Numerical solution of optimal control problems by an iterative scheme

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Abstract. This paper presents an iterative approach based on hybrid of perturbation and parametrization methods for obtaining approximate solutions of optimal control problems. The results reveal this method is very effective. and it produces approximate solutions with high precision.

Keywords: Optimal control problem (OCP); Ordinary differential equation; Perturbation method; nonlinear control.

1 Introduction

Optimal control problems arise in a wide variety of disciplines. Apart from traditional areas such as aerospace engineering, robotics and chemical engineering, optimal control theory has also been used with great success in areas as diverse as economics to biomedicine. It is well known that generally optimal control problems are difficult to solve. In particular, their analytical solutions are in many cases out of the question.Thus, numerical methods are needed for solving many of these real world problems. Some authors proposed new method for solving optimal control problem based on Pontryagin's maximum principle or Hamilton-Jacobi-Bellman equation. Such as the relaxed descent method, variation of Extermal, quasilinearization, gradiant projection method [1–5]. Some author prefers to transform the problem to new problem which is easy for solving. In [6] the problem is solved by converting the problem to differential inclusion form. In [7] the problem is converted to measure space and then solved and in [8] the problem is solved by genetic algorithm, Others deal with the optimal control problem directly. For example see [9, 11–14].
The current work intends to combine the method of parametrization, [9, 10, 15], and homotopy perturbation method, [17, 18], both are successful methods for solving

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some classes of optimal control problems and differential-integral equations, respectively. In this paper we solve the optimal control problem via combine perturbation and parametrization method, in which control variable is a continuous polynomial. The rest of the paper organized as follows: In section 2 we argue about method of solution . In section 3 we introduce algorithm of this scheme. In section 4 we express convergency of this scheme. In section 5 computational results have been represented. Section 6 is conclusion of the paper.

2 Method of solution

In this section we apply this method for providing an iterative scheme to find approximate solution of optimal control problems. First we consider $\{q_k(t)\}$ as a basis, which is dense in the space of $C([0,1])$. The continuous control function $u(t)$ can be approximated by a finite combination from elements of this basis [16]. Now we consider optimal control problem in following minimization problem:

$$\text{Minimize} \;\; J(x,u) = \int_0^T f_\circ(t, x(t), u(t))dt, \tag{2.1}$$

subject to:

$$\dot{x}(t) = f(t, x(t), u(t)). \tag{2.2}$$

Initial and final conditions are:

$$x(0) = x_0, \quad x(T) = x_T. \tag{2.3}$$

Where $f_\circ \in C([0,T] \times \mathbb{R} \times \mathbb{R})$. Thereafter, without loss of generality we suppose $T = 1$. For solving the optimal control problem by homotopy perturbation method we construct a convex homotopy [17] as follows:

$$(1-p)(\dot{X}(t) - \dot{x_0}(t)) + p(\dot{X}(t) - f(t, X(t), u(t))) = 0, \; p \in [0,1]. \tag{2.4}$$

Where the control in kth iteration express as follows:

$$u(t) = \sum_{j=0}^{k} c_j q_j(t), \tag{2.5}$$

and a power series

$$X(t) = X_0 + pX_1 + p^2 X_2 + \cdots , \tag{2.6}$$

where $X_i(t, c_0, c_1, \cdots, c_k), i = 1, 2, \cdots$, are unknown functions which must be determined. Thus the coefficients of p with the same power lead to

$$\begin{cases} p^0: \ \dot{X}_0(t) - \dot{x_0}(t) = 0, \\ p^1: \ \dot{X}_1(t, c_0, c_1, \cdots, c_k) - f(t, u(t), X_0(t)) = 0, \\ p^2: \ \dot{X}_2(t, c_0, c_1, \cdots, c_k) - f(t, u(t), X_1(t, c_0, c_1, \cdots, c_k)) = 0, \\ \vdots \end{cases} \tag{2.7}$$

