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A New Approximate Proximal Point Algorithm for General Variational Inequalities with Separable Operators

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Abstract. Based on the proximal algorithms for general variational inequalities, we present a new approximate proximal point algorithm (APPA) for general variational inequalities with separable operators in this paper. The resulting proximal subproblems are allowed to be solved approximately with summable accuracy. The global convergence of the proposed method is proved under mild conditions.

Keywords: General variational inequalities, APPA, separable operators, global convergence.

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1 Introduction

In recent years, variational inequalities have been extended and generalized in different directions. A basic generalization of variational inequalities is called general variational inequalities(GVI). Since the original work in [1], many researchers have concentrated on the development of GVI [2-4]. In this paper, we consider the following special GVI: general variational inequalities with separable operators, denote by GVI(F, G, Ω), which is to find $u^* \in \mathcal{R}^{m+n}$ such that

$$F(u^*) \in \Omega \;\text{ and }\; (u - F(u^*))^\top G(u^*) \geq 0, \quad \forall u \in \Omega, \tag{1}$$

where

$$u = \begin{pmatrix} x \\ y \end{pmatrix}, \; F(u) = \begin{pmatrix} f_1(x) \\ f_2(y) \end{pmatrix}, \; G(u) = \begin{pmatrix} g_1(x) \\ g_2(y) \end{pmatrix},$$

and

$$\Omega = \{(x, y) | x \in \mathcal{X}, y \in \mathcal{Y}, Ax + By = b\},$$

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$\mathcal{X} \subseteq \mathcal{R}^n$ and $\mathcal{Y} \subseteq \mathcal{R}^m$ are given nonempty closed convex sets; f_1, g_1 are given continuous operators from $\mathcal{R}^n$ into itself and f_2, g_2 are given continuous operators from $\mathcal{R}^m$ into itself; $A \in \mathcal{R}^{r\times n}$ and $B \in \mathcal{R}^{r\times m}$ are given matrices; $b \in \mathcal{R}^r$ is a given vector. It is easy to see that if $F(u) = u$, GVI(F, G, Ω) reduces to structured variational inequalities which have been well studied in the literatures [5-7].

As it is well known, Proximal Point Algorithm (PPA)[8] is an effective numerical approach to solving classical variational inequalities, however, the study on applying PPA to solve general variational inequalities does not receive much attention. To mention a few, [9] analyzed both the exact and inexact versions of PPA-based algorithms for general variational inequalities. [10] proposed some proximal algorithms for linearly constrained general variational inequalities. Without specific consideration of the special structure of GVI(F, G, Ω), the algorithms proposed in [9] are applicable to solving GVI(F, G, Ω), however, this treatment is obviously not beneficial to developing efficient numerical algorithms. In this paper, inspired by [9-10], we propose a new APPA-based algorithm for GVI(F, G, Ω).

The remainder of the paper is organized as follows. In Section 2, we summarize some basic concepts about variational inequalities. In Section 3, the new APPA method is described formally and its global convergence is proved. Some concluding remarks are given in the last section.

2 Preliminaries

In this section, we present some basic definitions and propositions which will be used in the following discussions. Recall that $P_{\mathcal{K}}[\cdot]$ denotes the projection onto a closed convex set $\mathcal{K} \subset \mathcal{R}^n$ in the Euclidean norm, i.e.,

$$P_{\mathcal{K}}[u] = \arg\min_{v\in\mathcal{K}} \|u - v\|.$$

Some fundamental properties are listed below without proof (see, e.g., [11]).

