AMO - Advanced Modeling and Optimization, Volume 12, Number 3, 2010

A Two-stage Descent Method with Optimal Step-sizes for Monotone Variational Inequality Problems

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Abstract. In this paper, we propose a two-stage descent method for monotone variational inequality problems, which only needs functional values for given variables in the solution process. Under certain conditions, the global convergence of the method is proved. Preliminary numerical experiments are included to illustrate the efficiency of the proposed method.

AMS(2000) Subject Classification: 90C25, 90C30

1 Introduction

A classical variational inequality, denoted by VI(f, S), which is to find a vector $x^* \in S$, such that

$$(x - x^*)^\top f(x^*) \geq 0 \quad \forall x \in S, \tag{1}$$

where $S \subseteq R^n$ is a nonempty closed convex subset of R^n and f is a continuous mapping from R^n into itself. The set S in VI(f, S) often has the following structure, see [3-5]:

$$S = \{x | Ax = b, x \in X\} \tag{2}$$

or

$$S = \{x | Ax \leq b, x \in X\} \tag{3}$$

where $A \in R^{m \times n}$,$b \in R^m$, and X is a simple closed convex subset of R^n.

In this paper, we focus our attention on the variational inequality problem (1) that S has the structure (2), which is also denoted by VI(f, S). This class of variational inequality problems arise frequently in

[1]This work was supported by the Foundation of Shandong Provincial Education Department (No.J10LA59)

***AMO-Advanced Modeling and optimization. ISSN: 1841-4311**

some practical applications. For example, in the traffic assignment problem, by representing x as the route flow variable, the demand constraint can by expressed in the form $Ax = b$.

By appending a Lagrange multiplier $y \in R^m$ to the linear constraints $Ax = b$, VI(f, S) can be translated to an enlarged but compact form(denoted by VI(F,Ω)): find $u^* \in \Omega$ such that

$$(u - u^*)^\top F(u^*) \geq 0, \forall\ u \in \Omega, \tag{4}$$

where $u = \begin{pmatrix} x \\ y \end{pmatrix}$, $F(u) = \begin{pmatrix} f(x) - A^\top y \\ Ax - b \end{pmatrix}$, $\Omega = X \times Y$.

Among powerful approaches to solving structured VI(F,Ω) is the alternating direction method(ADM), which was originally proposed in [1,2]. In particular, for the given $u^k = (x^k, y^k) \in X \times Y$, the new iterate $u^{k+1} = (x^{k+1}, y^{k+1})$ is generated by the following procedure: find $x^{k+1} \in X$, such that

$$(x' - x^{k+1})^\top f(x^{k+1}) - A^\top[y^k - (Ax^{k+1} - b)] \geq 0, \quad \forall x' \in X, \tag{5}$$

then update y via

$$y^{k+1} = y^k - (Ax^{k+1} - b).$$

The method is attractive for large-scale problems since it decomposes the original problem into a series of small-scale problems. However, note that its subproblem (5) is still a variational inequality problem, which is usually difficult to solve efficiently and exactly at each iteration.

Motivated by the above observation, this paper presents a two-stage descent method for monotone VI(f, S). Firstly, the separable structure of S is utilized to generate a descent direction; and an appropriate step size along this descent direction is identified to generate a temporal point. Then, an additional projection step is performed, and another optimal step size which is depended on the previous points is employed to generate next iterate.

The remainder of the paper is organized as follows: some definitions and properties used in this paper are presented in Section 2. In Section 3, a two-stage descent method is given and its global convergence is proved. Some preliminary computational results are given in Section 4.

2 Preliminaries

In this section, we first give some basic properties and related definitions used in the sequent sections.

First, we denote $\|x\| = \sqrt{x^\top x}$ as the Euclidean norm. For a given vector $x \in R^n$, the orthogonal projection of x onto the set X, is defined as the nearest vector $y \in X$ to x, i.e.,

$$P_K(x) = \text{argmin}\{\|y - x\| \mid y \in K\}.$$

Similarly, we denote $P_\Omega(\cdot)$ as the orthogonal projection mapping from R^{n+m} onto Ω. That is

$$P_\Omega(u) = P_\Omega \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} P_X(x) \\ y \end{pmatrix}, \qquad u = (x, y) \in R^{n+m}.$$

The projection mapping P_X has the following important properties, which will be used in the following.

