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Strictly sensitivity analysis for linear programming problems with upper bounds

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Abstract

In this paper, first we define strictly set and optimal partition for linear programming problems with upper bounds which are different from standard linear programming problems. Then, we study strictly set and optimal partition sensitivity analysis for these problems. We consider the case when variations occur in the right-hand-side of the constraints and the coefficients of the objective function simultaneously. We want to find the range of the parameter variations such that strictly set and optimal partition remain invariant and then present computable auxiliary problems to identify the invariancy intervals. We state the relation between simultaneous and independent perturbations for both sensitivity analysis. We illustrate the results by some numerical examples.

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1 Introduction

Sensitivity analysis is a basic tool for studying perturbations in optimization problems and it is still focus of research even for linear programming problems. Perturbations

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occur due to calculation errors or just to answer managerial questions " What if $\cdots$". Sensitivity analysis in simplex method is well developed on the foundation of optimal basis [3, 4, 11] and it is based on the non-degeneracy assumption of the optimal basis. However, in case of degeneracy, one gets different ranges due to alternative optimal bases [4]. In other hand, most interior point methods produce a solution which converges to an optimal solution. Some additional computations enable us to get an exact optimal basic or non-basic solution [10]. Interior point method has been studied for linear programming problems with bounded variables [2]. However, since sensitivity analysis using an optimal basis can not be applied to an optimal non-basic solution, another method for sensitivity analysis has been suggested as Strictly Sensitivity Analysis. In this context, one wants to find the range of parameter variations where for each parameter value in this range, an optimal solution exists with exactly the same set of positive variables as for the current optimal solution, which shows what variables can be chosen while optimality holds. Thus in this situation the variables suitable to the current position of the decision maker.

In this paper, we define strictly set for a solution of problem with upper bounds. Then we extend the Goldman-Tucker Theorem to these problems which guarantees a strictly complementary solution. So, we will studied the sensitivity analysis for the strictly set and the strictly complementary solution.

This paper is organized as follows. In section 2, we give some duality theorems and discuss existence of strictly complementary solution for linear programming with upper bounds. In section 3, we describe perturbed problem and give definitions of strictly set and optimal partition for linear programming problems with upper bounds. In section 4, we briefly investigate optimal basis invariancy. In section 5, we study strictly set sensitivity analysis with simultaneous and independent perturbations and their relations. In section 6, we state optimal partition sensitivity analysis with simultaneous and independent perturbations. In section 7, we demonstrate the results by some examples and we conclude the paper in section 8.

2 Preliminaries

Consider the following linear programming with upper bounds

$$\begin{array}{rl} \min & z = \mathbf{c}^T\mathbf{x} \\ \text{s.t.} & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & 0 \le \mathbf{x} \le \mathbf{u}, \end{array} \qquad (P)$$

where $\mathbf{c}, \mathbf{u}, \mathbf{x} \in \mathbb{R}^n; \mathbf{b} \in \mathbb{R}^m$ and $\mathbf{A} \in \mathbb{R}^{m\times n}$ is a matrix with full row rank and $\mathbf{u} \in \mathbb{R}^n$ is upper bound vector of the decision variables $\mathbf{x}$. Thus there exists an $m \times m$, sub-matrix $\mathbf{B}$ which is non-singular.

Let $\mathbf{A} = [A_{.1}, A_{.2}, \ldots, A_{.n}]$, which $A_{.j}$ denotes the jth column of matrix $\mathbf{A}$. Let $B = \{B_1, B_2, \ldots, B_m\} \subseteq \{1, 2, \ldots, n\}$ be a subset of the index set of the columns of matrix $\mathbf{A}$ such that $B = \{A_{.B_1}, A_{.B_2}, \ldots, A_{.B_m}\}$ is a non-singular matrix. Let

$N_1 \cup N_2 = \{1, 2, \ldots, n\} \setminus B$, where $N_1 = \{j : \ x_j = 0\}$ and $N_2 = \{j : \ x_j = u_j\}$ are called the index sets of non-basic variables which are at their lower and upper bounds respectively. So, the matrix $\mathbf{A}$ can be permuted as follows

$$\mathbf{A} = [\mathbf{A}_{.B}, \mathbf{A}_{.N_1}, \mathbf{A}_{.N_2}],$$

and correspondingly we can write

$$\mathbf{x} = [\mathbf{x}_B^T, \mathbf{x}_{N_1}^T, \mathbf{x}_{N_2}^T]^T \quad \text{and } \mathbf{c}^T = [\mathbf{c}_B^T, \mathbf{c}_{N_1}^T, \mathbf{c}_{N_2}^T].$$

Then a solution to $\mathbf{Ax} = \mathbf{b}$ is given by

$$\begin{aligned} \mathbf{x}_B &= \mathbf{B}^{-1}\mathbf{b} - \sum_{j \in N_2} \mathbf{B}^{-1} A_{.j} u_j \\ \mathbf{x}_{N_1} &= 0 \\ \mathbf{x}_{N_2} &= \mathbf{u}_{N_2}, \end{aligned} \tag{1}$$

and the corresponding objective value is equal to

$$\begin{aligned} \mathbf{c}^T\mathbf{x} &= \mathbf{c}_B^T\mathbf{x}_B + \mathbf{c}_{N_2}^T\mathbf{x}_{N_2} \\ &= \mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{b} + \sum_{j \in N_2}(c_j - \mathbf{c}_B^T\mathbf{B}^{-1}A_{.j})u_j. \end{aligned} \tag{2}$$

Definition 1. Any solution of the form (1) is called a generalized basic solution of $\mathbf{Ax} = \mathbf{b}$. Moreover, if $0 \leq \mathbf{x}_B \leq \mathbf{u}_B$, then the solution $\mathbf{x}$ is called a generalized basic feasible solution and is briefly denoted by $GBFS$.

