - x_n\|,$$

It follows from (2.9), (2.23) and the condition (f) that

$$\lim_{n\to\infty}\|x_n - Tx_n\| = 0.$$

In view of Lemma 1.4, we obtain that $q \in F(T)$. We, therefore, arrive at

$$\limsup_{n\to\infty}\langle u - \bar{x}, x_n - \bar{x}\rangle = \langle u - \bar{x}, q - \bar{x}\rangle \le 0. \tag{2.24}$$

Finally, we show that $x_n \to \bar{x}$ as $n \to \infty$. Note that

$$\begin{aligned}
&\|x_{n+1} - \bar{x}\|^2 \\
&= \langle \alpha_n u + \beta_n x_n + \gamma_n y_n - \bar{x}, x_{n+1} - \bar{x}\rangle \\
&= \alpha_n\langle u - \bar{x}, x_{n+1} - \bar{x}\rangle + \beta_n\langle x_n - \bar{x}, x_{n+1} - \bar{x}\rangle + \gamma_n\langle y_n - \bar{x}, x_{n+1} - \bar{x}\rangle \\
&\le \alpha_n\langle u - \bar{x}, x_{n+1} - \bar{x}\rangle + \beta_n\|x_n - \bar{x}\|\|x_{n+1} - \bar{x}\| + \gamma_n\|x_n - \bar{x}\|\|x_{n+1} - \bar{x}\| \\
&= \alpha_n\langle u - \bar{x}, x_{n+1} - \bar{x}\rangle + (1-\alpha_n)\|x_n - \bar{x}\|\|x_{n+1} - \bar{x}\| \\
&\le \frac{(1-\alpha_n)}{2}\|x_n - \bar{x}\|^2 + \frac{1}{2}\|x_{n+1} - \bar{x}\|^2 + \alpha_n\langle u - \bar{x}, x_{n+1} - \bar{x}\rangle,
\end{aligned}$$

which implies that

$$\|x_{n+1} - \bar{x}\|^2 \le (1-\alpha_n)\|x_n - \bar{x}\|^2 + \alpha_n\langle u - \bar{x}, x_{n+1} - \bar{x}\rangle. \tag{2.25}$$

From (2.24) and applying Lemma 1.2 to (2.25), we obtain from the condition (b) that

$$\lim_{n\to\infty}\|x_n - \bar{x}\| = 0.$$

This completes the proof.

Next, we give some real sequences which satisfy the restrictions (a), (b), (c), (d), (e), and (f). $\alpha_n = \frac{1}{n+4}$, $\beta_n = \frac{n+6}{2n+8}$, $\gamma_n = \frac{n}{n+4}$, $\delta_n = k + \frac{1-k}{n+1}$, $\eta_n = \frac{1}{n+1}$, $\lambda_n = \rho_n = \frac{a}{n+1} + b\frac{n}{n+1}$.

As corollaries of Theorem 2.1, we have the following results.

Corollary 2.2. *Let H be a real Hilbert space, C a nonempty closed convex subset of H. Let $A : C \to H$ be an α-inverse-strongly monotone mapping, $B : C \to H$ a β-inverse-strongly monotone mapping and*

$T : C \to C$ a nonexpansive mapping. Assume that $\mathcal{F} = F(T) \cap VI(C, A) \cap VI(C, B) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the following algorithm:

$$\begin{cases} x_0 \in C, \\ z_n = \eta_n x_n + (1 - \eta_n) P_C(x_n - \rho_n A x_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n T P_C(z_n - \lambda_n B z_n), \quad n \geq 0, \end{cases}$$

where u is fixed element in C, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\eta_n\}$ are sequences in $(0, 1)$ and $\{\lambda_n\}$ and $\{\rho_n\}$ are positive sequences. Assume that the above control sequences satisfy the following restrictions:

(a) $\alpha_n + \beta_n + \gamma_n = 1$, $\forall n \geq 0$;

(b) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(c) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(d) $\lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = \lim_{n\to\infty} |\rho_{n+1} - \rho_n| = 0$;

(e) $\{\lambda_n\}, \{\rho_n\} \in [a, b]$ *for some* a, b *with* $0 < a < b < \min\{2\alpha, 2\beta\}$;

(f) $\lim_{n\to\infty} \eta_n = 0$.

Then the sequence $\{x_n\}$ converges strongly to $\bar{x} \in \mathcal{F}$, where $\bar{x} = P_{\mathcal{F}} u$.

Further, putting $\eta_n = 0$ for each $n \geq 0$ in Corollary 2.2, we have the following.

