A Stochastic Variational Model for Multi-phase Soft Segmentation with Bias correction

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Abstract

Soft segmentation is more flexible than hard segmentation but the membership functions are sensitive to noise. In this paper, we proposed a multiphase soft segmentation model based on Gaussian mixture. Compared with previous models, the proposed energy functional is not only convex with respect to objective membership functions, but more robust to noise. With recently developed technique for solving simplex constraint, the model can be solved very fast.

I. Introduction

Soft segmentation has attracted increasing interest in recent studies, partly because of its success in seeking a global optimal solution in variational hard segmentation models. For example, Chan and Bresson et al [Chan, 2006; Bresson, 2007; Bresson, Chan, 2008] proved in their new variational frameworks that global minimum can be achieved in these models due to the convexity of the energy functional. Very recently, based on a variational convexification technique due to Pock et. al. [Pock, 2008], Brown et. al. [Brown, 2009], and Bae et. al. [Bae, 2009] provide different ways to find globally optimal solution for piecewise constant Mumford-Shah model (more generally called continuous Potts model). A common technique in these methods is the relaxation of characteristic functions to functions evaluated in $[0, 1]$ (called membership function or ownership function), transforming the original problem to a soft segmentation. It is then proved that one can find a global optimal solution at integrality value of membership functions in these models. As a result, this integrality global solution for soft segmentation is thereby a global solution for original hard segmentation. Another motivation for studying soft segmentation is partial volume effect in MRI brain image segmentation. Due to limited spacial resolution of the equipment, not all the voxels in a segmented region contain the same tissue type, especially near the boundary between different subregions. Using soft segmentation, one can find the percentage of each tissue contained in the partial volume.

There have been many soft segmentation methods [Chen, 2004; Li, 2005; Mory, Arden, 2007; Pham, 1998; Shen, 2006; Zhu, 1996]. For example, Mory and Ardon extended the original region competition model [Zhu, 1996] to a fuzzy region competition method [Mory, Arden, 2007]. Fuzzy C-mean (FCM) is a method developed for pattern classification and recognition. It can also be applied to image segmentation. The original FCM method is very sensitive to noise. Pham et.al [Pham, 1998] proposed an adaptive fuzzy C mean (AFCM) model, where the constant cluster centers used in the FCM model are substituted by spatially varying functions. AFCM is more robust to noise than the standard FCM. Another similar work is proposed by Zhang et. al. [Chen, 2004], where the fitting term is represented with a kernel function.

Another class of soft segmentations are based on stochastic approaches [Shen, 2006; Li, 2005]. For example, Li et. al. proposed a segmentation framework based on *maximum a posteriori* principle (MAP) for partial volume (PV) segmentation of magnetic resonance brain images [Li, 2005]. J. Shen [Shen, 2006] proposed a general multiphase stochastic variational fuzzy segmentation model combining stochastic principle and the Modica-Mortola's phase-transition theory. The intensity of images was modeled as a mixed Gaussian distribution.

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Among all multiphase segmentation methods, piecewise constant model is easier to implement [Chan, Vese, 2001]. However, even if an image is very ideal and should be piecewise constant, the image may appear quite different from piecewise constant due to the existence of noise and inhomogeneous illumination. Hence, bias correction based methods have been studied [Wells, 1996; Pham, 1998; Ahmed, 2002; Li, 2008]. These methods are usually integrated into soft segmentation schemes.

This paper is based on the work of Li et. al. [Li, 2005] and the work of Shen [Shen, 2006]. We proposed a stochastic variational model for multi-phase Mumford-Shah soft segmentation in the presence of intensity inhomogeneity, where the image intensity at each point is modeled as a mixed Gaussian distribution with means and variants to be optimized. Compared with J. Shen's work, our model does not assume that membership functions must be close to either 1 or 0. So, our model matches soft segmentation more precisely, and hence can be applied to partial volume segmentation. The new model is more robust to noise than the model in [Li, 2005]. The experiment shows that our model is also robust to bias.