Lemma 2.1 Let $\mathcal{K}$ be a nonempty closed convex subset of $\mathcal{R}^n$. For any $u, v \in \mathcal{R}^n$ and any $w \in \mathcal{K}$, the following properties hold:

$$(u - P_{\mathcal{K}}[u])^\top (w - P_{\mathcal{K}}[u]) \le 0. \tag{2}$$

$$\|(u - P_{\mathcal{K}}[u]) - (v - P_{\mathcal{K}}[v])\|^2 \le \|u - v\|^2 - \|P_{\mathcal{K}}[u] - P_{\mathcal{K}}[v]\|^2. \tag{3}$$

From (3), we have

$$\|(u - P_{\mathcal{K}}[u]) - (v - P_{\mathcal{K}}[v])\| \le \|u - v\|. \tag{4}$$

By attaching a Lagrange multiplier vector $\lambda \in \mathcal{R}^r$ to the linear equality constraint $Ax + By = b$, problem GVI(F, G, Ω) can be equivalently transformed into the following compact form, denoted by GVI$(M, N, \mathcal{W})$: Find $w^* \in \mathcal{R}^{m+n+r}$, such that

$$M(w^*) \in \mathcal{W} \;\text{ and }\; (w - M(w^*))^\top N(w^*) \ge 0, \quad \forall w \in \mathcal{W}, \tag{5}$$

where

$$w = \begin{pmatrix} x \\ y \\ \lambda \end{pmatrix}, M(w) = \begin{pmatrix} f_1(x) \\ f_2(y) \\ \lambda \end{pmatrix}, N(w) = \begin{pmatrix} g_1(x) - A^\top \lambda \\ g_2(y) - B^\top \lambda \\ Af_1(x) + Bf_2(y) - b \end{pmatrix}, \mathcal{W} = \mathcal{X} \times \mathcal{Y} \times \mathcal{R}^r.$$

It is well known[11] that problem GVI($M, N, \mathcal{W}$) is equivalent to finding the zero point of

$$e(w,\beta) = \begin{pmatrix} e_x(w,\beta) \\ e_y(w,\beta) \\ e_\lambda(w,\beta) \end{pmatrix} = \begin{pmatrix} f_1(x) - P_{\mathcal{X}}[f_1(x) - \beta(g_1(x) - A^\top \lambda)] \\ f_2(y) - P_{\mathcal{Y}}[f_2(y) - \beta(g_2(y) - B^\top \lambda)] \\ \beta[Af_1(x) + Bf_2(y) - b] \end{pmatrix}. \tag{6}$$

We need the following definitions concerning the functions.

Definition 2.1 Let f and g be mappings from $\mathcal{R}^n$ into itself. Then

(a). The mapping f is said to be $g-$monotone if and only if

$$(f(s) - f(t))^\top (g(s) - g(t)) \geq 0, \quad \forall s, t \in \mathcal{R}^n;$$

(b). The mapping f is said to be uniformly $g-$strongly monotone if there exists a positive constant $\mu > 0$ such that

$$(f(s) - f(t))^\top (g(s) - g(t)) \geq \mu \|s - t\|^2, \quad \forall s, t \in \mathcal{R}^n;$$

(c). The mapping f is said to be uniformly $g-$Lipschitz continuous on if there exists a constant $L > 0$ such that

$$\|f(s) - f(t)\| \leq L\|g(s) - g(t)\|, \quad \forall s, t \in \mathcal{R}^n.$$

Throughout this paper, we make the following assumptions:

Assumptions. (A). The mapping f_i is g_i-monotone and g_i-Lipschitz continuous with modulus L_i, $i = 1, 2$.

(B). The solution set of GVI($M, N, \mathcal{W}$), denoted by $\mathcal{W}^*$, is nonempty.

(C). $\mathcal{X}$ and $\mathcal{Y}$ are simple closed convex sets in the sense that the projection onto them is easy to compute. (e.g. the positive orthant, boxed set, ball).

(D). The mapping g_i is nonsingular, i.e., there exists a positive constant ρ_i ($i = 1, 2$) such that

$$\|g_1(u) - g_1(\tilde{u})\| \geq \rho_1 \|u - \tilde{u}\|, \quad \forall u, \tilde{u} \in \mathcal{R}^n,$$

and

$$\|g_2(v) - g_2(\tilde{v})\| \geq \rho_2 \|v - \tilde{v}\|, \quad \forall v, \tilde{v} \in \mathcal{R}^m.$$