Lemma 2.1. For any $x \in R^n$, $y \in X$, the following inequalities hold

$$(x - P_X(x))^\top (y - P_X(x)) \le 0; \tag{6}$$

$$||P_X(x) - y||^2 \le ||x - y||^2 - \|x - P_X(x)\|^2. \tag{7}$$

Many numerical methods for solving variational inequality problems are based on the following well-known result due to Eaves[6], that is, VI(F, Ω) is equivalent to a projection equation

$$u = P_\Omega[u - \beta F(u)],$$

where β is an arbitrary but fixed positive constant. Let

$$r(u, \beta) = \begin{pmatrix} r_1(u, \beta) \\ r_2(u, \beta) \end{pmatrix} = u - P_\Omega[u - \beta F(u)] = \begin{pmatrix} x - P_X[x - \beta(f(x) - A^\top y)] \\ \beta(Ax - b) \end{pmatrix}$$

denote the residual function of the projection equation. VI(F, Ω) is equivalent to finding a zero point of the residual function $r(u, \beta)$.

The following lemma plays an important role in the global convergent analysis of our algorithm.

Lemma 2.2. For all $u \in R^{m+n}$ and $\rho_1 > \rho_2 > 0$, it holds that

$$\|r(u, \rho_1)\| \ge \|r(u, \rho_2)\|. \tag{8}$$

$$\frac{\|r(u, \rho_1)\|}{\rho_1} \le \frac{\|r(u, \rho_2)\|}{\rho_2}. \tag{9}$$

Definition 2.1. A mapping $f : R^n \to R^n$ is said to be monotone if

$$(x - y)^\top (f(x) - f(y)) \ge 0, \qquad \forall x, y \in R^n.$$

Throughout of this paper, we assume the solution set of VI(f, S), denoted by S^*, is nonempty, and the solution set of VI(F, Ω), denoted by Ω^*, is also nonempty.

3 Main results

For simplicity, set $r_i = r_i(u, \beta), i = 1, 2,\ g = f(x) - A^\top y$.

Lemma 3.1. Let $u^* = (x^*, y^*) \in \Omega^*$ be an arbitrary solution of VI(F, Ω), and

$$d(u,\beta) := \begin{pmatrix} r_1 - \beta f(x) + \beta f(x - r_1) + \beta A^\top r_2 \\ r_2 - \beta A r_1 \end{pmatrix},$$

then for any $u = (x, y) \in R^{n+m}$, $\beta > 0$, we have

$$(u - u^*)^\top d(u, \beta) \geq \|r_1\|^2 + \|r_2\|^2 - \beta r_1^\top (f(x) - f(x - r_1)).$$

Proof. The proof is similar to that of Lemma 1 of [3].

Lemma 3.2 If $u \in \Omega$ is not a solution of VI(F, Ω), then for any $\delta \in (0, 1)$, there exist $\tilde{\beta}(u) > 0$, such that $\forall \beta \in (0, \tilde{\beta}(u)]$, we have

$$\beta \|f(x) - f(x - r_1)\| \leq \delta \|r(u, \beta)\|. \tag{10}$$

Proof. See Lemma 3 of [4].

If $u \in \Omega$ is not a solution of VI(F, Ω), from Lemma 3.1 and Lemma 3.2, there is a positive $\beta > 0$, such that

$$(u - u^*)^\top d(u, \beta) \geq (1 - \delta)\|r(u, \beta)\|^2. \tag{11}$$

which means that $-d(u, \beta)$ is a descent direction of the merit function $\|u - u^*\|^2/2$ whenever u is not a solution of VI(F, Ω). This motivates us to construct the following algorithm.

Algorithm 3.1 Two-stage descent method for VI(f,S).

Step 0: Given $\varepsilon > 0$. Choose $u^0 \in \Omega$, and positive parameters $\mu \in (0, 1)$, $\gamma_1, \gamma_2 \in [1, 2)$, $\beta = 1.0$, $\delta \in (0, 1)$, $v \in (0, 1)$ and and a nonnegative sequence $\{\mu_k\}$, satisfying $\sum_{k=1}^{+\infty} \mu_k < +\infty$. Set $k := 0$;

Step 1: Set $\beta_k = \beta$. If $\|r(u^k, \beta_k)\| < \varepsilon$, then stop; else, find that smallest nonnegative integer m_k, such that $\beta_k = \beta\mu^{m_k}$ satisfying

$$\beta_k \|f(x^k) - f(P_X[x^k - \beta_k(f(x^k) - A^\top y^k)])\| \leq \delta \|r(u^k, \beta_k)\|. \tag{12}$$