The rest of the paper is organized as follows. Section II is the development of the new model. The implementation scheme is presented in Section III. In Section IV, we demonstrate the efficiency of the model with synthetic images and authentic images.

II. Model Development

Let I be an image defined in an open bounded domain Ω. Suppose I contains K phases, and make the following stochastic assumption: the intensity $I(x)$ of each pixel is a sample of Gaussian mixture of K Gaussian probability density functions (PDF) $f_i(x)$, defined as

$$f_i(x)) = g(I(x), |c_i, \sigma_i) := \frac{1}{\sqrt{2\pi}\sigma_i} e^{-\frac{(I(x)-c_i)^2}{2\sigma_i^2}} \tag{1}$$

with mixture coefficient field $p_i(x)$, $(1 \leq i \leq K)$. Then the pdf of the mixture image I at any pixel x is given by

$$P(I(x)) = \sum_{i=1}^{K} f_i(x) p_i(x). \tag{2}$$

Further, we assume that all those samples $\{I(x), x \in \Omega\}$ are independent. Then the likelihood (joint pdf) is

$$\prod_{x\in\Omega} \sum_{i=1}^{K} f_i(x) p_i(x) \tag{3}$$

Ideal segmentation should maximize the likelihood or minimize the negative log-likelihood as follows.

$$E[I|P, U] = -\int_\Omega log \left(\sum_{i=1}^{K} f_i(x) p_i(x) \right), \tag{4}$$

Adding total variation of $p_i(x)$ as regularity terms for membership functions $p_i(x)$ respectively leads to the following stochastic variational segmentation model with Gaussian mixture. We add to the energy functional the L^2- norm of ∇ and total variation of $p_i(x)$ as regularity terms for bias field $b(x)$ and membership functions $p_i(x)$ respectively. This leads to the following stochastic variational segmentation model with Gaussian mixture and bias correction.

$$\begin{aligned} E(p, b, c) =& \lambda \int_\Omega -log[\sum_{i=1}^{K} \frac{1}{\sqrt{2\pi}\sigma_i} exp(-\frac{(I(x) - b(x)c_i)^2}{2\sigma_i^2}) p_i] dx \\ &+ \mu \int_\Omega |\nabla b|^2 dx + \sum_{i=1}^{K} \int_\Omega |\nabla p_i| dx \end{aligned} \tag{5}$$

Simply write it as

$$E(p,b,c) = -\lambda \int_\Omega log(\sum f_i(x)p_i(x))dx + \mu \int_\Omega |\nabla b|^2 dx + \sum_{i=1}^{K} \int_\Omega |\nabla p_i| dx \tag{6}$$

where

$$f_i(x) = \frac{1}{\sqrt{2\pi}\sigma_i} exp(-\frac{(I(x) - b(x)c_i)^2}{2\sigma_i^2}) \tag{7}$$

We use two different regularization methods for bias field b and membership functions $p_i(x)$ (L^2-norm for bias field and TV for membership function). The reason is that we require the bias field to be smooth in the entire domain (there is no obvious edge in b). So we use L^2-norm since it is isotropic. However, although we assume all membership functions are defined in the entire domain, each p_i should be close to 1 in the region corresponding to the object and be close to 0 in other regions. Just as in partial volume segmentation, only near the boundary of different objects, the value may neither close to 1 nor close to 0. Hence, each membership function may contains obvious edge. Using TV for its regularization can protect edges since L^1-norm is anisotropic.