Corollary 2.3. *Let H be a real Hilbert space, C a nonempty closed convex subset of H. Let $A : C \to H$ be an α-inverse-strongly monotone mapping, $B : C \to H$ a β-inverse-strongly monotone mapping and $T : C \to C$ a nonexpansive mapping. Assume that $\mathcal{F} = F(T) \cap VI(C, A) \cap VI(C, B) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the following algorithm:*

$$\begin{cases} x_0 \in C, \\ z_n = P_C(x_n - \rho_n A x_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n T P_C(z_n - \lambda_n B z_n), \quad n \geq 0, \end{cases}$$

where u is fixed element in C, $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0, 1)$ and $\{\lambda_n\}$ and $\{\rho_n\}$ are positive sequences. Assume that the above control sequences satisfy the following restrictions:

(a) $\alpha_n + \beta_n + \gamma_n = 1$, $\forall n \geq 0$;

(b) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(c) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(d) $\lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = \lim_{n\to\infty} |\rho_{n+1} - \rho_n| = 0$;

(e) $\{\lambda_n\}, \{\rho_n\} \in [a, b]$ *for some* a, b *with* $0 < a < b < \min\{2\alpha, 2\beta\}$.

Then the sequence $\{x_n\}$ *converges strongly to* $\bar{x} \in \mathcal{F}$*, where* $\bar{x} = P_{\mathcal{F}}u$.

Remark 2.4. Corollary 2.3 is reduced to Theorem 3.1 of Yao and Yao [15] if $A = B$ and $\lambda_n = \rho_n$ for each $n \geq 0$; see [15] for more details.

Recall that a mapping $T : C \to C$ is said to be a k-strict pseudo-contraction if there exists a constant $k \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C.$$

Put $A := I - T$, where $T : C \to C$ is a k-strict pseudo-contraction. Then A is $\frac{1-k}{2}$-inverse-strongly monotone; see [1] for more details.

As applications of Theorem 2.1, we obtain the following results on the class of strict pseudocontractions.

Theorem 2.5. *Let* H *be a real Hilbert space,* C *a nonempty closed convex subset of* H*. Let* $T_A : C \to C$ *be an* k_α*-strict pseudocontraction,* $T_B : C \to C$ *a* k_β*-strict pseudocontraction and* $T : C \to C$ *a* k*-strict pseudocontraction. Assume that* $\mathcal{F} = F(T) \cap F(T_A) \cap F(T_B) \neq \emptyset$*. Let* $\{x_n\}$ *be a sequence generated by the following algorithm:*

$$\begin{cases} x_0 \in C, \\ z_n = (1 - \rho_n)x_n + \rho_n T_A x_n, \\ y_n = (1 - \lambda_n)z_n + \lambda_n T_B z_n, \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n[\delta_n y_n + (1 - \delta_n)Ty_n], \quad n \geq 0, \end{cases}$$

where u *is fixed element in* C*,* $\{\alpha_n\}$*,* $\{\beta_n\}$*,* $\{\gamma_n\}$*,* $\{\delta_n\}$ *and* $\{\eta_n\}$ *are sequences in* $(0, 1)$ *and* $\{\lambda_n\}$ *and* $\{\rho_n\}$ *are positive sequences. Assume that above the control sequences satisfy the following restrictions:*

(a) $\alpha_n + \beta_n + \gamma_n = 1$, $\forall n \geq 0$;

(b) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(c) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(d) $\lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = \lim_{n\to\infty} |\delta_{n+1} - \delta_n| = \lim_{n\to\infty} |\rho_{n+1} - \rho_n| = 0$;

(e) $\{\lambda_n\}, \{\rho_n\} \in [a, b]$ *for some* a, b *with* $0 < a < b < \min\{(1 - k_\alpha), (1 - k_\beta)\}$;

(f) $k \leq \delta_n \leq e < 1$*, where* $e \in [k, 1)$ *is some constant.*

Then the sequence $\{x_n\}$ *converges strongly to* $\bar{x} \in \mathcal{F}$*, where* $\bar{x} = P_{\mathcal{F}}u$.

Proof. Putting $A = I - T_A$ and $B = I - T_B$, we see that A is $\frac{1-k_\alpha}{2}$-inverse-strongly monotone and B is $\frac{1-k_\beta}{2}$-inverse-strongly monotone. We also have $F(T_A) = VI(C, A)$ and $F(T_B) = VI(C, B)$. Note that

$$P_C(x_n - \rho_n Ax_n) = (1 - \rho_n)x_n + \rho_n T_B x_n$$

and

$$P_C(z_n - \lambda_n Bz_n) = (1 - \lambda_n)z_n + \lambda_n T_B z_n.$$

Putting $\eta_n = 0$ for each $n \geq 0$, we can conclude the desired conclusion easily from Theorem 2.1.

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