With the assumption (A), it is easily verify that

$$\begin{aligned} & (M(w)-M(\tilde{w}))^\top(N(w)-N(\tilde{w})) \\ = & \begin{pmatrix} f_1(x)-f_1(\tilde{x}) \\ f_2(y)-f_2(\tilde{y}) \\ \lambda-\tilde{\lambda} \end{pmatrix}^\top \begin{pmatrix} (g_1(x)-g_1(\tilde{x}))-A^\top(\lambda-\tilde{\lambda}) \\ (g_2(y)-g_2(\tilde{y}))-B^\top(\lambda-\tilde{\lambda}) \\ A(f_1(x)-f_1(\tilde{x}))+B(f_2(y)-f_2(\tilde{y})) \end{pmatrix} \\ = & (f_1(x)-f_1(\tilde{x}))^\top(g_1(x)-g_1(\tilde{x}))+(f_2(y)-f_2(\tilde{y}))^\top(g_2(y)-g_2(\tilde{y})) \\ \geq & \ 0, \end{aligned}$$

which implies that the mapping M in GVI$(M,N,\mathcal{W})$ is $N-$monotone.

3 APPA Method and Global Convergence

In this section, we present the APPA method for solving GVI$(M,N,\mathcal{W})$ and show its global convergence. We first denote

$$R(w)=\begin{pmatrix} \beta(g_1(x)-A^\top\lambda) \\ \beta(g_2(y)-B^\top\lambda) \\ \lambda \end{pmatrix}, \tag{7}$$

and

$$P_k(w)=\begin{pmatrix} f_1(x)+\beta[g_1(x)-g_1(x^k)-A^\top(\lambda-\lambda^k)] \\ f_2(y)+\beta[g_2(y)-g_2(y^k)-B^\top(\lambda-\lambda^k)] \\ \lambda \end{pmatrix}, Q_k(w)=\begin{pmatrix} \beta(g_1(x)-A^\top\lambda) \\ \beta(g_2(y)-B^\top\lambda) \\ \lambda-\lambda^k+\beta(Af_1(x)+Bf_2(y)-b) \end{pmatrix}. \tag{8}$$

According to the assumption (D), it is easy to verify that R is a continuous and nonsingular mapping, i.e., there exists a positive constant ρ_R such that

$$\|R(w)-R(\tilde{w})\| \geq \rho_R\|w-\tilde{w}\|, \quad \forall w,\tilde{w}\in\mathcal{W}. \tag{9}$$

Now we introduce the APPA method as follows:

Algorithm 3.1. The APPA method with summable accuracy

Step 0. Given $\varepsilon>0$, choose $w^0=(x^0,y^0,\lambda^0)^\top\in\mathcal{W}$, $\beta>0$ and set k:=0.

Step 1. If $\|e(w^k,\beta)\|\leq\varepsilon$, then stop. Otherwise, goto Step 2.

Step 2. Generate the next iterate w^{k+1} such that

$$P_k(w^{k+1})\in\mathcal{W}, \quad [w-(P_k(w^{k+1})-\xi^k)]^\top Q_k(w^{k+1})\geq 0, \quad \forall w\in\mathcal{W}, \tag{10}$$

with the inexactness criterion

$$\|\xi^k\|\leq\delta_k\|R(w^k)-R(w^{k+1})\|, \tag{11}$$

where

$$\xi^k = \begin{pmatrix} \xi_x^k \\ \xi_y^k \\ 0 \end{pmatrix}, \quad \delta_k > 0, \quad \sum_{k=0}^{\infty} \delta_k^2 < +\infty. \tag{12}$$

Step 3. With $k = k+1$, go to Step 1.

Lemma 3.1. The GVI (10) is uniformly strongly monotone GVI under the assumption that f_i is g_i-monotone, $i = 1, 2$.