III. Numerical Implementation

Note that the energy functional is convex with respect to all parameters except for the variances. For fixed variances, global minimization can be achieved for any initialization. The Euler-Lagrange equations of variance, means and bias are as follows.

$$\frac{\partial E}{\partial \sigma_i^2} = -\lambda \int_\Omega \frac{f_i p_i((I - bc_i)^2 - \sigma_i^2)}{2\sigma_i^4 \sum_{i=1}^{K} f_i p_i} dx \tag{8}$$

$$\frac{\partial E}{\partial c_i} = -\frac{\lambda}{\sigma_i^2} \int_\Omega \frac{f_i p_i (Ib - c_i b^2)}{\sum f_i p_i} dx = 0 \tag{9}$$

$$\frac{\partial E}{\partial b} = -\mu \Delta b - \lambda \frac{\sum f_i p_i (I - bc_i) c_i}{\sigma_i^2 \sum f_i p_i} \tag{10}$$

The only challenge of the implementation is the optimization of membership functions $p_i(x)$ because of the constraints.

$$1 \geq p_i(x) \geq 0$$

and

$$\sum_{i=1}^{K} p_i(x) = 1.$$

The constraint $1 \geq p_i(x) \geq 0$ can be achieved by adding term

$$\gamma \int_\Omega \nu(p_i) dx \tag{11}$$

where ν is defined by

$$\nu(\xi) = max\{0, 2|\xi - \frac{1}{2}| - 1\} \tag{12}$$

and γ should be big enough so that the added term is an exact penalty [Chan, 2006].

The constraint $\sum_{i=1}^{K} p_i(x) = 1$ can be achieved by using Lagrange multiplier method. So, the final energy functional is

$$\begin{aligned}E(p,b,c) =&\lambda\int_\Omega -log[\sum_{i=1}^K f_ip_i]dx\\&+\mu\int_\Omega|\nabla b|^2dx+\sum_{i=1}^K\int_\Omega|\nabla p_i|dx\\&+\int_\Omega\alpha(\sum_{i=1}^K p_i-1)dx+\gamma\sum_{i=1}^K\int_\Omega\nu(p_i)dx\end{aligned} \tag{13}$$

Considering we have fast algorithm for some variational functional, we add some auxiliary variables $v_1,...,v_K$, and consider the dual projection problem [Bresson, 2007; Huang, 2008]

$$\begin{aligned}E(p,v,b,c) =&\lambda\int_\Omega -log[\sum_{i=1}^K f_ip_i]dx\\&+\mu\int_\Omega|\nabla b|^2dx+\sum_{i=1}^K\int_\Omega|\nabla v_i|dx+\frac{1}{2\theta}\sum_{i=1}^K\int_\Omega(p_i-v_i)^2dx\\&+\int_\Omega\alpha(\sum_{i=1}^K p_i-1)dx+\gamma\sum_{i=1}^K\int_\Omega\nu(p_i)dx\end{aligned} \tag{14}$$

The v-minimization problem can be efficiently solved by fast duality projection algorithm [Chambolle, 2004]. The solution is given by

$$v_i=p_i-\theta div(d_i),\ 1\le i\le K \tag{15}$$

where the vector d_i can be solved by fixed point method: Initializing $d_i^0=0$ and iterating

$$d_i^{n+1}=\frac{d_i^n+\tau\nabla(div(d_i^n)-p_i/\theta)}{1+\tau|\nabla(div(d_i^n)-p_i/\theta)|} \tag{16}$$

with $\tau\le 1/8$ to ensure convergence.

The derivative of the energy functional with respect to p_i is

$$\frac{\partial E}{\partial p_i}=-\lambda\frac{f_i}{\sum f_ip_i}+\frac{1}{\theta}(p_i-v_i)+\alpha=0 \tag{17}$$

The p_i can not be solved explicitly. We use the following iteration which is also very fast.

$$p_i^{(n+1)}=v_i^{(n)}+\theta\left\{\lambda\frac{f_i}{\sum f_ip_i}-\alpha\right\}^{(n)} \tag{18}$$

With the constraint that $\sum p_i=1$ we get

$$\alpha=\frac{\sum v_i-1}{\theta K}+\frac{\lambda\sum f_i}{K\sum f_ip_i} \tag{19}$$

