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Global Optimization Approach to Nonzero Sum n Person Game

S.Batbileg[1], R.Enkhbat[2]

[1] *National University of Mongolia, Ulaanbaatar*
email:sbatbileg@gmail.com
[2] *National University of Mongolia, Ulaanbaatar*
email: enkhbat46@gmail.com

Abstract

The nonzero sum n person game has been considered. We show that the game can be reduced to global optimization problem. We derive necessary and sufficient conditions for a point to be Nash point.

Key words: Nash equilibrium , mixed strategies.

1 Introduction

Game theory plays an important role in applied mathematics, economics and decision theory. There are many works devoted to game theory[2-7]. Most of them deals with zero sum two person games or nonzero sum two person games. Also, two person non zero sum game was studied in [5] by reducing it to D.C programming. This paper considers nonzero sum n person game. The paper is organized as follows. In Section 2, we formulate non zero sum n person game and show that it can be formulated as a global optimization problem with polynom constraints. We formulate the problem of finding a Nash equilibrium for non zero sum n-person games as a nonlinear programming problem.

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2 Nonzero Sum n-person Game

Consider the n-person game in mixed strategies with matrices ($A_q,\ q = 1, 2, \dots, n$) for players $1, 2, \dots, n$.

$$A_q = \left(a^q_{i_1 i_2 \dots i_n}\right), q = 1, 2, \dots, n$$

$$i_1 = 1, 2, \dots, k_1,\ \dots, i_n = 1, 2, \dots, k_n,$$

Denote by D_q the set

$$D_p = \{u \in R^p \,|\sum_{i=1}^{p} u_i = 1,\ u_i \geq 0,\ i = 1, \dots, p\}$$

$$p = k_1, k_2, \dots, k_n$$

A mixed strategy for player 1 is a vector $x^1 = (x^1_{i_1}, x^1_{i_2}, \dots, x^1_{i_{k_1}}) \in D_{k_1}$ representing the probability that player 1 uses a strategy i. Similarly, the mixed strategies for q-th player is $x^q = (x^q_{i_1}, x^q_{i_2}, \dots, x^q_{i_{k_1}}) \in D_{k_q},\ q = 1, 2, \dots, n$. Their expected payoffs are given by for 1-th person :

$$f_1(x^1, x^2, \dots, x^n) = \sum_{i_1=1}^{k_1} \sum_{i_2=1}^{k_2} \dots \sum_{i_n=1}^{k_n} a^1_{i_1 i_2 \dots i_n} x^1_{i_1} x^2_{i_2} \dots x^n_{i_n}.$$

and for q-th person

$$f_q(x^1, x^2, \dots, x^n) = \sum_{i_1=1}^{k_1} \sum_{i_2=1}^{k_2} \dots \sum_{i_n=1}^{k_n} a^q_{i_1 i_2 \dots i_n} x^1_{i_1} x^2_{i_2} \dots x^n_{i_n},$$

$$q = 1, 2, \dots, n$$

Definition 2.1 *A vector of mixed strategies* $\tilde{x}^q \in D_{k_q},\ q = 1, 2, \dots, n$ *is a Nash equilibrium if*

$$\begin{cases} f_1(\tilde{x}^1, \tilde{x}^2, \dots, \tilde{x}^n) \geq f_1(x^1, \tilde{x}^2, \dots, \tilde{x}^n),\ \forall x^1 \in D_{k_1} \\ \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ f_q(\tilde{x}^1, \tilde{x}^2, \dots, \tilde{x}^n) \geq f_q(x^1, \tilde{x}^2, \dots, \tilde{x}^n),\ \forall x^q \in D_{k_q} \\ \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ f_n(\tilde{x}^1, \tilde{x}^2, \dots, \tilde{x}^n) \geq f_n(x^1, \tilde{x}^2, \dots, \tilde{x}^n),\ \forall x^n \in D_{k_n}. \end{cases}$$

It is clear that

$$\begin{array}{l} f_1(\tilde{x}^1, \tilde{x}^2, \ldots, \tilde{x}^n) = \max_{x^1 \in D_{k_1}} f_1(x^1, \tilde{x}^2, \ldots \quad \ldots, \tilde{x}^n), \\ \ldots \quad \ldots \quad \ldots \quad \ldots \\ f_q(\tilde{x}^1, \tilde{x}^2, \ldots, \tilde{x}^n) = \max_{x^q \in D_{k_q}} f_q(\tilde{x}^1, \tilde{x}^2, \ldots, \ldots, \tilde{x}^{q-1}, x^q, \ldots, \tilde{x}^{q+1}, \ldots, \tilde{x}^n), \\ \ldots \quad \ldots \quad \ldots \quad \ldots \\ f_n(\tilde{x}^1, \tilde{x}^2, \ldots, \tilde{x}^n) = \max_{x^n \in D_{k_n}} f_n(\tilde{x}^1, \tilde{x}^2, \ldots, \tilde{x}^{n-1}, x^n). \end{array}$$