Proof. It follows from (7)-(9) that

$$\begin{aligned}
& (P_k(w) - P_k(\tilde{w}))^\top (Q_k(w) - Q_k(\tilde{w})) \\
= & \begin{pmatrix} f_1(x) - f_1(\tilde{x}) + \beta[g_1(x) - g_1(\tilde{x}) - A^\top(\lambda - \tilde{\lambda})] \\ f_2(y) - f_2(\tilde{y}) + \beta[g_2(y) - g_2(\tilde{y}) - B^\top(\lambda - \tilde{\lambda})] \\ \lambda - \tilde{\lambda} \end{pmatrix}^\top \begin{pmatrix} \beta[g_1(x) - g_1(\tilde{x}) - A^\top(\lambda - \tilde{\lambda})] \\ \beta[g_2(y) - g_2(\tilde{y}) - B^\top(\lambda - \tilde{\lambda})] \\ \lambda - \tilde{\lambda} + \beta[A(f_1(x) - f_1(\tilde{x})) + B(f_2(y) - f_2(\tilde{y}))] \end{pmatrix} \\
= & \beta[f_1(x) - f_1(\tilde{x})]^\top [g_1(x) - g_1(\tilde{x})] + \beta^2 \|g_1(x) - g_1(\tilde{x}) - A^\top(\lambda - \tilde{\lambda})\|^2 \\
& + \beta[f_2(y) - f_2(\tilde{y})]^\top [g_2(y) - g_2(\tilde{y})] + \beta^2 \|g_2(y) - g_2(\tilde{y}) - B^\top(\lambda - \tilde{\lambda})\|^2 + \|\lambda - \tilde{\lambda}\|^2 \\
\geq & \|R(w) - R(\tilde{w})\|^2 \\
\geq & \rho_R^2 \|w - \tilde{w}\|^2,
\end{aligned}$$

which implies the assertion.

Remark 3.1. Therefore, each iteration of Algorithm 3.1 for solving GVI$(M, N, \mathcal{W})$ consists of solving the uniformly strongly monotone GVI (10), which is easier than the original GVI (5) and can be solved by existing methods, such as [3].

Remark 3.2. According to (6), the GVI (10) is characterized by the projection equation:

$$P_k(w^{k+1}) - \xi^k = P_{\mathcal{W}}[P_k(w^{k+1}) - \xi^k - Q_k(w^{k+1})],$$

which is equivalent to the equalities

$$\left\{ \begin{array}{l} f_1(x^{k+1}) + \beta[g_1(x^{k+1}) - g_1(x^k) - A^\top(\lambda^{k+1} - \lambda^k)] - \xi_x^k = P_{\mathcal{X}}[f_1(x^{k+1}) - \beta(g_1(x^k) - A^\top \lambda^k) - \xi_x^k], \\ f_2(y^{k+1}) + \beta[g_2(y^{k+1}) - g_2(y^k) - B^\top(\lambda^{k+1} - \lambda^k)] - \xi_y^k = P_{\mathcal{Y}}[f_2(y^{k+1}) - \beta(g_2(y^k) - B^\top \lambda^k) - \xi_y^k], \\ \lambda^{k+1} = \lambda^k - \beta[A f_1(x^{k+1}) + B f_2(y^{k+1}) - b] \end{array} \right. \tag{13}$$

In the following we study some convergence properties of the proposing APPA method. To prove the convergence conveniently, we let $v \in (0, 1)$ and denote $\eta_k = \delta_k / v$. Then, we have

$$\delta_k = v\eta_k \ \text{ with } \ v \in (0, 1), \ \ \eta_k > 0 \ \text{ and } \ \sum_{k=1}^{\infty} \eta_k^2 < +\infty. \tag{14}$$

Lemma 3.2. Let$\{w^k\}$ be the sequence generated by Algorithm 3.1. Then, for any $w^* \in \mathcal{W}^*$, we have

$$\begin{aligned} & \|R(w^{k+1}) - R(w^*)\|^2 \\ \leq \ & (1 + 2\eta_k^2)\|R(w^k) - R(w^*)\|^2 - (1 - v^2 - 2\eta_k^2)\|R(w^k) - R(w^{k+1})\|^2. \end{aligned} \tag{15}$$