Recall that the problem (P) can be solved by using the upper bound simplex method without increasing the size of the problem. In this way, a generalized basic feasible solution $\mathbf{x} = [\mathbf{x}_B^T, \mathbf{x}_{N_1}^T, \mathbf{x}_{N_2}^T]^T$ is optimal if

$$\begin{aligned} c_j - \mathbf{y}A_{.j} &= c_j - \mathbf{c}_B^T\mathbf{B}^{-1}A_{.j} = 0, \quad \forall j \in B, \\ c_j - \mathbf{y}A_{.j} &= c_j - \mathbf{c}_B^T\mathbf{B}^{-1}A_{.j} \geq 0, \quad \forall j \in N_1, \\ c_j - \mathbf{y}A_{.j} &= c_j - \mathbf{c}_B^T\mathbf{B}^{-1}A_{.j} \leq 0, \quad \forall j \in N_2. \end{aligned}$$

Consider the dual of the problem (P)

$$\begin{array}{ll} \max & \mathbf{b}^T\mathbf{v} - \mathbf{u}^T\mathbf{w} \\ \text{s.t.} & \mathbf{A}^T\mathbf{v} - \mathbf{w} \leq \mathbf{c} \\ & \mathbf{w} \geq 0, \end{array} \qquad (D)$$

where $\mathbf{v} \in \mathbb{R}^m$ and $\mathbf{w} \in \mathbb{R}^n$ are the dual variables. The weak, strong duality and complementary slackness theorems can easily be generalized to the problems (P) and (D) and are stated without proof.

Theorem 1. (**Weak Duality**) If $\mathbf{x}$ is a feasible solution of (P) and $(\mathbf{v}, \mathbf{w})$ is a feasible solution of (D), then

$$\mathbf{c}^T\mathbf{x} \geq \mathbf{b}^T\mathbf{v} - \mathbf{u}^T\mathbf{w}.$$

Theorem 2. (**Strong Duality**) If the primal and the dual problems (P) and (D) are feasible, then both problems have optimal solutions and the optimal objective values are equal.

Theorem 3. (**Complementary Slackness Theorem**) $\mathbf{x}^*$ and $(\mathbf{v}^*, \mathbf{w}^*)$ are optimal solution to (P) and (D) respectively if and only if

$$x_j^* s_j^* = 0, \quad w_j^*(u_j - x_j^*) = 0, \quad j = 1, 2, \ldots, n,$$

where $\mathbf{s}$ is the slack vector of (D).

Consider the primal problem in standard form

$$\begin{array}{rl} \min & \mathbf{c}^T\mathbf{x} \\ \text{s.t.} & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq 0, \end{array} \qquad (P_1)$$

and its dual

$$\begin{array}{rl} \max & \mathbf{b}^T\mathbf{y} \\ \text{s.t.} & \mathbf{A}^T\mathbf{y} + \mathbf{s} = \mathbf{c} \\ & \mathbf{s} \geq 0. \end{array} \qquad (D_1)$$

According to Goldman-Tucker Theorem [5], there exists a primal-dual optimal solution $(\mathbf{x}^*, \mathbf{y}^*, \mathbf{s}^*)$ which is strictly complementary; that is,

$$x_j^* s_j^* = 0, \quad x_j^* + s_j^* > 0, \quad j = 1, 2, \ldots, n.$$

Now by using the following theorem for the skew-symmetric matrix K, we extend Goldman-Tucker Theorem to the problems (P) and (D).

Theorem 4 ([12]). *The system $K\mathbf{x} \geq 0,\ \mathbf{x} \geq 0$ has a solution $\mathbf{x}^*$ such that $K\mathbf{x}^* + \mathbf{x}^* > 0$.*

Theorem 5. *If the problems (P) and (D) are feasible, then there exists an optimal solution $(\mathbf{x}^*, \mathbf{v}^*, \mathbf{w}^*, \mathbf{s}^*)$ such that*

$$x_j^* + s_j^* > 0, \quad w_j^* + (u_j - x_j^*) > 0, \quad j = 1, 2, \ldots, n,$$

which is referred to as strictly complementary solution.

Proof. Consider the embedding self-dual system of primal and dual problems as follows

$$\begin{array}{ccccl} \mathbf{A}\mathbf{x} & & & -\mathbf{b}t & = 0 \\ & \mathbf{A}^T\mathbf{v} & -\mathbf{w} & -\mathbf{c}t & \leq 0 \\ \mathbf{c}^T\mathbf{x} & -\mathbf{b}^T\mathbf{v} & +\mathbf{u}^T\mathbf{w} & & \leq 0 \\ \mathbf{x} & & & -\mathbf{u}t & \leq 0. \end{array}$$

By using Theorem 4 to the skew-symmetric matrix K

$$K = \begin{bmatrix} 0 & -\mathbf{A}^T & \mathbf{A}^T & I & \mathbf{c} \\ \mathbf{A} & 0 & 0 & 0 & -\mathbf{b} \\ -\mathbf{A} & 0 & 0 & 0 & \mathbf{b} \\ -I & 0 & 0 & 0 & \mathbf{u} \\ -\mathbf{c}^T & \mathbf{b}^T & -\mathbf{b}^T & -\mathbf{u}^T & 0 \end{bmatrix},$$

there exists a solution $(\mathbf{x}^*, \mathbf{v}^*, \mathbf{w}^*, t^*)$ such that

$$\begin{array}{llll} (1.a) & \mathbf{A}\mathbf{x}^* - \mathbf{b}t^* = 0, & (2.a) & \mathbf{A}^T\mathbf{v}^* - \mathbf{w}^* \leq \mathbf{c}t^*, \\ (3.a) & \mathbf{b}^T\mathbf{v}^* - \mathbf{u}^T\mathbf{w}^* \geq \mathbf{c}^T\mathbf{x}^*, & (4.a) & \mathbf{A}^T\mathbf{v}^* - \mathbf{w}^* < \mathbf{c}t^* + \mathbf{x}^*, \\ (5.a) & -\mathbf{x}^* + \mathbf{w}^* > -\mathbf{u}t^*, & (6.a) & \mathbf{c}^T\mathbf{x}^* < \mathbf{b}^T\mathbf{y}^* - \mathbf{u}^T\mathbf{w}^* + t^*. \end{array}$$

For $t^* > 0$, the relations (1.a) and (2.a) show that $\frac{\mathbf{x}^*}{t^*}$ and $(\frac{\mathbf{v}^*}{t^*}, \frac{\mathbf{w}^*}{t^*})$ are feasible solutions to (P) and (D) respectively. The relation (3.a) and Theorem 1 show that these solutions are optimal. The relations (4.a) and (5.a) imply that

$$(\mathbf{u} - \frac{\mathbf{x}^*}{t^*}) + \frac{\mathbf{w}^*}{t^*} > 0,$$

and

$$\frac{\mathbf{x}^*}{t^*} + \frac{\mathbf{s}^*}{t^*} > 0.$$

We set $t^* = 1$ which completes the proof. □

Note 1. *For $t^* = 0$, either both problems are infeasible, or at least one of the problems is infeasible and another is unbounded [13]. This is beyond our discussion and is omitted.*