Step 2: Calculate $d(u^k, \beta_k)$ by the expression of $d(u, \beta)$ in Lemma 3.1 and

$$\rho_k = (1 - \delta)\|r(u^k, \beta_k)\|^2 / \|d(u^k, \beta_k)\|^2. \tag{13}$$

Then calculate the temporal iterate $\tilde{u}^k = P_\Omega[u^k - \gamma_1 \rho_k d(u^k, \beta_k)]$.

Step 3: Calculate the next iterate $u^{k+1} = P_\Omega[u^k - \gamma_2 \lambda_k (u^k - \tilde{u}^k)]$, where the step length λ_k is defined by

$$\lambda_k = \frac{\|u^k - \tilde{u}^k\|^2 + \gamma_1(2 - \gamma_1)\rho_k \|r(u^k, \beta_k)\|^2}{2\|u^k - \tilde{u}^k\|^2}. \tag{14}$$

Step 4:(adjust β[4]) If

$$\beta_k\|f(x^k)-f(P_X[x^k-\beta_k(f(x^k)-A^\top y^k)])\|\geq v\|r(u^k,\beta_k)\|$$

then set $\beta=(1+\mu_k)\beta_k$, else set $\beta=\beta_k$. Set $k:=k+1$, go to Step 1.

Remark 3.1 It follows from Lemma 3.2 that at each k, if $u^k\in\Omega$ is not a solution of $\mathrm{VI}(F,\Omega)$, then the line search procedure is well defined, the algorithm is therefore well defined.

Remark 3.2 From $\sum_{k=1}^{+\infty}\mu_k<+\infty$ and it is nonnegative, there is a positive integer M, such that

$$\prod_{k=1}^{+\infty}(\mu_k+1)<M.$$

So we have $\beta_k<M,\quad \forall\ k=1,2,\cdots$.

In the following, we assume that the algorithm does not stop in finite steps and an infinite sequence $\{u^k\}$ is generated.

We first investigate the technique of identifying the optimal step sizes along the descent directions $d(u^k,\beta_k)$. To justify the strategy of choosing the step size ρ_k as in Step 2, we use

$$\widetilde{u}^k(\rho):=P_\Omega[u^k-\rho d(u^k,\beta_k)].$$

to denote the temporary point taking ρ as the step size along $d(u^k,\beta_k)$, the the following lemma motivates us to identify the optimal step size along this direction.

Lemma 3.3. For given u^k and $\beta_k>0$, we have

$$\Theta_k(\rho):=\|u^k-u^*\|^2-\|\widetilde{u}^k(\rho)-u^*\|\geq\Phi_k(\rho),$$

where

$$\Phi_k(\rho)=-\rho^2\|d(u^k,\beta_k)\|^2+2\rho(1-\delta)\|r(u^k,\beta_k)\|^2.$$

Proof. Because $\widetilde{u}^k(\rho):=P_\Omega[u^k-\rho d(u^k,\beta_k)]$, by setting $x=u^k-\rho d(u^k,\beta_k)$ and $y=u^*$ in (7), we obtain

$$\|\widetilde{u}^k(\rho)-u^*\|^2\leq\|u^k-\rho d(u^k,\beta_k)-u^*\|^2-\|u^k-\rho d(u^k,\beta_k)-\widetilde{u}^k(\rho)\|^2,$$

and consequently

$$\Theta_k(\rho)\geq\|u^k-\widetilde{u}^k(\rho)\|^2+2\rho(u^k-u^*)^\top d(u^k,\beta_k)-2\rho(u^k-\widetilde{u}^k(\rho))^\top d(u^k,\beta_k),$$

Since u^* is a solution, it follows from (11) that

$$\begin{aligned}&\Theta_k(\rho)\\ \geq\ &\|u^k-\widetilde{u}^k(\rho)\|^2+2\rho(1-\delta)\|r(u^k,\beta_k)\|^2-2\rho(u^k-\widetilde{u}^k(\rho))^\top d(u^k,\beta_k)\\ =\ &\|u^k-\widetilde{u}^k(\rho)-\rho d(u^k,\beta_k)\|^2+\rho^2\|d(u^k,\beta_k)\|^2+2\rho(1-\delta)\|r(u^k,\beta_k)\|^2\\ \geq\ &-\rho^2\|d(u^k,\beta_k)\|^2+2\rho(1-\delta)\|r(u^k,\beta_k)\|^2:=\Phi_k(\rho)\end{aligned}$$

The assertion follows from the above inequality directly. The proof is completed.