Proof. By using the first equation of (13) and $e_x(w^*, \beta) = 0$, we have

$$\begin{aligned} & \beta(g_1(x^{k+1}) - A^\top \lambda^{k+1}) \\ = \ & P_{\mathcal{X}}[f_1(x^{k+1}) - \beta(g_1(x^k) - A^\top \lambda^k) - \xi_x^k] - [f_1(x^{k+1}) - \beta(g_1(x^k) - A^\top \lambda^k) - \xi_x^k] \end{aligned} \tag{16}$$

and

$$\begin{aligned} & -\beta(g_1(x^*) - A^\top \lambda^*) \\ = \ & -P_{\mathcal{X}}[f_1(x^*) - \beta(g_1(x^*) - A^\top \lambda^*)] + [f_1(x^*) - \beta(g_1(x^*) - A^\top \lambda^*)] \end{aligned} \tag{17}$$

Setting $u = f_1(x^*) - \beta(g_1(x^*) - A^\top \lambda^*)$ and $v = f_1(x^{k+1}) - \beta(g_1(x^k) - A^\top \lambda^k) - \xi_x^k$ in (3), it follows from (16) and (17) that

$$\begin{aligned} & \beta^2\|g_1(x^{k+1}) - g_1(x^*) - A^\top(\lambda^{k+1} - \lambda^*)\|^2 \\ \leq \ & \|\beta(g_1(x^k) - g_1(x^*) - A^\top(\lambda^k - \lambda^*)) + \xi_x^k - (f_1(x^{k+1}) - f_1(x^*))\|^2 \\ & -\|P_{\mathcal{X}}[f_1(x^*) - \beta(g_1(x^*) - A^\top \lambda^*)] - P_{\mathcal{X}}[f_1(x^{k+1}) - \beta(g_1(x^k) - A^\top \lambda^k) - \xi_x^k]\|^2 \\ = \ & \|\beta(g_1(x^k) - g_1(x^*) - A^\top(\lambda^k - \lambda^*)) + \xi_x^k - (f_1(x^{k+1}) - f_1(x^*))\|^2 \\ & -\|(f_1(x^*) - f_1(x^{k+1})) + \beta[(g_1(x^k) - g_1(x^{k+1})) - A^\top(\lambda^k - \lambda^{k+1})] + \xi_x^k\|^2 \\ \leq \ & \|\beta[g_1(x^k) - g_1(x^*) - A^\top(\lambda^k - \lambda^*)] + \xi_x^k\|^2 - \|\beta[(g_1(x^k) - g_1(x^{k+1})) - A^\top(\lambda^k - \lambda^{k+1})] + \xi_x^k\|^2 \\ & -2\beta(f_1(x^{k+1}) - f_1(x^*))^\top A^\top(\lambda^* - \lambda^{k+1}) \\ \leq \ & [(1 + 2\eta_k^2)\|\beta[g_1(x^k) - g_1(x^*) - A^\top(\lambda^k - \lambda^*)]\|^2 + (1 + \tfrac{1}{2\eta_k^2})\|\xi_x^2\|^2] - [(1 - \tfrac{1}{2\eta_k^2})\|\xi_x^2\|^2 \\ & -(1 - 2\eta_k^2)\|\beta[(g_1(x^k) - g_1(x^{k+1})) - A^\top(\lambda^k - \lambda^{k+1})]\|^2] - 2\beta(f_1(x^{k+1}) - f_1(x^*))^\top A^\top(\lambda^* - \lambda^{k+1}) \end{aligned} \tag{18}$$

where the second inequality follows from the g_1-monotonicity of f_1, and the last inequality follows from Cauchy-Schwartz inequality. Similarly, we obtain

$$\begin{aligned} & \beta^2\|g_2(y^{k+1}) - g_2(y^*) - B^\top(\lambda^{k+1} - \lambda^*)\|^2 \\ \leq \ & [(1 + 2\eta_k^2)\|\beta[g_2(y^k) - g_2(y^*) - B^\top(\lambda^k - \lambda^*)]\|^2 + (1 + \tfrac{1}{2\eta_k^2})\|\xi_y^2\|^2] - [(1 - \tfrac{1}{2\eta_k^2})\|\xi_y^2\|^2 \\ & -(1 - 2\eta_k^2)\|\beta[(g_2(y^k) - g_2(y^{k+1})) - B^\top(\lambda^k - \lambda^{k+1})]\|^2] - 2\beta(f_2(y^{k+1}) - f_2(y^*))^\top B^\top(\lambda^* - \lambda^{k+1}) \end{aligned} \tag{19}$$