3 Perturbed problem and sensitivity analysis

To address sensitivity analysis on the data of problem (P), we consider the following perturbed problem:

$$\begin{array}{ll} \min & z = (\mathbf{c} + \lambda\Delta\mathbf{c})^T\mathbf{x} \\ \text{s.t.} & \mathbf{A}\mathbf{x} = \mathbf{b} + \lambda\Delta\mathbf{b} \\ & 0 \leq \mathbf{x} \leq \mathbf{u}, \end{array} \qquad (P_\lambda)$$

and its dual

$$\begin{array}{ll} \max & (\mathbf{b} + \lambda\Delta\mathbf{b})^T\mathbf{v} - \mathbf{u}^T\mathbf{w} \\ \text{s.t.} & \mathbf{A}^T\mathbf{v} - \mathbf{w} + \mathbf{s} = \mathbf{c} + \lambda\Delta\mathbf{c} \\ & \mathbf{w},\ \mathbf{s} \geq 0, \end{array} \qquad (D_\lambda)$$

where $\Delta\mathbf{c} \in \mathbb{R}^n$ and $\Delta\mathbf{b} \in \mathbb{R}^m$ are perturbation vectors and λ is a parameter. In special cases, one of the vectors $\Delta\mathbf{b}$ or $\Delta\mathbf{c}$ may be zero. Let $\mathcal{P}_\lambda$ and $\mathcal{D}_\lambda$ denote the feasible sets of problems (P_λ) and (D_λ) respectively. Their optimal solution sets are denoted by ${\mathcal{P}_\lambda}^*$ and ${\mathcal{D}_\lambda}^*$ correspondingly. For $\lambda = 0$ we denote them simply by $\mathcal{P}^*$ and $\mathcal{D}^*$.

Definition 2. For a given vector $\mathbf{x} \in \mathbb{R}^n$, $0 \leq \mathbf{x} \leq \mathbf{u}$, we define

$$\sigma(\mathbf{x}) = \{j :\ 0 < x_j < u_j\},$$

and refer to it as strictly set of $\mathbf{x}$.

According to Definition 2, the index set $\{1, 2, \ldots, n\}$ can be partitioned in three subsets

$$\begin{aligned}
\mathcal{B} &= \{j :\ 0 < x_j^* < u_j \ \text{ for some } \mathbf{x}^* \in \mathcal{P}^*\},\\
\mathcal{N} &= \{j :\ s_j^* > 0 \ \text{ for some } (\mathbf{v}^*, \mathbf{w}^*, \mathbf{s}^*) \in \mathcal{D}^*\},\\
\mathcal{M} &= \{j :\ w_j^* > 0 \ \text{ for some } (\mathbf{v}^*, \mathbf{w}^*, \mathbf{s}^*) \in \mathcal{D}^*\},
\end{aligned}$$

which is called optimal partition and denoted by $\pi = (\mathcal{B}, \mathcal{N}, \mathcal{M})$. The uniqueness of the optimal partition is a direct consequence of the convexity of $\mathcal{P}^*$ and $\mathcal{D}^*$.

Let us consider the problem (P_λ). One wants to know what happens to the given optimal solution if such perturbation occurs. Such questions occurred soon after the simplex method was introduced and the related research area is known as basis invariancy sensitivity analysis [11, 12]. Here, we briefly state this sensitivity analysis for the problems with upper bounds.

3.1 Basis invariancy sensitivity analysis

Let B be the index set of basic variables and $N_1,\ N_2$ be index sets of non-basic variables which are at their lower and upper bounds respectively. Let $\mathbf{x}^*$ and $(\mathbf{v}^*, \mathbf{w}^*, \mathbf{s}^*)$ be optimal basic solution of (P) and (D) and assume that $[\mathbf{A}_{.B}, \mathbf{A}_{.N_1}, \mathbf{A}_{.N_2}]$ be the corresponding matrix to $\mathbf{x}^*$ called a basis matrix. We want to compute the set of parameter values λ such that the given optimal basis remains optimal. This study is in the domain of the simplex method and based on the non-degeneracy assumption of the optimal basis. We denote this set by $\Upsilon_{\mathbf{B}}(\mathbf{x}^*)$. It is easy to verify that $\Upsilon_{\mathbf{B}}(\mathbf{x}^*)$ is given by

$$\begin{aligned}
\Upsilon_{\mathbf{B}}(\mathbf{x}^*) = \{\lambda :\ & 0 \leq \mathbf{B}^{-1}\mathbf{b} - \sum_{j \in N_2} \mathbf{B}^{-1} A_{.j} u_j + \lambda \mathbf{B}^{-1} \Delta b \leq \mathbf{u}_B,\\
& \mathbf{c}_{N_1} - \mathbf{c}_B^T \mathbf{B}^{-1} \mathbf{A}_{.N_1} + \lambda(\Delta \mathbf{c}_{N_1} - \Delta \mathbf{c}_B^T \mathbf{B}^{-1} \mathbf{A}_{.N_1}) \geq 0,\\
& \mathbf{c}_{N_2} - \mathbf{c}_B^T \mathbf{B}^{-1} \mathbf{A}_{.N_2} + \lambda(\Delta \mathbf{c}_{N_2} - \Delta \mathbf{c}_B^T \mathbf{B}^{-1} \mathbf{A}_{.N_2}) \leq 0 \ \ \}.
\end{aligned}$$

4 Strictly set invariancy sensitivity analysis

Let $\mathbf{x}^*$ and $(\mathbf{v}^*, \mathbf{w}^*, \mathbf{s}^*)$ be optimal solutions of problems (P) and (D) respectively. Further, let $P := \sigma(\mathbf{x}^*)$. Thus, the index set $\{1, 2, \ldots, n\}$ can be partitioned in (P, Z_1, Z_2), where $Z_1 \cup Z_2 := \{1, 2, \ldots, n\} \setminus P$, $Z_1 := \{j : \ x_j^* = 0\}$ and $Z_2 := \{j : \ x_j^* = u_j\}$. Strictly set sensitivity analysis deals with determining the range of parameter variations λ, in which the perturbed problem (P_λ) has an optimal solution with partition (P, Z_1, Z_2). Note that the given optimal solution is not necessarily a basic feasible solution. We denote the corresponding set of this sensitivity analysis by $\Upsilon_P(\mathbf{x}^*)$; i.e.

$$\Upsilon_P(\mathbf{x}^*) = \{\lambda : \ \exists(\mathbf{x}(\lambda), \mathbf{v}(\lambda), \mathbf{w}(\lambda), \mathbf{s}(\lambda)) \in \mathcal{P}_\lambda{}^* \times \mathcal{D}_\lambda{}^*, \\ \sigma(\mathbf{x}(\lambda)) = P, \ \mathbf{x}_{Z_1}(\lambda) = 0, \ \mathbf{x}_{Z_2}(\lambda) = \mathbf{u}_{Z_2}\}.$$

The following lemma shows that the set $\Upsilon_P(\mathbf{x}^*)$ is a convex set.