The approximate solutions of (2.2), with initial condition which are dependent on the parameters $c_j, \ j = 0, 1, \cdots, k$, can be obtained by setting $p = 1$ as follows:

$$x(t, c_0, c_1, \cdots, c_k) = \lim_{p \to 1} X = X_0 + X_1 + X_2 + \cdots. \tag{2.8}$$

With substituting (2.5) and (2.8) in (2.1) we obtain approximate solution of optimal control problem as follows:

$$\min_{(c_0, c_1, \cdots, c_k)} \quad J_k(c_0, c_1, \cdots, c_k) = \int_0^T f_\circ(t, x(t, c_0, c_1, \cdots, c_k), \sum_{j=0}^{k} c_j q_j(t))dt, \tag{2.9}$$

subject to:

$$x(T, c_0, c_1, \cdots, c_k) = x_T.$$

Assuming J_k^* as optimal value of (2.9) in kth iteration, a stopping criteria may be considered as follows:

$$|J_k^* - J_{k-1}^*| < \epsilon, \tag{2.10}$$

for a prescribed small positive number ϵ that should be chosen according to the accuracy desired. The above results has been summarized in an algorithm. This algorithm is presented in two stages, initialization step and main steps.

3 Algorithm of the scheme

In this section, we are going to propose an algorithm on the basis of the above discussions. This algorithm is based on in two stages, initialization step and main steps.

Initialization step: Choose $\epsilon > 0$ for the accuracy desired and set $k = 1$, and go to the main steps.

Main steps:

Step 1. Set $u(t)$ by (2.5), and go to Step 2.

Step 2. Compute $X_n(t, c_0, c_1, \cdots, c_k)$ by (2.7) and go to Step 3.

Step 3. Compute $J_k^* = \inf J_k$ in (2.9) by (2.8), if $k = 1$ go to step 5, otherwise go to step 4 .
Step 4. If the stopping criteria (2.10) holds, stop; Otherwise, go to step 5.
step 5. $k = k + 1$ and go step 1.

4 Convergency of the scheme

Definition: the pair(x, u) is called an admissible pair, if it satisfies in (2.2)-(2.3). We define φ as the set of admissible control functions and ξ as the set of admissible pairs.
Define ξ^n and ξ_k^n as follows:

$$\xi^n = \{(x_n(t, u), u) | u \in \varphi\}, \tag{4.1}$$

$$\xi_k^n = \{(x_n(t, u), u) | u = \sum_{i=0}^{k} c_i t^i, \quad u \in \varphi, \quad x_n(t, u) = \sum_{i=0}^{n} X_i, c_i \in \mathbb{R}\}. \tag{4.2}$$

The coefficients $\{c_i\}_{i=0}^k$ are unknown and computed in the Step 3 by minimization of J on ξ_k^n, from (4.1) we define $x_n(t, u) = \sum_{i=0}^n X_i$ which is the obtained solution by homotopy perturbation method from equation below:

$$\dot{x} = f(t, x, u), \tag{4.3}$$

$$x(0) = x_0.$$

At first we express the following theorems.

Theorem 4.1. Suppose that X and Y be Banach space and $N : X \to Y$ is a contraction nonlinear mapping, that is

$$\forall \;\; v, \tilde{v}; \qquad \| N(v) - N(\tilde{v}) \| \leq \gamma \| v - \tilde{v} \|, \quad 0 < \gamma < 1.$$

Which according to Banach's fixed point theorem, having the fixed point x, that is $N(x) = x$.

The sequence generated by the homotopy perturbation method will be regarded as

$$x_n = N(x_{n-1}), \quad x_{n-1} = \Sigma_{i=0}^{n-1} X_i, \quad n = 1, 2, 3 \cdots$$

and suppose that $X_0 = x_0 \in B_r(x)$ where $B_r(x) = \{y \in X | \; |y - x| < r\}$, then we have the following statements:

$$(i): \quad \| x_n - x \| \leq \gamma^n \| x_0 - x \|$$

$$(ii): \quad x_n \in B_r(x)$$

$$(iii):\ lim_{n\to\infty} x_n = x$$

If we define $\alpha_k^n = \inf_{\xi_k^n} J$ and also assume $\inf_{\xi^n} J$ is finite, unique and equal to α^n, then we will have:

Theorem 4.2. The following relation is hold:

$$\alpha_1^n \geq \alpha_2^n \geq \cdots \geq \alpha_k^n \geq \cdots \geq \alpha^n = \inf_{\xi^n} J.$$

Proof: By definition ξ_k^n

$$\xi_1^n \subseteq \xi_2^n \subseteq, \cdots, \subseteq \xi_k^n \cdots \subseteq \xi^n. \square$$

Theorem 4.3. $lim_{k\to\infty}\ \alpha_k^n = \alpha^n$ in which $\alpha^n = \inf_{\xi^n} J$.
Proof: Since $\{\alpha_k^n\}$ is a non-increasing and bounded sequence, then it is convergent. Assume $\tilde{\alpha}$ be the limit non-increasing sequence $\{\alpha_k^n\}$, if $\tilde{\alpha} > \alpha_n$, then $\epsilon = \frac{\tilde{\alpha}-\alpha_n}{2} > 0$ and there exist $(x_n(.), u(.)) \in \xi_k^n$, such that $\tilde{\alpha} > J(x_n(.), x(.))$ which is contradiction with the continuity of f and density of polynomials in $C(I)$.$\square$

Theorem 4.4. $\lim_{n\to\infty}\ lim_{k\to\infty}\alpha_k^n = \alpha,\ inwhich\ \alpha = inf_{(x,u)\in\xi} J(x,u) \equiv J(x^*, u^*)$

Proof: Since the polynomials are dense in $C(I)$, if we define $u_k = \Sigma_{i=0}^k \bar{c}_i t^i$ therefore we have

$$lim_{k\to\infty} \mid u_k - u^* \mid = 0.$$

Also by theorem 4.1 and continuity J we will have

$$lim_{n\to\infty} \mid x_n - x^* \mid = 0.$$

and by the continuity of, f

$$lim_{n\to\infty}\ lim_{k\to\infty} \inf \int_0^T f(t, x_n, u_k)dt = \int_0^T f(t, x^*, u^*)dt.\ \square$$

5 Computational results

In this section we apply hybrid perturbation and parametrization method for obtaining approximate solutions of optimal control problems. In all examples, monomial functions $\{t^k\}$ have been considered as dense basis of $C([0,1])$. all computations were carried out by MATLAB 7.5.
Example 5.1. Consider the following optimal control problem [19] which is minimization of the functional

$$J(x,u) = \int_0^1 (x(t)^2 + u(t)^2)dt,$$

subject to:

$$\dot{x} = u(t).$$

In which $x(0) = 1$ but $x(1)$ is undetermined.
The exact optimal trajectory and control functions are $x(t) = \frac{\cosh(1-t)}{\cosh 1}$ and $u(t) = -\frac{\sinh(1-t)}{\cosh 1}$, respectively, and the exact objective value is $J^*(x,u) = 0.7616$.
By applying the proposed perturbation method we consider that $X_i = 0, i = 3, 4, \cdots$, considering $\epsilon = 0.0003$, the computed results of applying algorithm have been shown in Table.1. Also the obtained approximate and exact optimal control and trajectory have been shown in Figures 1 and 2.

k	n	J_k^*
1	2	0.7618
2	2	0.7616

Table.1. Numerical results in Example 5.1.