Clearly, $\Theta_k(\rho)$ means the progress made by the temporal point $\tilde{u}^k(\rho)$ at the kth iteration. Therefore, in order to accelerate the convergence, it is reasonable to choose

$$\rho_k = (1-\delta)\|r(u^k, \beta_k)\|^2 / \|d(u^k, \beta_k)\|^2,$$

i.e., the optimal value of ρ maximizing the quadratic function $\Phi_k(\rho)$ which provides a lower bound function of $\Theta_k(\rho)$. Based on numerical experiences, we prefer to attach a relax factor $\gamma_1 \in [1, 2)$ to ρ_k, that is, $\tilde{u}^k = P_\Omega[u^k - \gamma_1 \rho_k d(u^k, \beta_k)]$, and simple calculation show that

$$\Phi_k(\gamma_1 \rho_k) = \gamma_1(2-\gamma_1)\Phi_k(\rho_k) = \gamma_1(2-\gamma_1)\rho_k(1-\delta)\|r(u^k, \beta_k)\|^2. \tag{15}$$

We now consider the criteria of λ_k, which ensures that u^{k+1} is closer to the solution set than u^k. For this purpose, we define

$$\Gamma_k(\lambda) := \|u^k - u^*\|^2 - \|u^{k+1}(\lambda) - u^*\|^2, \tag{16}$$

where $u^{k+1}(\lambda) = P_\Omega[u^k - \lambda(u^k - \tilde{u}^k)]$.

Lemma 3.4. Let $u^* \in \Omega^*$. Then we have

$$\Gamma_k(\lambda) \geq \lambda\{\|u^k - \tilde{u}^k\|^2 + \|u^k - u^*\|^2 - \|\tilde{u}^k - u^*\|^2\} - \lambda^2\|u^k - \tilde{u}^k\|^2. \tag{17}$$

Proof. It follows from (7) and (16) that

$$\begin{aligned} & \Gamma_k(\lambda) \\ \geq\ & \|u^k - u^*\|^2 - \|u^k - \lambda(u^k - \tilde{u}^k) - u^*\|^2 \\ =\ & 2\lambda(u^k - u^*)^\top(u^k - \tilde{u}^k) - \lambda^2\|u^k - \tilde{u}^k\|^2 \\ =\ & 2\lambda\{\|u^k - \tilde{u}^k\|^2 - (u^* - \tilde{u}^k)^\top(u^k - \tilde{u}^k)\} - \lambda^2\|u^k - \tilde{u}^k\|^2 \end{aligned}$$

Using the following identity

$$(u^* - \tilde{u}^k)^\top(u^k - \tilde{u}^k) = \tfrac{1}{2}(\|\tilde{u}^k - u^*\|^2 - \|u^k - u^*\|^2) + \tfrac{1}{2}\|u^k - \tilde{u}^k\|^2.$$

we obtain (17), the required result. The proof is completed.

Using Lemma 3.3, (15) and (17), we get

$$\Gamma_k(\lambda) \geq \Upsilon_k(\lambda) = \lambda\{\|u^k - \tilde{u}^k\|^2 + \Lambda_k\} - \lambda^2\|u^k - \tilde{u}^k\|^2, \tag{18}$$

where

$$\Lambda_k = \gamma_1(2-\gamma_1)\rho_k(1-\delta)\|r(u^k, \beta_k)\|^2. \tag{19}$$

The above inequality tells us how to choose a suitable λ_k. Since $\Upsilon_k(\lambda)$ is a quadratic function of λ and it reaches its maximum at

$$\lambda_k = \frac{\|u^k - \tilde{u}^k\|^2 + \Lambda_k}{2\|u^k - \tilde{u}^k\|^2},$$

and

$$\Upsilon_k(\lambda_k) = \frac{\lambda_k[\|u^k - \tilde{u}^k\|^2 + \Lambda_k]}{2}. \tag{20}$$