Lemma 6. *The set $\Upsilon_P(\mathbf{x}^*)$ is a convex set.*

Proof. Let λ_1 and λ_2 be two elements of $\Upsilon_P(\mathbf{x}^*)$. Let $(\mathbf{x}(\lambda_1), \mathbf{v}(\lambda_1), \mathbf{w}(\lambda_1), \mathbf{s}(\lambda_1))$ and $(\mathbf{x}(\lambda_2), \mathbf{v}(\lambda_2), \mathbf{w}(\lambda_2), \mathbf{s}(\lambda_2))$ be the corresponding optimal solutions to λ_1 and λ_2, respectively. By assumption, we have $\sigma(\mathbf{x}(\lambda_1)) = \sigma(\mathbf{x}(\lambda_2)) = P$, $\mathbf{x}_{Z_1}(\lambda_1) = \mathbf{x}_{Z_1}(\lambda_2) = 0$ and $\mathbf{x}_{Z_2}(\lambda_1) = \mathbf{x}_{Z_2}(\lambda_2) = \mathbf{u}_{Z_2}$. For any $\lambda = \theta\lambda_1 + (1-\theta)\lambda_2$, where $\theta \in (0, 1)$, we define

$$\mathbf{x}(\lambda) = \theta\mathbf{x}(\lambda_1) + (1-\theta)\mathbf{x}(\lambda_2),$$
$$\mathbf{v}(\lambda) = \theta\mathbf{v}(\lambda_1) + (1-\theta)\mathbf{v}(\lambda_2),$$
$$\mathbf{w}(\lambda) = \theta\mathbf{w}(\lambda_1) + (1-\theta)\mathbf{w}(\lambda_2),$$
$$\mathbf{s}(\lambda) = \theta\mathbf{s}(\lambda_1) + (1-\theta)\mathbf{s}(\lambda_2).$$

One can easily checks that $(\mathbf{x}(\lambda), \mathbf{v}(\lambda), \mathbf{w}(\lambda), \mathbf{s}(\lambda))$ is a primal-dual feasible solution of (P_λ) and (D_λ). It is obvious that $\sigma(\mathbf{x}(\lambda)) = P$, $\mathbf{x}_{Z_1}(\lambda) = 0$ and $\mathbf{x}_{Z_2}(\lambda) = \mathbf{u}_{Z_2}$. Moreover, we have

$$(\mathbf{x}(\lambda))^T(\mathbf{s}(\lambda)) = 0, \quad (\mathbf{w}(\lambda))^T(\mathbf{u} - \mathbf{x}(\lambda)) = 0.$$

Therefore, $\lambda \in \Upsilon_P(\mathbf{x}^*)$ and this completes the proof. □

The Lemma 6 shows that $\Upsilon_P(\mathbf{x}^*)$ is an interval of the real line that contains zero. We refer to $\Upsilon_P(\mathbf{x}^*)$ as the strictly set invariancy interval of the problem P_λ with respect to the partition (P, Z_1, Z_2).

The following theorem presents two auxiliary problems to identify the end points of the interval $\Upsilon_P(\mathbf{x}^*)$.

Theorem 7. *Let* $\mathbf{x}^*$ *and* $(\mathbf{v}^*, \mathbf{w}^*, \mathbf{s}^*)$ *be optimal solutions of the primal and dual problems* (P) *and* (D) *respectively, where* $P := \sigma(\mathbf{x}^*)$*. Then,* λ_ℓ *and* λ_u*, the end points of* $\overline{\Upsilon}_P(\mathbf{x}^*)$*, are optimal values of the following problems respectively:*

$$\begin{aligned} &\min(\max)\ \lambda \\ &\text{s.t.}\quad \mathbf{A}_P\mathbf{x}_P - \lambda\Delta\mathbf{b} = \mathbf{b} - \mathbf{A}_{Z_2}\mathbf{u}_{Z_2} \\ &\qquad \mathbf{A}_P^T\mathbf{v} - \lambda\Delta\mathbf{c}_P = \mathbf{c}_P \\ &\qquad \mathbf{A}_{Z_1}^T\mathbf{v} + \mathbf{s}_{Z_1} - \lambda\Delta\mathbf{c}_{Z_1} = \mathbf{c}_{Z_1} \\ &\qquad \mathbf{A}_{Z_2}^T\mathbf{v} - \mathbf{w}_{Z_2} - \lambda\Delta\mathbf{c}_{Z_2} = \mathbf{c}_{Z_2} \\ &\qquad \mathbf{x}_P \leq \mathbf{u}_P \\ &\qquad \mathbf{s}_{Z_1}, \mathbf{w}_{Z_2}, \mathbf{x}_P \geq 0, \end{aligned} \tag{3}$$

where, $\overline{\Upsilon}_P(\mathbf{x}^*)$ *denotes the closure of the interval* $\Upsilon_P(\mathbf{x}^*)$*.*

Proof. First we establish the inclusion $[\lambda_\ell, \lambda_u] \subseteq \overline{\Upsilon}_P(\mathbf{x}^*)$. If $\lambda_\ell = \lambda_u = 0$, then it is trivial. Without loss of generality, let $\lambda_\ell < 0$ and $\lambda \in (\lambda_\ell, 0)$ be given. It is obvious that $\sigma(\mathbf{x}_P(\lambda_\ell)) \subseteq P$. Now, for $\theta = \dfrac{\lambda}{\lambda_\ell} \in (0, 1)$, we define

$$\mathbf{x}(\lambda) = \theta\mathbf{x}(\lambda_\ell) + (1-\theta)\mathbf{x}^*, \tag{4}$$

$$\mathbf{v}(\lambda) = \theta\mathbf{v}(\lambda_\ell) + (1-\theta)\mathbf{v}^*, \tag{5}$$

$$\mathbf{w}(\lambda) = \theta\mathbf{w}(\lambda_\ell) + (1-\theta)\mathbf{w}^*, \tag{6}$$

$$\mathbf{s}(\lambda) = \theta\mathbf{s}(\lambda_\ell) + (1-\theta)\mathbf{s}^*. \tag{7}$$