Substitute α in previous formula, we get the iteration

$$p_i^{(n+1)}=v_i^{(n)}+\theta\left\{\lambda\frac{f_i}{\sum f_ip_i}-\frac{\sum v_i-1}{\theta K}-\frac{\lambda\sum f_i}{K\sum f_ip_i}\right\}^{(n)} \tag{20}$$

Finally, we take the max-min operation to make sure that $0\le p_i(x)\le 1$.

$$p_i(x)=min\{max\{p_i(x),0\},1\} \tag{21}$$

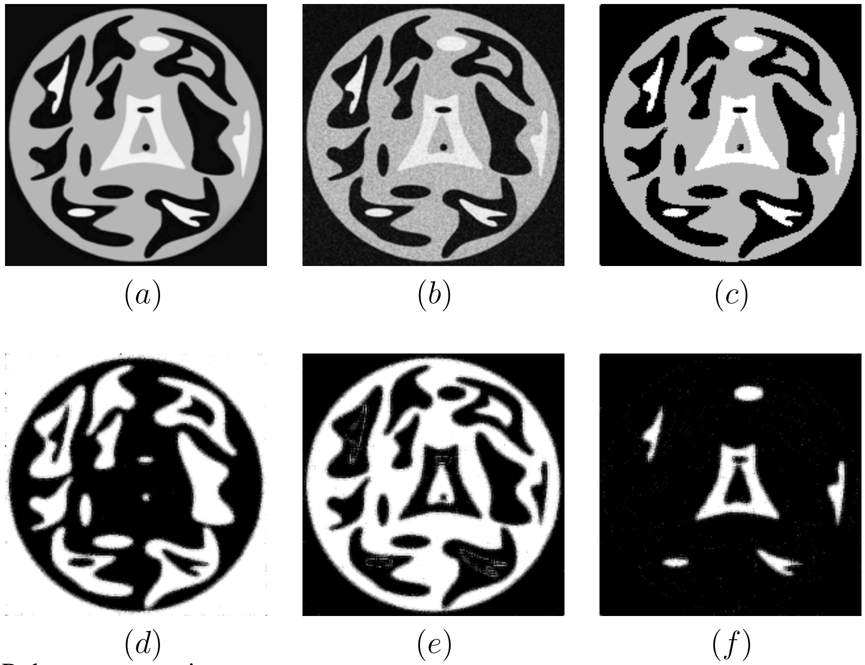

Fig. 1. Experiment 1: Robustness to noise

IV. Experiment and Discussion

In this part, we demonstrate the good performance of the proposed model with several experiments in different aspects. Since the paper is discussing soft segmentation model, we present all membership functions in each experiment, where at each pixel, if the intensity is very bright, then it means that the corresponding membership function at that point is very close to 1; on the contrary, if the intensity is very dark, then it means that the membership function has a value close to 0. For some examples (Fig.3), we also show hard segmentation by thresholding.

The first experiment aims at the robustness to noise. In Fig.1, (a) is the original synthetic image, (b) is obtained by adding Gaussian noise $G(\mu, \sigma^2)$ to (a) with $\mu = 0$ and $\sigma^2 = 0.02$. The result is shown in (c) for hard segmentation and (d)-(f) for membership functions.

The second experiment aims at the robustness to bias.

In Fig.2, (a) is the original image, (b) is a vertical bias field. (c) is the biased image obtained by adding multiplicative bias (b) to image (a). $(d), (e)$ and (f) are the three membership functions obtained using the proposed model. Clearly, there is no bias in the membership functions.

Fig.3 gives a segmentation for real MRI brain image. (a) is the original real MRI brain image. $(c), (d), (e)$ are membership functions of CSF, gray matter and white matter, respectively. (b) is the hard segmentation after thresholding.