Since $\lambda^{k+1} = \lambda^k - \beta[Af_1(x^{k+1}) + Bf_2(y^{k+1}) - b]$, we have

$$\begin{aligned} & \|\lambda^{k+1} - \lambda^*\|^2 \\ = \ & \|(\lambda^k - \lambda^*) - \beta[Af_1(x^{k+1}) + Bf_2(y^{k+1}) - b]\|^2 - \|(\lambda^k - \lambda^{k+1}) - \beta[Af_1(x^{k+1}) + Bf_2(y^{k+1}) - b]\|^2 \\ = \ & \|\lambda^k - \lambda^*\|^2 - \|\lambda^k - \lambda^{k+1}\|^2 - 2\beta(Af_1(x^{k+1}) + Bf_2(y^{k+1}) - b)^\top(\lambda^{k+1} - \lambda^*) \end{aligned} \tag{20}$$

Adding (18)-(19) and using $Af_1(x^*)+Bf_2(y^*)=b$, we have

$$\begin{aligned}
&\beta^2\|g_1(x^{k+1})-g_1(x^*)-A^\top(\lambda^{k+1}-\lambda^*)\|^2+\beta^2\|g_2(y^{k+1})-g_2(y^*)-B^\top(\lambda^{k+1}-\lambda^*)\|^2\|\lambda^{k+1}-\lambda^*\|^2\\
\le\;& (1+2\eta_k^2)\|R(w^k)-R(w^*)\|^2-(1-2\eta_k^2)\|R(w^k)-R(w^{k+1})\|^2+\frac{1}{\eta_k^2}(\|\xi_x^2\|^2+\|\xi_y^2\|^2)\\
\le\;& (1+2\eta_k^2)\|R(w^k)-R(w^*)\|^2-(1-v^2-2\eta_k^2)\|R(w^k)-R(w^{k+1})\|^2,
\end{aligned}$$

where the first inequality follows from the definition of R, and the second inequality follows from the inexactness criterion (12), (14). The assertion is a compact form of the above inequality and then the proof is complete.

Lemma 3.3. Let $\{w^k\}$ be the sequence generated by Algorithm 3.1. Then, we have

$$\|e(w^{k+1},\beta)\|\le 3(1+v\eta_k)\|R(w^k)-R(w^{k+1})\|. \tag{21}$$

Proof. From the definition of $e(w,\beta)$, we know that

$$\begin{aligned}
&\|e_x(w^{k+1},\beta)\|\\
=\;& \|f_1(x^{k+1})-P_{\mathcal{X}}[f_1(x^{k+1})-\beta(g_1(x^{k+1})-A^\top\lambda^{k+1})]\|\\
=\;& \|P_{\mathcal{X}}[f_1(x^{k+1})-\beta(g_1(x^k)-A^\top\lambda^k)-\xi_x^k]+\xi_x^k+\beta[g_1(x^k)-g_1(x^{k+1})-A^\top(\lambda^k-\lambda^{k+1})]\\
&-P_{\mathcal{X}}[f_1(x^{k+1})-\beta(g_1(x^{k+1})-A^\top\lambda^{k+1})]\|\\
\le\;& \|\xi_x^k+\beta[g_1(x^k)-g_1(x^{k+1})-A^\top(\lambda^k-\lambda^{k+1})]\|\\
\le\;& \|\xi_x^k\|+\|\beta[g_1(x^k)-g_1(x^{k+1})-A^\top(\lambda^k-\lambda^{k+1})]\|\\
\le\;& (1+\delta_k)\|R(w^k)-R(w^{k+1})\|.
\end{aligned}$$

where the first inequality follows from (4). Similarly, we have

$$\begin{aligned}
&\|e_y(w^{k+1},\beta)\|\\
\le\;& (1+\delta_k)\|R(w^k)-R(w^{k+1})\|.
\end{aligned}$$

From the third equality of (13), we obtain

$$\begin{aligned}
&\|e_\lambda(w^{k+1},\beta)\|\\
=\;& \beta\|Af_1(x^{k+1})+Bf_2(y^{k+1})-b\|\\
=\;& \|\lambda^{k+1}-\lambda^k\|\\
\le\;& \|R(w^k)-R(w^{k+1})\|.
\end{aligned}$$

The assertion is followed by the above three inequalities and this complete the proof.