It is easy to verify that $(\mathbf{x}(\lambda), \mathbf{v}(\lambda), \mathbf{w}(\lambda), \mathbf{s}(\lambda))$ is a primal-dual feasible solution of problems (P_λ) and (D_λ). Moreover, we have

$$\mathbf{x}(\lambda)^T\mathbf{s}(\lambda) = 0, \qquad \mathbf{w}(\lambda)^T(\mathbf{u} - \mathbf{x}(\lambda)) = 0,$$

and

$$\sigma(\mathbf{x}(\lambda)) = \sigma(\mathbf{x}(\lambda_\ell)) \cup \sigma(\mathbf{x}^*) = P, \quad \mathbf{x}_{Z_1} = 0, \quad \mathbf{x}_{Z_2} = \mathbf{u}_{Z_2}.$$

Therefore, $\lambda \in \overline{\Upsilon}_P(\mathbf{x}^*)$.

Now we prove that $\overline{\Upsilon}_P(\mathbf{x}^*) \subseteq [\lambda_\ell, \lambda_u]$. Let $\tilde{\lambda} \in \overline{\Upsilon}_P(\mathbf{x}^*)$ but $\tilde{\lambda} \notin [\lambda_\ell, \lambda_u]$. Without loss of generality, assume that $\tilde{\lambda} < \lambda_\ell$. In this way, for all $\lambda \in [\tilde{\lambda}, \lambda_\ell)$, we have $\lambda \in \overline{\Upsilon}_P(\mathbf{x}^*)$ that contradicts the optimality of λ_ℓ. Thus, $\tilde{\lambda} \in [\lambda_\ell, \lambda_u]$ and the proof is complete. □

Remark 8. If $\lambda_\ell = \lambda_u = 0$, then it is not possible to perturb the right-hand-side of the constraints and the coefficients of the objective function of problem (P_λ) in directions $\Delta\mathbf{b}$ and $\Delta\mathbf{c}$ while strictly set remains invariant. In this case, $\Upsilon_P(\mathbf{x}^*)$ is the singleton $\{0\}$. On the other hand, if one of the problems in Theorem 7 is unbounded, then the interval $\Upsilon_P(\mathbf{x}^*)$ is infinity.

Remark 9. If $\sigma(\mathbf{x}(\lambda_\ell)) = P, \mathbf{x}_{Z_1}(\lambda_\ell) = 0$ and $\mathbf{x}_{Z_2}(\lambda_\ell) = \mathbf{u}_{Z_2}$ then $\lambda_\ell \in \Upsilon_P(\mathbf{x}^*)$; that is, the interval $\Upsilon_P(\mathbf{x}^*)$ is closed from the left. Analogous discussion is valid for the right end point λ_u.

In the rest of this section, we discuss the results for non-simultaneous perturbation, when either $\Delta\mathbf{c}$ or $\Delta\mathbf{b}$ is zero.

4.1 Perturbation of the vector b

In this case $\Delta\mathbf{c} = 0$ and the primal and dual perturbed problems are as follows

$$\begin{array}{ll} \min & z = \mathbf{c}^T\mathbf{x} \\ \text{s.t.} & \mathbf{A}\mathbf{x} = \mathbf{b} + \lambda\Delta\mathbf{b} \\ & 0 \le \mathbf{x} \le \mathbf{u}, \end{array} \qquad (P_\lambda(\Delta\mathbf{b}))$$

and

$$\begin{array}{ll} \max & (\mathbf{b} + \lambda\Delta\mathbf{b})^T\mathbf{v} - \mathbf{u}^T\mathbf{w} \\ \text{s.t.} & \mathbf{A}^T\mathbf{v} - \mathbf{w} + \mathbf{s} = \mathbf{c} \\ & \mathbf{w},\ \mathbf{s} \ge 0. \end{array} \qquad (D_\lambda(\Delta\mathbf{b}))$$

Let $P^*_\lambda(\Delta\mathbf{b})$ and $D^*_\lambda(\Delta\mathbf{b})$ denote the optimal solution set of the problems $(P_\lambda(\Delta\mathbf{b}))$ and $(D_\lambda(\Delta\mathbf{b}))$ respectively. Suppose that $\Upsilon_P(\mathbf{x}^*, \Delta b)$ denotes the strictly set invariancy interval of these problems in $\mathbf{x}^*$. In this way, the problem (3) becomes as follows

$$\begin{array}{rcl} \multicolumn{3}{l}{\min(\max)\ \lambda} \\ \text{s.t.}\quad \mathbf{A}_P\mathbf{x}_P - \lambda\Delta\mathbf{b} & = & \mathbf{b} - \mathbf{A}_{Z_2}\mathbf{u}_{Z_2} \\ \mathbf{A}_P^T\mathbf{v} & = & \mathbf{c}_P \\ \mathbf{A}_{Z_1}^T\mathbf{v} + \mathbf{s}_{Z_1} & = & \mathbf{c}_{Z_1} \\ \mathbf{A}_{Z_2}^T\mathbf{v} - \mathbf{w}_{Z_2} & = & \mathbf{c}_{Z_2} \\ \mathbf{x}_P & \le & \mathbf{u}_P \\ \mathbf{s}_{Z_1}, \mathbf{w}_{Z_2}, \mathbf{x}_P & \ge & 0 \end{array} \qquad (8)$$

Obviously, the corresponding constraints in dual are independent from λ and $\mathbf{x}_P$, and hence every feasible solution $(\mathbf{v}, \mathbf{s}_{Z_1}, \mathbf{w}_{Z_2})$ will be complementary with the solution set of the system

$$\begin{array}{rcl} \mathbf{A}_P\mathbf{x}_P - \lambda\Delta\mathbf{b} & = & \mathbf{b} - \mathbf{A}_{Z_2}\mathbf{u}_{Z_2} \\ 0 \le \mathbf{x}_P \le \mathbf{u}_P. & & \end{array} \qquad (9)$$

Thus, we only need to find the range of parameter variations in λ such that the system (9) has a feasible solution. The following lemma summarizes these discussions. Its proof is similar to the proof of Theorem 7 and is omitted.