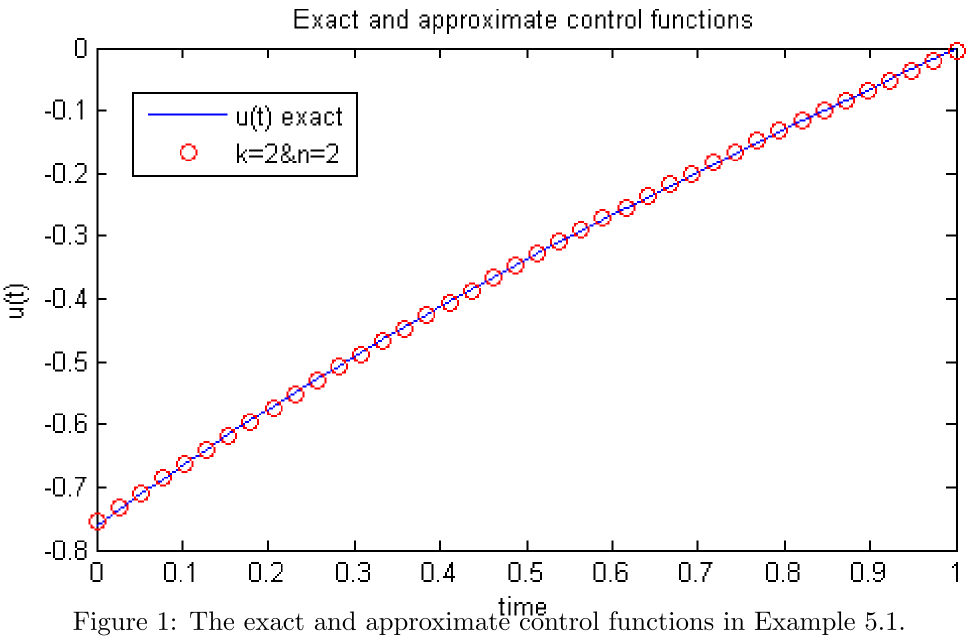

Figure 1: The exact and approximate control functions in Example 5.1.

Example 5.2. Consider the following optimal control problem [8]:

$$\text{Minimize} \quad J(x,u) = \int_0^1 u^2(t)dt$$

subject to:

$$\dot{x} = x^2(t) + u(t).$$

Initial and final conditions are:

$$x(0) = 0, x(1) = 0.5.$$

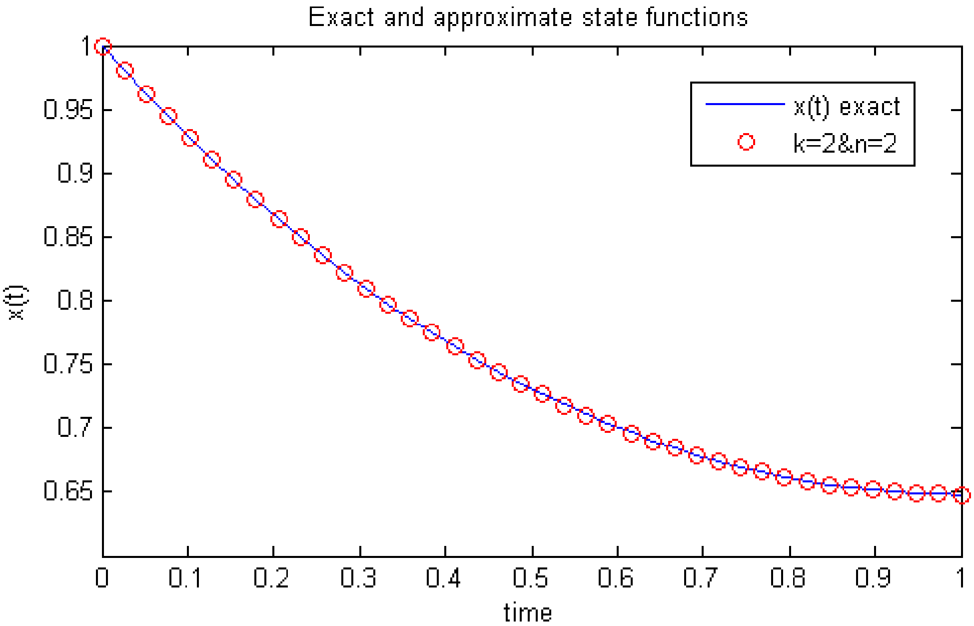

Figure 2: The exact and approximate state functions in Example 5.1.