Denote by

$$\sum_{i_1=1}^{k_1} \sum_{i_2=1}^{k_2} \ldots \sum_{i_n=1}^{k_n} a^q_{i_1 i_2 \ldots i_n} x^1_{i_1} x^2_{i_2} \ldots x^n_{i_n} \triangleq \sum_{i_1 i_2 \ldots i_n=1}^{k_1, k_2, \ldots, k_n} a^q x^1 x^2 \ldots x^n \triangleq$$

$$\triangleq \sum_{i_j=1}^{k_j} a^q \left(\prod x^j \right) \triangleq \sum_{i_1 i_2 \ldots i_n=1}^{k_1, k_2, \ldots, k_n} a^q x \triangleq f_q(x^1, x^2, \ldots, x^n) \triangleq f_q(x), \quad q = 1, 2, \ldots, n$$

and

$$\sum_{i_1=1}^{k_1} \sum_{i_2=1}^{k_2} \ldots \sum_{i_{q-1}=1}^{k_{q-1}} \sum_{i_{q+1}=1}^{k_{q+1}} \ldots \sum_{i_n=1}^{k_n} a^q_{i_1 i_2 \ldots i_n} x^1_{i_1} x^2_{i_2} \ldots x^{q-1}_{i_{q-1}} x^{q+1}_{i_{q+1}} \ldots x^n_{i_n} \triangleq$$

$$\sum_{i_1, \ldots i_{q-1}, i_{q-1}, \ldots i_n=1}^{k_1, \ldots, k_{q-1}, k_{q+1}, \ldots, k_n} a^q x^1 \ldots x^{q-1} x^{q+1} \ldots x^n \triangleq \sum_{i_j=1 \quad j \neq q}^{k_j} a^q x^1 \ldots x^{q-1} x^{q+1} \ldots x^n \triangleq$$

$$\triangleq \sum_{i_j=1}^{k_j} a^q \left(\prod_{j=1,\ j \neq q}^{n} x^j \right) \triangleq f_q(x^1 x^2 \ldots x^{q-1} x^{q+1} \ldots x^n) \triangleq f_q(x \backslash x^j), \ \ j, q = 1, 2, \ldots, n.$$

For further purpose, it is useful to formulate the following statement.

Theorem 2.1 *A vector strategy $(\tilde{x}^1, \tilde{x}^2, \ldots, \tilde{x}^n)$ is a Nash equilibrium if and only if*

$$\sum_{i_j=1}^{k_j} a^q \left(\prod \tilde{x}^j \right) \geq \sum_{i_j=1}^{k_j} a^q \left(\prod_{j=1,\ j \neq q}^{n} \tilde{x}^j \right) \tag{1}$$

for

$$\begin{aligned} i_j &= 1, 2, \ldots, k_j, \\ j &= 1, 2, \ldots, n, \\ q &= 1, 2, \ldots, n. \end{aligned}$$

Proof. Necessity: Assume that $\tilde{x}$ is a Nash equilibrium. Then by definition 1.1, we have

$$\sum_{i_1=1}^{k_1}\sum_{i_2=1}^{k_2}\cdots\sum_{i_n=1}^{k_n} a^q_{i_1 i_2\ldots i_n}\tilde{x}^1_{i_1}\ldots\tilde{x}^n_{i_n} \geq$$
$$\geq \sum_{i_1=1}^{k_1}\sum_{i_2=1}^{k_2}\cdots\sum_{i_{q-1}=1}^{k_{q-1}}\sum_{i_{q+1}=1}^{k_{q+1}}\cdots\sum_{i_n=1}^{k_n} a^q_{i_1 i_2\ldots i_n}\tilde{x}^1_{i_1}\ldots\tilde{x}^{q-1}_{i_{q-1}}x^q_{i_q}\tilde{x}^{q+1}_{i_{q+1}}\ldots\tilde{x}^n_{i_n} \tag{2}$$