Lemma 10. *Let* $\mathbf{x}^*$ *and* $(\mathbf{v}^*, \mathbf{w}^*, \mathbf{s}^*)$ *be optimal solutions of the primal and dual problems* (P) *and* (D) *respectively, where* $P := \sigma(\mathbf{x}^*)$. *Then,* λ_ℓ *and* λ_u*, the end points of* $\overline{\Upsilon}_P(\mathbf{x}^*, \Delta\mathbf{b})$*, are optimal values of the problems*

$$\begin{aligned} &\min(\max)\ \lambda \\ &\text{s.t. } \mathbf{A}_P\mathbf{x}_P - \lambda\Delta\mathbf{b} = \mathbf{b} - \mathbf{A}_{Z_2}\mathbf{u}_{Z_2} \\ &\quad 0 \le \mathbf{x}_P \le \mathbf{u}_P, \end{aligned} \tag{10}$$

respectively, where $\overline{\Upsilon}_P(\mathbf{x}^*, \Delta\mathbf{b})$ *denotes the closure of the interval* $\Upsilon_P(\mathbf{x}^*, \Delta\mathbf{b})$.

Remark 11. If $\sigma(\mathbf{x}(\lambda_\ell)) = P, \mathbf{x}_{Z_1}(\lambda_\ell) = 0$ and $\mathbf{x}_{Z_2}(\lambda_\ell) = \mathbf{u}_{Z_2}$ then $\lambda_\ell \in \Upsilon_P(\mathbf{x}^*, \Delta\mathbf{b})$; that is, the interval $\Upsilon_P(\mathbf{x}^*, \Delta\mathbf{b})$ is closed from the left. Analogous discussion is valid for the right end point λ_u.

4.2 Perturbation in the coefficients of objective function

In this case $\Delta\mathbf{b} = 0$ and the primal and dual perturbed problems are as follows

$$\begin{aligned} \min \quad & z = (\mathbf{c} + \lambda\Delta\mathbf{c})^T\mathbf{x} \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \qquad\qquad (P_\lambda(\Delta\mathbf{c})) \\ & 0 \le \mathbf{x} \le \mathbf{u}, \end{aligned}$$

and

$$\begin{aligned} \max \quad & \mathbf{b}^T\mathbf{v} - \mathbf{u}^T\mathbf{w} \\ \text{s.t.} \quad & \mathbf{A}^T\mathbf{v} - \mathbf{w} + \mathbf{s} = \mathbf{c} + \lambda\Delta\mathbf{c} \qquad\qquad (D_\lambda(\Delta\mathbf{c})) \\ & \mathbf{w},\ \mathbf{s} \ge 0. \end{aligned}$$

Let $P^*_\lambda(\Delta\mathbf{c})$ and $D^*_\lambda(\Delta\mathbf{c})$ denote the optimal solution set of the problems $(P_\lambda(\Delta\mathbf{c}))$ and $(D_\lambda(\Delta\mathbf{c})$ respectively. Suppose that $\Upsilon_P(\mathbf{x}^*, \Delta c)$ be the strictly set invariancy interval of these problems in $\mathbf{x}^*$. In this way, by the same discussion as in subsection 4.1, the end points of $\Upsilon_P(\mathbf{x}^*, \Delta\mathbf{c})$ are determined by two auxiliary problems given in the following lemma.

Lemma 12. *Let* $\mathbf{x}^*$ *and* $(\mathbf{v}^*, \mathbf{w}^*, \mathbf{s}^*)$ *be optimal solutions of the primal and dual problems* (P) *and* (D) *respectively, where* $P := \sigma(\mathbf{x}^*)$. *Then,* λ_ℓ *and* λ_u*, the end points of* $\overline{\Upsilon}_P(\mathbf{x}^*, \Delta\mathbf{c})$*, are optimal values of the following problems respectively:*

$$\begin{aligned} &\min(\max)\ \lambda \\ &\text{s.t.} \quad \mathbf{A}_P^T\mathbf{v} - \lambda\Delta\mathbf{c}_P = \mathbf{c}_P \\ &\qquad \mathbf{A}_{Z_1}^T\mathbf{v} + \mathbf{s}_{Z_1} - \lambda\Delta\mathbf{c}_{Z_1} = \mathbf{c}_{Z_1} \\ &\qquad \mathbf{A}_{Z_2}^T\mathbf{v} - \mathbf{w}_{Z_2} - \lambda\Delta\mathbf{c}_{Z_2} = \mathbf{c}_{Z_2} \\ &\qquad \mathbf{s}_{Z_1}, \mathbf{w}_{Z_2} \ge 0, \end{aligned} \tag{11}$$

where, $\overline{\Upsilon}_P(\mathbf{x}^*, \Delta\mathbf{c})$ *denotes the closure of the interval* $\Upsilon_P(\mathbf{x}^*, \Delta\mathbf{c})$.

4.3 The relation between simultaneous and independent perturbations

Let us assume that both $\Delta\mathbf{c}$ and $\Delta\mathbf{b}$ are not zero. The following theorem presents the relation between the intervals $\Upsilon_P(\mathbf{x}^*, \Delta\mathbf{c})$, $\Upsilon_P(\mathbf{x}^*, \Delta\mathbf{b})$ and $\Upsilon_P(\mathbf{x}^*)$. It allows us to identify $\Upsilon_P(\mathbf{x}^*)$ when $\Upsilon_P(\mathbf{x}^*, \Delta\mathbf{c})$ and $\Upsilon_P(\mathbf{x}^*, \Delta\mathbf{b})$ are known. It is a direct consequence of the relation between feasible solution sets of problems (3), (10) and (11).

Theorem 13. $\Upsilon_P(\mathbf{x}^*) = \Upsilon_P(\mathbf{x}^*, \Delta\mathbf{c}) \cap \Upsilon_P(\mathbf{x}^*, \Delta\mathbf{b})$.

5 Optimal partition invariancy sensitivity analysis

Let $\pi = (\mathcal{B}, \mathcal{N}, \mathcal{M})$ be the optimal partition of (P) and (D). Sensitivity analysis aims to find the range of parameter variations λ such that the optimal partition π remains invariant for any λ in this range. In other hand, sensitivity analysis proceeds to find the range of λ within which optimal partition of (P_λ) and (D_λ) is equal to π and denote this set by Υ_π; that is,

$$\Upsilon_\pi = \{\lambda : \ \pi = \pi(\lambda) = (\mathcal{B}(\lambda), \mathcal{N}(\lambda), \mathcal{M}(\lambda))\}.$$

Several papers have been published based on the concept of optimal partition. These studies focused on finding the range of the parameter variations for which the optimal partition remains invariant. Adler and Monteiro [1] are studied this concept to independent perturbed linear programming problems to standard form. Simultaneous perturbations of the right-hand-side and the objective function for the primal and dual problems in canonical form are studied by Greenberg[7]. For the case of perturbation in right-hand-side of constraints and objective function coefficients in two different directions with two different parameters has been studied by Kheirfam and Mirina [9]. Here we study the optimal partition in linear programming with upper bounds which is different from the optimal partition of standard linear programming problems.