The exact objective value is $J^*(x,u) = 0.1790$. Choosing $\epsilon = 0.0002$, results of applying the given algorithm are presented in Table.2. Also, the approximate optimal control and trajectory may be seen in Fig.3 and Fig.4.

k	n	J_k^*
1	7	0.1797
2	7	0.1793
3	7	0.1792

Table.2. Numerical results in Example 5.2.

Example 5.3. In this example the following optimal control problem is considered

$$\text{Minimize} \quad J(x,u) = \int_0^1 u^2(t)dt$$

subject to:

$$\dot{x} = \frac{1}{2}x^2(t)\sin(x) + u(t).$$

Initial and final conditions are:

$$x(0) = 0, x(1) = 0.5.$$

The results of applying the algorithm have been shown in Table.3. The approximate optimal control and trajectory may be seen in Fig.5 and Fig.6.

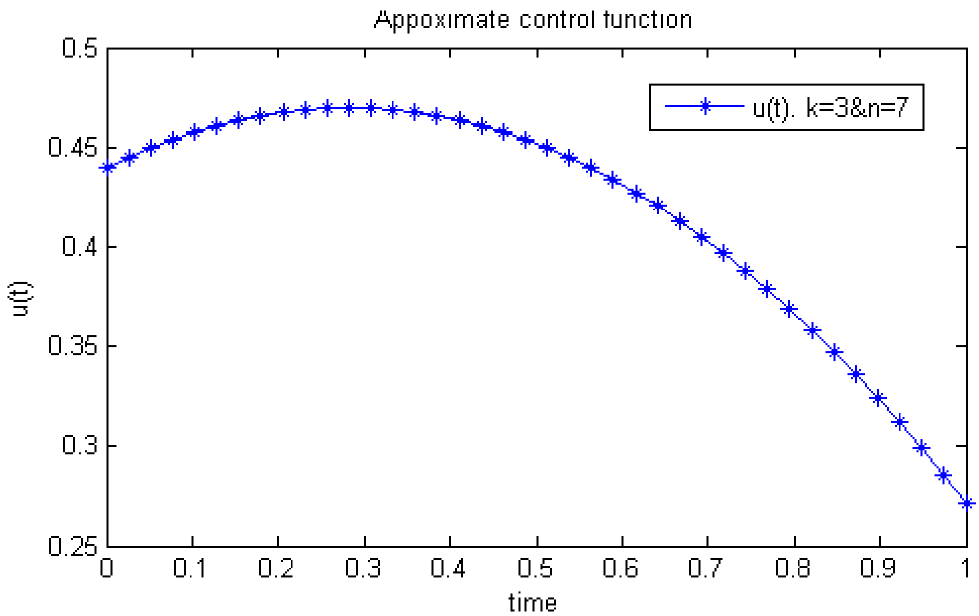

Figure 3: The approximate control function in Example 5.2.