V. Conlusion

In this paper, we proposed a stochastic variational model for multiphase soft segmentation based on Gaussian mixture. Compared with previous models, the proposed model is more robust to noise and bias. Several experiments are presented to demonstrate the efficiency of our model.

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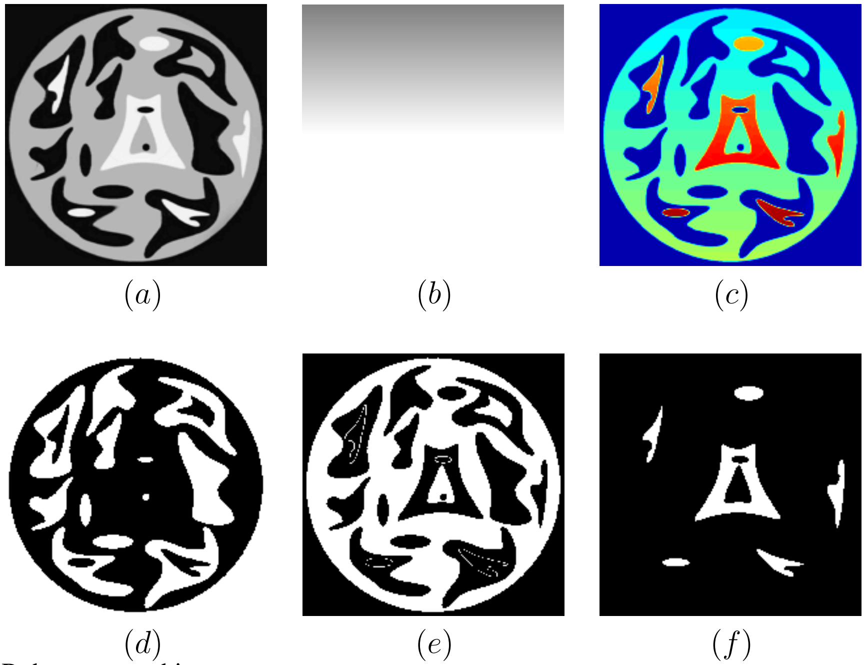

Fig. 2. Experiment 2: Robustness to bias.

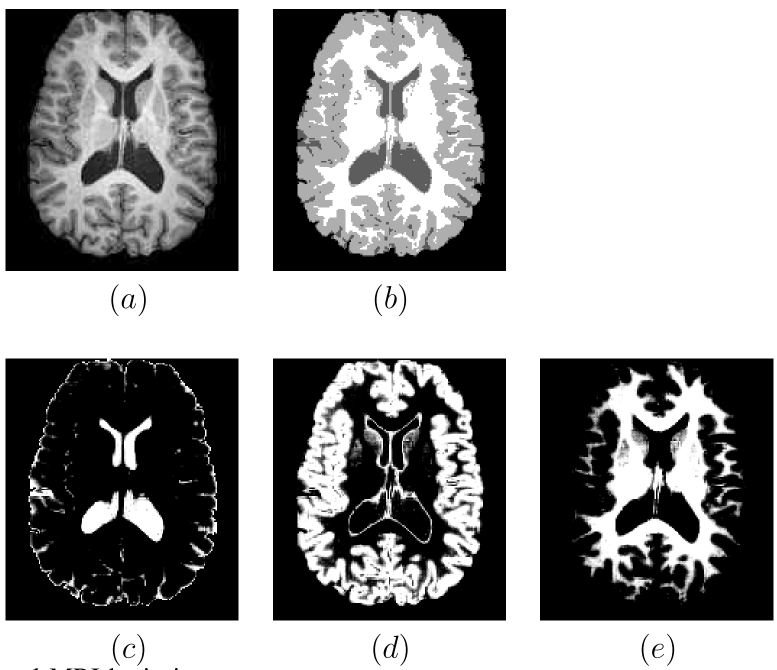

Fig. 3. Comparison using real MRI brain image

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