The following lemma shows that the set Υ_π is a convex set.

Lemma 14. *The set Υ_π is a convex set.*

Proof. Let $\lambda_1, \lambda_2 \in \Upsilon_\pi$. Further, let $(\mathbf{x}(\lambda_1), \mathbf{v}(\lambda_1), \mathbf{w}(\lambda_1), \mathbf{s}(\lambda_1))$ and $(\mathbf{x}(\lambda_2), \mathbf{v}(\lambda_2), \mathbf{w}(\lambda_2), \mathbf{s}(\lambda_2))$ be the corresponding strictly complementary solutions to λ_1 and λ_2, respectively. For $\lambda = \theta\lambda_1 + (1-\theta)\lambda_2$, where $\theta \in (0,1)$ we define

$$\mathbf{x}(\lambda) = \theta\mathbf{x}(\lambda_1) + (1-\theta)\mathbf{x}(\lambda_2),$$

$$\mathbf{v}(\lambda) = \theta\mathbf{v}(\lambda_1) + (1-\theta)\mathbf{v}(\lambda_2),$$

$$\mathbf{w}(\lambda) = \theta\mathbf{w}(\lambda_1) + (1-\theta)\mathbf{w}(\lambda_2),$$
$$\mathbf{s}(\lambda) = \theta\mathbf{s}(\lambda_1) + (1-\theta)\mathbf{s}(\lambda_2).$$

One can easily verifies that $(\mathbf{x}(\lambda), \mathbf{v}(\lambda), \mathbf{w}(\lambda), \mathbf{s}(\lambda))$ is a feasible solution for P_λ and D_λ. In the other hand, $\sigma(\mathbf{x}(\lambda)) = \mathcal{B}, \sigma(\mathbf{s}(\lambda)) = \mathcal{N}$ and $\sigma(\mathbf{w}(\lambda)) = \mathcal{M}$. So $(\mathbf{x}(\lambda), \mathbf{v}(\lambda), \mathbf{w}(\lambda), \mathbf{s}(\lambda))$ is a strictly complementary optimal solution for problems P_λ and D_λ, as well as the invariancy of the optimal partition $\pi = (\mathcal{B}, \mathcal{N}, \mathcal{M})$ at λ. This implies the lemma. □

The above Lemma shows that Υ_π is an interval on the real line that contains zero. We refer it as the actual invariancy interval. The following theorem gives two auxiliary problems to obtain the end points of the invariancy interval Υ_π. We also state a relation between these intervals in simultaneous and independent perturbation cases. The proofs are analogous with the strictly set case when the given solution is strictly complementary and they are omitted.

Theorem 15. *Consider the primal and dual problems (P_λ) and (D_λ) respectively. Further, let $\pi = (\mathcal{B}, \mathcal{N}, \mathcal{M})$ be a optimal partition of problems (P) and (D). Then, λ_ℓ and λ_u, the end points of the interval $\overline{\Upsilon}_\pi$ are optimal solutions of the following problems respectively:*

$$\begin{aligned} &\min(\max)\ \lambda \\ &\text{s.t.} \quad \mathbf{A}_\mathcal{B}\mathbf{x}_\mathcal{B} - \lambda\Delta\mathbf{b} = \mathbf{b} - \mathbf{A}_\mathcal{M}\mathbf{u}_\mathcal{M} \\ &\qquad \mathbf{A}_\mathcal{B}^T\mathbf{v} - \lambda\Delta\mathbf{c}_\mathcal{B} = \mathbf{c}_\mathcal{B} \\ &\qquad \mathbf{A}_\mathcal{N}^T\mathbf{v} + \mathbf{s}_\mathcal{N} - \lambda\Delta\mathbf{c}_\mathcal{N} = \mathbf{c}_\mathcal{N} \\ &\qquad \mathbf{A}_\mathcal{M}^T\mathbf{v} - \mathbf{w}_\mathcal{M} - \lambda\Delta\mathbf{c}_\mathcal{M} = \mathbf{c}_\mathcal{M} \\ &\qquad \mathbf{x}_\mathcal{B} \le \mathbf{u}_\mathcal{B} \\ &\qquad \mathbf{s}_\mathcal{N}, \mathbf{w}_\mathcal{M}, \mathbf{x}_\mathcal{B} \ge 0, \end{aligned} \tag{12}$$

where, $\overline{\Upsilon}_\pi$ denotes the closure of the interval Υ_π.

Lemma 16. *Consider the primal and dual problems ($P_\lambda(\Delta\mathbf{b})$) and ($D_\lambda(\Delta\mathbf{b})$) respectively. Further, let $\pi = (\mathcal{B}, \mathcal{N}, \mathcal{M})$ be a optimal partition of problems (P) and (D). Then, λ_ℓ^p and λ_u^p, the end points of the optimal partition invariancy interval π are optimal solutions of the following problems respectively:*

$$\begin{aligned} &\min(\max)\ \lambda \\ &\text{s.t.}\ \mathbf{A}_\mathcal{B}\mathbf{x}_\mathcal{B} - \lambda\Delta\mathbf{b} = \mathbf{b} - \mathbf{A}_\mathcal{M}\mathbf{u}_\mathcal{M} \\ &\qquad 0 \le \mathbf{x}_\mathcal{B} \le \mathbf{u}_\mathcal{B}. \end{aligned} \tag{13}$$

Lemma 17. *Consider the primal and dual problems ($P_\lambda(\Delta\mathbf{c})$) and ($D_\lambda(\Delta\mathbf{c})$) respectively. Further, let $\pi = (\mathcal{B}, \mathcal{N}, \mathcal{M})$ be a optimal partition of problems (P) and (D).*