Theorem 3.1. The sequence$\{w^k\}$ generated by Algorithm 3.1 converges to a point in $\mathcal{W}^*$.

Proof. Since $\lim_{k\to\infty}\eta_k = 0$, without loss of the generality, we can assume that $v\eta_k < 1/3$ and $\eta_k^2 \le (1-v^2)/4$ for all k. Thus, (15) and (21) can be modified to

$$\|R(w^{k+1}) - R(w^*)\|^2 \le (1+2\eta_k^2)\|R(w^k) - R(w^*)\|^2 - \frac{1-v^2}{2}\|R(w^k) - R(w^{k+1})\|^2, \tag{22}$$

and

$$\|e(w^{k+1},\beta)\| \le 4\|R(w^k) - R(w^{k+1})\|, \tag{23}$$

respectively. Setting $\zeta_k = 2\eta_k^2$, then by (12), we have

$$\sum_{k=0}^{\infty}\zeta_k < +\infty.$$

Therefore we have $\prod_{k=0}^{\infty}(1+\zeta_k) < +\infty$. We denote

$$C_s = \sum_{k=0}^{\infty}\zeta_k \quad \text{and} \quad C_p = \prod_{k=0}^{\infty}(1+\zeta_k).$$

Let $w^* \in \mathcal{W}^*$. From (22), we obtain

$$\|R(w^{k+1}) - R(w^*)\|^2 \le \prod_{i=j}^{k}(1+\zeta_i)\|R(w^j) - R(w^*)\|^2 \le C_p\|R(w^j) - R(w^*)\|^2, \quad \forall j \le k. \tag{24}$$

Therefore, there exists a constant $C > 0$ such that

$$\|R(w^k) - R(w^*)\| \le C, \quad \forall k \ge 0, \tag{25}$$

thus by the nonsingularity of R (see(9)), it is easy to verify that the sequence $\{w^k\}$ is bounded. Combining (22) and (25), we have

$$\begin{aligned} & \frac{1-v^2}{2}\|R(w^k) - R(w^{k+1})\|^2 \\ \le\ & \|R(w^0) - R(w^*)\|^2 + \sum_{k=0}^{\infty}\zeta_k\|R(w^k) - R(w^*)\|^2 \\ \le\ & C + C\sum_{k=0}^{\infty}\zeta_k \\ \le\ & (1+C_s)C. \end{aligned}$$

Recall that $v \in (0,1)$. Consequently, we have

$$\lim_{k\to\infty}\|R(w^k) - R(w^{k+1})\| = 0. \tag{26}$$

From (23) and (26), we obtain

$$\lim_{k\to\infty} e(w^k,\beta) = 0. \tag{27}$$

Let $\tilde{w}$ be a cluster point of $\{w^k\}$ and the subsequence $\{w^{k_j}\}$ converges to $\tilde{w}$. Because $e(w, \beta)$ is continuous, we have

$$e(\tilde{w}, \beta) = \lim_{j \to \infty} e(w^{k_j}, \beta) = 0.$$

So $\tilde{w} \in \mathcal{W}^*$. By a similar analysis as Theorem 2.1 of [9], we can show that the whole sequence $\{w^k\}$ converges to $\tilde{w}$, a solution of GVI$(M, N, \mathcal{W})$. This completes the proof.

4 Conclusions

In this paper, we presented a new APPA method for general variational inequalities with separable operators. Each iteration of the method consists of solving a proximally regularized subproblem, and this subproblem is allowed to be solved approximately subject to some inexactness criterion. Global convergence of the new method is proved under mild assumptions.

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