$$q = 1, 2, \ldots, n$$

In the inequality (2), successively choose $x^i = (0, 0, \ldots, 1, \ldots, 0)$ with 1 in each of the k_i spots. We can easily see that

$$f_q(\tilde{x}) = \sum_{i_j=1}^{k_j} a^q \left(\prod \tilde{x}^j\right) \geq \sum_{i_j=1}^{k_j} a^q \left(\prod_{j=1,\ j\neq q}^{n} \tilde{x}^j\right), \quad for \quad i_j = 1, 2, \ldots, k_j;$$

$$j = 1, 2, \ldots, n, \quad q = 1, 2, \ldots, n.$$

Sufficiency: Suppose that for a vector $\tilde{x} \in D_{k_1} \times D_{k_2} \times \ldots \times D_{k_n}$, conditions (1) are satisfied. We choose $x_q \in D_{k_q},\ q = 1, 2, \ldots, n$ and multiply (1) by x respectively. We obtain

$$\sum_{j=1}^{k_q} x_j \left[\sum_{i_j=1}^{k_j} a^q \left(\prod \tilde{x}^j\right)\right] \geq \sum_{i_1=1}^{k_1}\cdots\sum_{i_q=1}^{k_q}\cdots\sum_{i_n=1}^{k_n} a^q_{i_1 i_2\ldots i_n}\tilde{x}^1_{i_1}\ldots\tilde{x}^{q-1}_{i_{q-1}}x^q_{i_q}\tilde{x}^{q+1}_{i_{q+1}}\ldots\tilde{x}^n_{i_n}$$

$$q = 1, 2, \ldots, n$$

Taking into account that $\sum_{i=1}^{k_q} x^q_i = 1, \quad q = 1, 2, \ldots, n.$ we have

$$f_q(\tilde{x}^1, \tilde{x}^2, \ldots, \tilde{x}^n) \geq f_q(\tilde{x}^1, \ldots, \tilde{x}^{q-1}, x^q, \tilde{x}^{q+1} \ldots \tilde{x}^n), \quad \forall x^q \in D_{k_q}$$

$$q = 1, 2, \ldots, n$$

which shows that $\tilde{x}$ is a Nash equilibrium. The proof is complete.□

Theorem 2.2 *A mixed strategy $\tilde{x}$ is a Nash equilibrium for the nonzero sum n-person game if and only if there exists vector $\tilde{p}$ such* *that vector $(\tilde{x}, \tilde{p})$ is a solution to the following nonlinear programming problem :*

$$\max_{(x,p)} F(x,p) = \sum_{i_1=1}^{k_1} \sum_{i_2=1}^{k_2} \cdots \sum_{i_n=1}^{k_n} \left(\sum_{q=1}^{n} a^q_{i_1 i_2 \ldots i_n} \right) x^1_{i_1} x^2_{i_2} \ldots x^n_{i_n} - \sum_{q=1}^{n} p_q \tag{3}$$

subject to :

$$\sum_{i_j=1}^{k_j} a^q \left(\prod_{j=1,\ j\neq q}^{n} x^j \right) \leq p_q, \quad \forall i_q = 1, 2, \ldots, k_q, \tag{4}$$

$$\sum_{i=1}^{k_q} x^q_i = 1, \quad q = 1, 2, \ldots, n. \tag{5}$$

Proof. Necessity: Now suppose that $\tilde{x}$ is a Nash point. Choose vector $\tilde{p}$ as : $\tilde{p}_q = f_q(\tilde{x}),\ q = 1, 2, \ldots, n$
We show that $(\tilde{x}, \tilde{p})$ is a solution to problem (3)-(5).First, we show that $(\tilde{x}, \tilde{p})$ is a feasible point for problem (3).
By Theorem 1.1, the equivalent characterization of a Nash point, we have

$$\sum_{i_j=1 \quad j\neq q}^{k_j} a^q \tilde{x}^1 \ldots \tilde{x}^{q-1} \tilde{x}^{q+1} \ldots \tilde{x}^n \geq f_q(\tilde{x}^1, \ldots, \tilde{x}^n), \quad q = 1, 2, \ldots, n.$$

The rest of the constraints are satisfied because $\tilde{x}^q \in D_{kq}, \quad q = 1, 2, \ldots, n.$ It meant that $(\tilde{x}, \tilde{p})$ is a feasible point. Choose any $x^q \in D_{kq}, \quad q = 1, 2, \ldots, n.$
Multiply (4) by $x^q_i,\ q = 1, 2, \ldots, n.$ respectively. If we have sum up these inequalities, we obtain

$$f_q(x, y, z) = \sum_{i_j=1}^{k_j} a^q \left(\prod x^j \right) \leq p_q, \quad q = 1, 2, \ldots, n.$$