Then, λ_ℓ^d and λ_u^d, the end points of the optimal partition invariancy interval π are optimal solutions of the following problems respectively:

$$\begin{aligned}
&\min(\max)\ \lambda \\
&\text{s.t. } \mathbf{A}_{\mathcal{B}}^T\mathbf{v} - \lambda\Delta\mathbf{c}_{\mathcal{B}} = \mathbf{c}_{\mathcal{B}} \\
&\qquad \mathbf{A}_{\mathcal{N}}^T\mathbf{v} + \mathbf{s}_{\mathcal{N}} - \lambda\Delta\mathbf{c}_{\mathcal{N}} = \mathbf{c}_{\mathcal{N}} \\
&\qquad \mathbf{A}_{\mathcal{M}}^T\mathbf{v} - \mathbf{w}_{\mathcal{M}} - \lambda\Delta\mathbf{c}_{\mathcal{M}} = \mathbf{c}_{\mathcal{M}} \\
&\qquad\quad \mathbf{s}_{\mathcal{N}}, \mathbf{w}_{\mathcal{M}} \geq 0.
\end{aligned} \tag{14}$$

The following lemma is a direct consequence of the feasible solution sets of problems (12), (13) and (14) and shows the relation between simultaneous and independent perturbations.

Lemma 18. $\Upsilon_\pi = (\lambda_\ell^p, \lambda_u^p) \cap (\lambda_\ell^d, \lambda_u^d)$.

6 Examples

Example 1. *Consider the primal problem*

$$\begin{aligned}
\min z &= 2x_1 + 6x_2 - x_3 - 4x_4 + x_5 \\
\text{s.t.}\quad & 2x_1 + x_2 + 4x_3 + x_4 + x_5 = 8 \\
& 3x_1 + 8x_2 - 3x_3 + x_4 \quad = -2 \\
& 0 \leq x_1 \leq 3, \\
& 0 \leq x_2 \leq 3 \\
& 0 \leq x_3 \leq 8 \\
& 0 \leq x_4 \leq 1 \\
& 0 \leq x_5 \leq 20,
\end{aligned}$$

with its dual

$$\begin{aligned}
\max\quad & 8v_1 - 2v_2 - 3w_1 - 3w_2 - 8w_3 - w_4 - 20w_5 \\
\text{s.t.}\quad & 2v_1 + 3v_2 - w_1 \leq 2 \\
& v_1 + 8v_2 - w_2 \leq 6 \\
& 4v_1 - 3v_2 - w_3 \leq -1 \\
& v_1 + v_2 - w_4 \leq -4 \\
& v_1 - w_5 \leq 1 \\
& w_1, w_2, w_3, w_4, w_5 \geq 0.
\end{aligned}$$

One can easily verifies that $\mathbf{x}^* = (\frac{1}{2}, 0, \frac{3}{2}, 1, 0)^T, \mathbf{v} = (\frac{1}{6}, \frac{5}{9})^T, \mathbf{w} = (0, 0, 0, \frac{85}{18}, 0)^T$ and $\mathbf{s} = (0, \frac{25}{18}, 0, 0, \frac{5}{6})^T$ is a primal-dual optimal solution with strictly set $\sigma(\mathbf{x}) = \{1, 3\}, Z_1 = \{2, 5\}$ and $Z_2 = \{4\}$. Let us consider the perturbation vectors $\Delta b = (2, 1)^T$ and $\Delta c = (-3, -1, 2, -1, 1)^T$. The invariancy intervals $\Upsilon_P(\mathbf{x}^*, \Delta\mathbf{b}), \Upsilon_P(\mathbf{x}^*, \Delta\mathbf{c})$ and $\Upsilon_P(\mathbf{x}^*)$ are $(-0.9, 4.5), [-0.2212, 85)$ and $[-0.2212, 4.5)$ respectively.

Example 2. *Consider the following primal problem*

$$\begin{aligned}\max z = -2x_1 - x_2 - x_3 + 2x_4 - 2x_5 + x_6 + x_7 - 3x_8 \\ \text{s.t.} \quad x_1 + 3x_3 + x_4 - 5x_5 - 2x_6 + 4x_7 - 6x_8 = 7 \\ x_2 - 2x_3 - x_4 + 4x_5 + x_6 - 3x_7 + 5x_8 = -3 \\ 0 \le x_1 \le 8, \quad 0 \le x_2 \le 6, \quad 0 \le x_3 \le 4, \\ 0 \le x_4 \le 15, \quad 0 \le x_5 \le 2, \quad 0 \le x_6 \le 10, \\ 0 \le x_7 \le 10, \quad 0 \le x_8 \le 3.\end{aligned}$$

One can easily verifies that $x^* = (0,6,0,15,0,1,0,1)^T, \tilde{x} = (4,6,0,15,0,6,0,0)^T, \bar{x} = (\frac{20}{3},6,0,15,0,10,\frac{4}{3},0)^T$ *and* $\hat{x} = (6,6,0,15,2,10,4,0)^T$ *are primal basic optimal solutions and* $v = (2,3)^T, w = (0,2,0,1,0,0,0,0)^T$ *and* $s = (0,0,1,0,0,0,0,0)^T$ *is a dual optimal solution.Therefore, the optimal partition is as follows*

$$(\mathcal{B},\mathcal{N},\mathcal{M}) = \Big(\{1,5,6,7,8\},\{3\},\{2,4\}\Big).$$

Now consider the non-basic optimal solution

$$\check{x} = (2,6,0,15,0,\frac{7}{2},0,\frac{1}{2})^T,$$

which $\sigma(\check{x}) = \{1,6,8\}, Z_1 = \{3,5,7\}$ *and* $Z_2 = \{2,4\}$*. Let us consider the perturbation vectors* $\Delta b = (3,-2)^T$ *and* $\Delta c = (0,3,0,1,\frac{3}{4},-3,-3,0)^T$*. In this way, the invariancy interval* $\Upsilon_P(\check{x},\triangle c) = 0$*,* $\Upsilon_P(\check{x},\triangle b) = (-9.5,3)$ *and* $\Upsilon_P(\check{x}) = 0$*.*

Example 3. *Consider the Example 2. For this example, invariancy interval of optimal partition is equal to zero, that is;* $\Upsilon_\pi = 0$*.*

7 Conclusions

In this paper, we have proved existence of strictly complementary optimal solution for linear programming problems with upper bounds which leads to study of optimal partition sensitivity analysis for such problems. Also strictly set sensitivity analysis is studied and its main advantage that can be performed to any optimal solution which is a basic or non-basic optimal solution. We have developed computational procedures to calculate the invariancy intervals. We investigated the case when perturbation occurs in the vectors **b** or **c**, and also when both **b** and **c** are perturbed simultaneously.

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