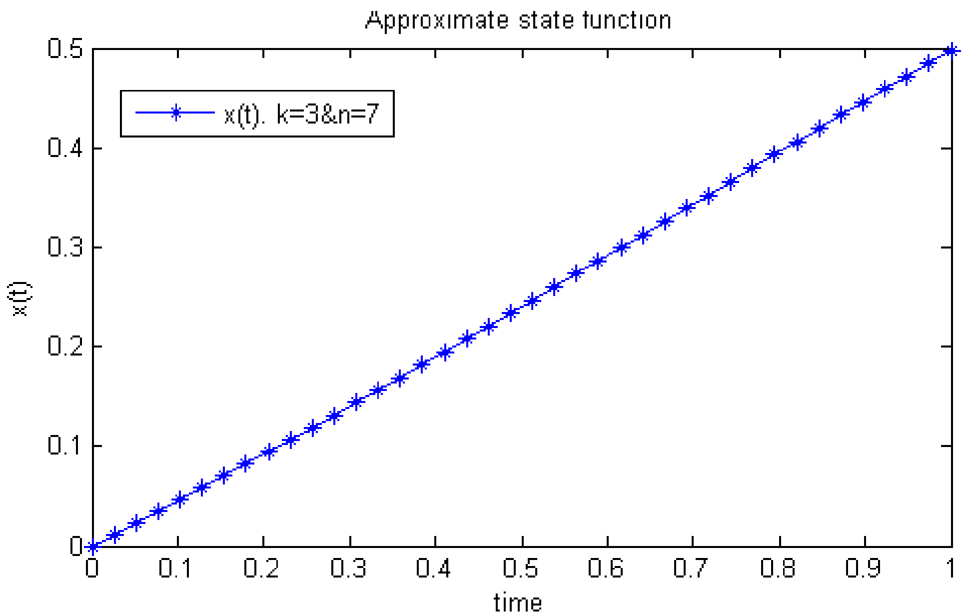

Figure 4: The approximate state function in Example 5.2.

k	n	J_k^*
1	4	0.2236
2	4	0.2235
3	4	0.2235

Table.3. Numerical results in Example 5.3.

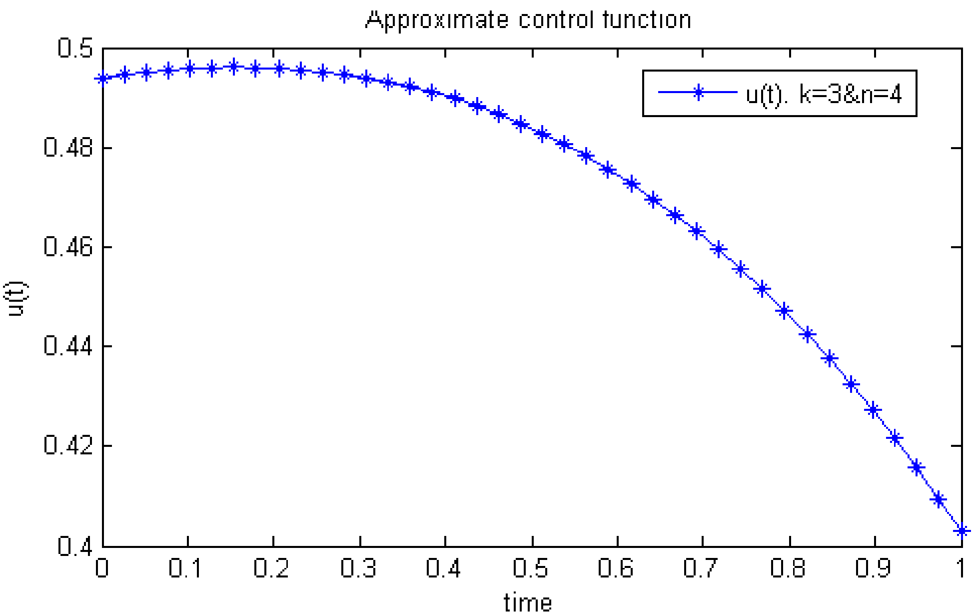

Figure 5: The approximate control function in Example 5.3.

Example 5.4. In this example a system of optimal control problem is considered [20] as follows:

$$\text{Minimize} \quad J(x,u) = \int_0^1 u^2(t)dt$$

subject to:

$$\dot{x} = y(t),$$

$$\dot{y} = u(t).$$

Initial and final conditions are:

$$x(0) = 0, \quad y(0) = 0,$$

$$x(1) = 0.1, \; y(1) = 0.3.$$

The approximate objective value is $J(x,u) = 0.1268$ which obtained by measure theory, by applying the perturbation method we consider that $X_i = Y_i = 0, i =$