Hence, we get

$$F(x,p) = \sum_{i_1=1}^{k_1} \sum_{i_2=1}^{k_2} \cdots \sum_{i_n=1}^{k_n} \left(\sum_{q=1}^{n} a^q_{i_1 i_2 \ldots i_n} \right) x^1_{i_1} x^2_{i_2} \ldots x^n_{i_n} - \sum_{q=1}^{n} p_q \leq 0$$

for all $x^q \in D_q, \quad q = 1, 2, \ldots, n.$
But with $\tilde{p_q} = f_q(\tilde{x})$, we have $F(\tilde{x}, \tilde{p}) = 0$ Hence, the point $(\tilde{x}, \tilde{p})$ is a solution to the problem (3)-(5).
Sufficiency: Now we have to show reverse, namely, that any solution of problem (3)-(5) must be a Nash point. Let $(\bar{x}, \bar{p})$ be any solution of problem (3)-(5). Let $\tilde{x}$ be a Nash point for the game, and set $\tilde{p_q} = f_q(\tilde{x})$.
We will show that $\bar{x}$ must be a Nash equilibrium of the game. Since $(\bar{x}, \bar{p})$ is a feasible point, we have

$$\sum_{i_j=1}^{k_j} a^q \left(\prod_{j=1,\ j\neq q}^{n} \bar{x}^j \right) \leq \bar{p}_q \quad \forall j = 1, 2, \ldots, k_q, \quad q = 1, 2, \ldots, n. \quad (6)$$

Hence, we have

$$\sum_{i_j=1}^{k_j} a^q \left(\prod \bar{x}^j \right) \leq \bar{p}_q, \quad q = 1, 2, \ldots, n.$$

Adding these inequalities, we obtain

$$F(\bar{x}, \bar{p}) = \sum_{i_1=1}^{k_1} \sum_{i_2=1}^{k_2} \cdots \sum_{i_n=1}^{k_n} \left(\sum_{q=1}^{n} a^q_{i_1 i_2 \ldots i_n} \right) \bar{x}^1 \bar{x}^2 \ldots \bar{x}^n - \sum_{q=1}^{n} p_q \leq 0 \quad (7)$$

We know that at a Nash equilibrium $F(\tilde{x}, \tilde{p}) = 0$. Since $(\bar{x}, \bar{p})$ is also a solution, $F(\bar{x}, \bar{p})$ be equal to zero:

$$F(\bar{x}, \bar{p}) = \sum_{i_1=1}^{k_1} \sum_{i_2=1}^{k_2} \cdots \sum_{i_n=1}^{k_n} \left(\sum_{q=1}^{n} a^q_{i_1 i_2 \ldots i_n} \right) \bar{x}^1 \bar{x}^2 \ldots \bar{x}^n - \sum_{q=1}^{n} p_q = 0 \quad (8)$$

Consequently,

$$\sum_{i_j=1}^{k_j} a^q \left(\prod \bar{x}^j \right) = \bar{p}_q, \quad q = 1, 2, \ldots, n.$$

Since a point $(\bar{x}, \bar{p})$ feasible, we can write the constrains (6) as follows:

$$\sum_{i_j=1}^{k_j} a^q \left(\prod \bar{x}^j \right) \geq \sum_{i_j=1}^{k_j} a^q \left(\prod_{j=1,\ j\neq q}^{n} \bar{x}^j \right), \ for\ i_j = 1, 2, \ldots, k_j,\ q = 1, 2, \ldots, n.$$

3 Computational Experiments

Let $A = (a_{ijk})_{2\times2\times2}$ and $B = (b_{ijk})_{2\times2\times2}$, $C = (ijk)_{2\times2\times2}$
Three problems of type (5) − (9) have been solved numerically on "MATLAB" for dimensions $2 \times 2 \times 2$. In all cases, Nash points were found successfully. These problems were:
Problem 1. Let $a_{111} = 2, a_{112} = 3, a_{121} = -1, a_{122} = 0, a_{211} = 1, a_{212} = -2, a_{221} = 4, a_{222} = 3,\ b_{111} = 1, b_{112} = 2, b_{121} = 0, b_{122} = -1, b_{211} = -1, b_{212} = 0, b_{221} = 2, b_{222} = 1$, and $c_{111} = 3, c_{112} = 2, c_{121} = 1, c_{122} = -3, c_{211} = 0, c_{212} = 2, c_{221} = -1, c_{222} = 2$.
Then we have the problem:

$$F(x, y, z, p, q, t) = 6x_1y_1z_1 + 7x_1y_1z_2 - 3x_1y_2z_2 + 5x_2y_1z_2+$$

$$+6x_2y_2z_2 - p - q - t \rightarrow max$$

$$\begin{cases} 2y_1z_1 + 3y_1z_2 - y_2z_1 - p & \leq 0 \\ y_1z_1 - 2y_1z_2 + 4y_2z_1 + 3y_2z_2 - p & \leq 0 \\ x_1z_1 + 2x_1z_2 - x_2z_1 - q & \leq 0 \\ -1x_1z_2 + 2x_2z_1 + x_2z_2 - q & \leq 0 \\ 3x_1y_1 + x_1y_2 - x_2y_2 - t & \leq 0 \\ 2x_1y_1 - 3x_1y_2 + 2x_2y_1 + 2x_2y_2 - t & \leq 0 \\ x_1 + x_2 & = 1 \\ y_1 + y_2 & = 1 \\ z_1 + z_2 & = 1 \\ x_1 \geq 0\ , x_2 \geq 0\ , y_1 \geq 0\ , y_2 \geq 0 & \\ z_1 \geq 0\ , z_2 \geq 0\ , p \geq 0\ , q \geq 0\ t \geq 0 & \end{cases}$$

Solution is $F^* = -2.2204e - 016$, $x^* = (0.5191; 0.4809)^T$, $y^* = (0.5888; 0.4112)^T$ and $z^* = (0.5382; 0.4618)^T$. $p^* = 1.2281$, $q^* = 0.5$ and $t^* = 0.9327$
Problem 2. Let $a_{111} = 5, a_{112} = 3, a_{121} = 6, a_{122} = 7, a_{211} = 0, a_{212} = 8, a_{221} = 2, a_{222} = 1,\ b_{111} = 2, b_{112} = 4, b_{121} = -1, b_{122} = 0, b_{211} = 3, b_{212} = 5, b_{221} = 4, b_{222} = 9$, and $c_{111} = 2, c_{112} = 0, c_{121} = -4, c_{122} = -1, c_{211} = -2, c_{212} = 6, c_{221} = 8, c_{222} = 9$.
Solution is $F^* = -0.00986$, $x^* = (0.8; 0.2)^T$, $y^* = (0.1; 0)^T$ and $z^* = (0.5; 0.5)^T$. $p^* = -2.2204e - 016$, $q^* = 3.2$ and $t^* = 1.2$
Problem 3. Let $a_{111} = 3, a_{112} = 2, a_{121} = 1, a_{122} = 5, a_{211} = 8, a_{212} = 4, a_{221} = 1, a_{222} = 3,\ b_{111} = 3, b_{112} = 2, b_{121} = 4, b_{122} = 0, b_{211} = 1, b_{212} = 8, b_{221} = 6, b_{222} = 6$, and $c_{111} = 3, c_{112} = 1, c_{121} = 9, c_{122} = 2, c_{211} = 4, c_{212} = 7, c_{221} = 2, c_{222} = 3$.

Solution is $F^* = 0$, $x^* = (1;0)^T$, $y^* = (1;0)^T$ and $z^* = (0;1)^T$. $p^* = 4$, $q^* = 8$ and $t^* = 7$

Now taking into account the above results, by Theorem 2.2 we conclude that $\bar{x}$ is a Nash point which a completes the proof.□

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References

[1] Strekalovsky A.S.(1998) *Global Optimality Conditions for Nonconvex Optimization*, Journal of Global Optimization 12 415-434.

[2] Melvin Dresher(1981) *The Mathematics of Games of Strategy* , Dover Publications.

[3] Vorobyev,N.N.(1984) *Noncooperative games*, Nauka .

[4] Germeyer,YU.B.(1976) *Introduction to Operation Reseach*,Nauka .

[5] Strekalovsky,A.S and Orlov,A.V.(2007) *Bimatrix Game and Bilinear Programming*, Nauka

[6] Owen,G. (1971) *Game Theory*, Nauka.

[7] Gubbons,R.(1992) *Game Theory for Applied Economists*, Princeton University, Press

[8] Horst,R. and Tuy,H. (1993) *Global Optimization* , Springer-Verlag.

[9] Enkhbat,R. (1999) Quasiconvex Programming and its Applications *Lambert Publisher*, Germany