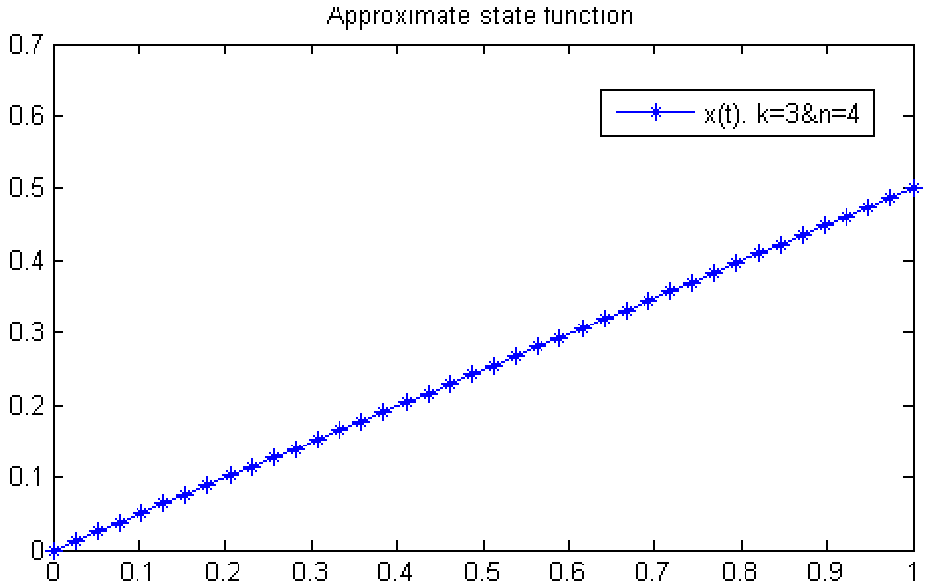

Figure 6: The approximate state function in Example 5.3.

$3, 4, \cdots$, the computed results of applying algorithm have been shown in Table.4. Also, one can observe the approximate optimal trajectory and control functions which is obtained in some iterations of the given algorithm in Fig.7,8 and Fig.9.

k	n	J_k^*
1	2	0.1200
2	2	0.1200

Table.4. Numerical results in Example 5.4.

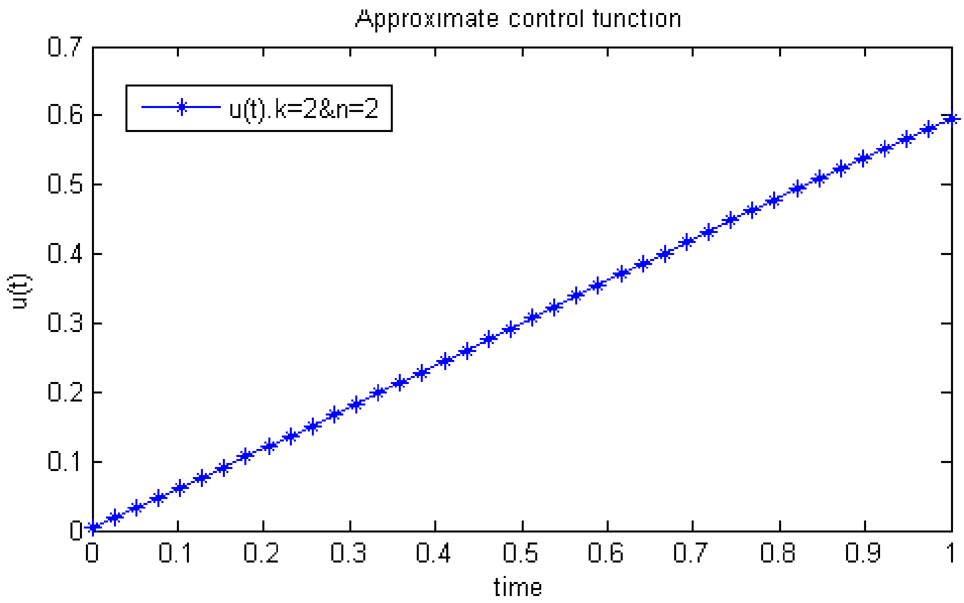

Figure 7: The approximate control function in Example 5.4.

6 Conclusion

In this article, the perturbation method and parametrization approach are combined for solving optimal control problems. We suggest the procedure which is simple and effective and some numerical results show that the given scheme can produce the approximate solutions with high precision.

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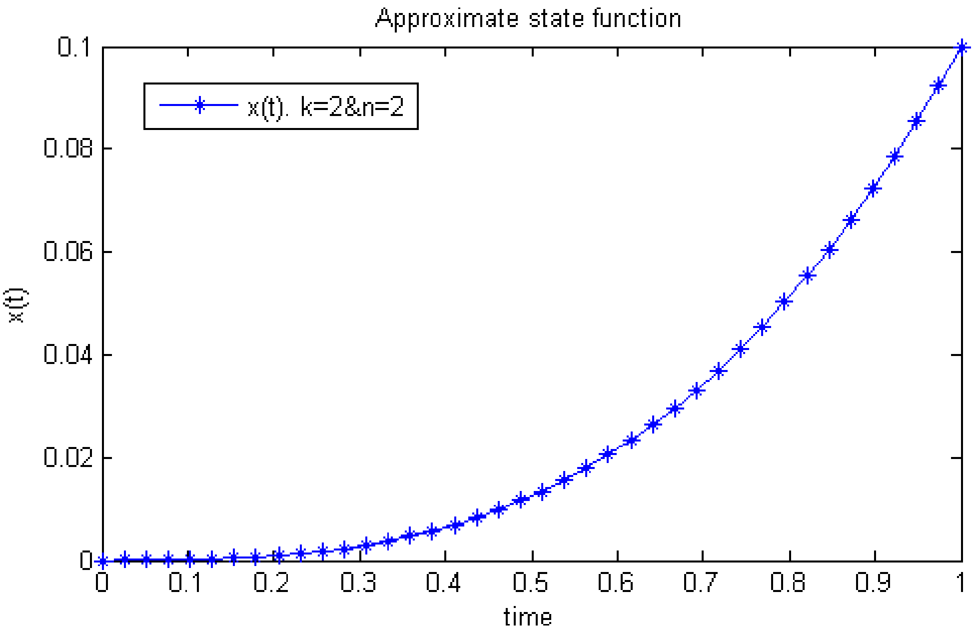

Figure 8: The approximate state function in Example 5.4.

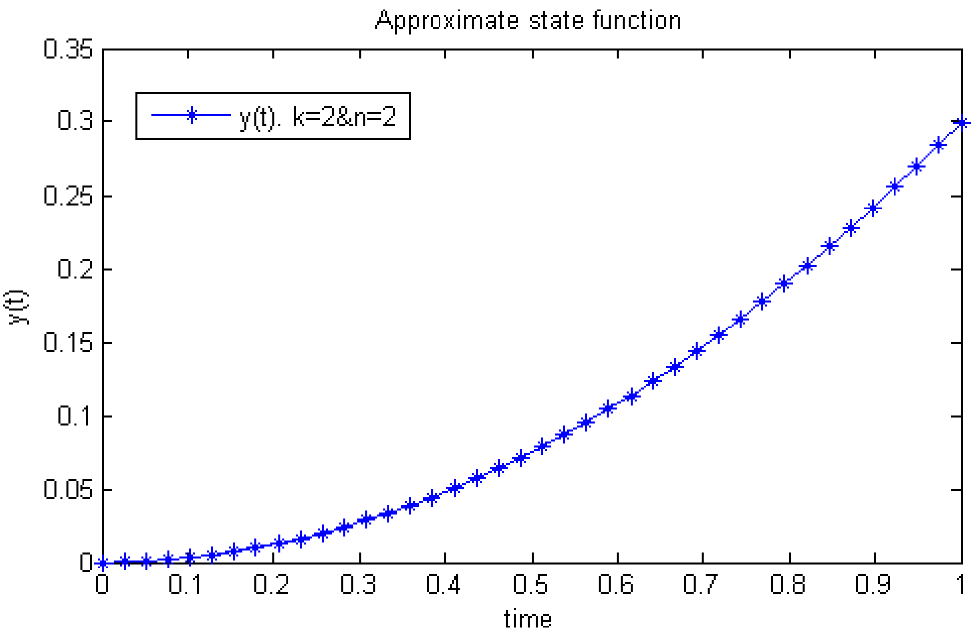

Figure 9: The approximate state function in Example 5.4.

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