In addition, from $\gamma_1 \in [1, 2)$ and $\delta \in (0, 1)$, we have

$$\Lambda_k \geq 0, \quad \lambda_k \geq \frac{1}{2},$$

and from (18) (20), we get

$$\Gamma_k(\lambda_k) \geq \Upsilon_k(\lambda_k) \geq \frac{\Lambda_k}{4}.$$

For fast convergence, we take a relax factor $\gamma_2 \in [1, 2)$ to λ_k, and then

$$\Gamma_k(\gamma_2\lambda_k) \geq \Upsilon_k(\gamma_2\lambda_k) = \gamma_2(2-\gamma_2)\Upsilon_k(\lambda_k) \geq \gamma_2(2-\gamma_2)\frac{\Lambda_k}{4}. \tag{21}$$

Theorem 3.1. Suppose that the operator $f(\cdot)$ is continuous and monotone. Then the sequence of $\{u^k\} = \{(x^k, y^k)\}$ generated by algorithm 3.1 is bounded.

Proof. From (21), it follows that

$$\|u^{k+1} - u^*\|^2 \leq \|u^k - u^*\|^2 - \gamma_2(2-\gamma_2)\Lambda_k/4. \tag{22}$$

From $\gamma_2 \in [1, 2)$, $\Lambda_k \geq 0$, we obtain

$$\|u^{k+1} - u^*\| \leq \|u^k - u^*\| \leq ... \leq \|u^0 - u^*\|.$$

This implies that the sequence $\{u^k\}$ is bounded. The proof is completed.

From Theorem 3.1, the proposed method is a projection and contraction method because the new iterate u^{k+1} is closer to the solution set Ω^* than u^k.

Lemma 3.4 Suppose that the operator $f(x)$ is continuous, then there is $\tau > 0$, such that

$$\rho_k \geq \frac{1-\delta}{\tau^2} > 0. \tag{23}$$

Proof. The proof is quite easy, so is omitted.

Now, we are in the stage to prove the convergence of the proposed method.

Theorem 3.2 Suppose that the assumptions in Theorem 3.1 hold. Then, the whole sequence $\{u^k\}$ converges to a solution of VI(F, Ω).

Proof. It follows from (22) that

$$\sum_{k=0}^{\infty} \Lambda_k < \infty.$$

which means that

$$\lim_{k\to\infty} \Lambda_k = 0. \tag{24}$$

From (19), (23) and (24), we have

$$\lim_{k\to\infty} \|r(u^k, \beta_k)\| = 0. \tag{25}$$

It follows from Lemma 2.2 that

$$\|r(u^k, \beta_L)\| \geq \min\{1, \beta_k\}\|r(u^k, 1)\|.$$

This together with (25) means that

$$\lim_{k\to\infty} \beta_k \|r(u^k, 1)\| = 0. \tag{26}$$

We consider the two possible cases. Firstly, suppose that

$$\limsup_{k\to\infty} \beta_k > 0.$$

It follows from (26) that

$$\liminf_{k\to\infty} \|r(u^k, 1)\| = 0.$$

Since $\{u^k\}$ is bounded, it has a cluster point $\bar{u} \in \Omega$ such that $\|r(\bar{u}, 1)\| = 0$. That is, $\bar{u}$ is a solution of $\mathrm{VI}(F, \Omega)$. Since u^* is an arbitrary solution, we can just take $u^* = \bar{u}$ in Theorem 3.1 and we have

$$\|u^{k+1} - u^*\| \leq \|u^k - u^*\|.$$

The whole sequence $\{u^k\}$ therefore converges to $\bar{u}$, a solution of $\mathrm{VI}(F, \Omega)$.

Now, we consider the other possible case that

$$\lim_{k\to\infty} \beta_k = 0.$$

By the choice of β_k we know that (12) was not satisfied for $m_k - 1$. That is,

$$\|f(x^k) - f(x^k - r_1(u^k, \beta_k/\mu))\| > \delta\mu\|r(u^k, \beta_k/\mu)\|/\beta_k.$$

Combining the above inequality and (8), we have

$$\|f(x^k) - f(x^k - r_1(u^k, \beta_k/\mu))\| > \delta\|r(u^k, 1)\|. \tag{27}$$

Suppose $\tilde{u}$ is a cluster point of $\{u^k\}$, there exists a subsequence $\{u^{k_j}\}$ converging to it. Taking limit along such a sequence in (27) and using the continuity of $r(\cdot, \beta)$, we have

$$\|r(\tilde{u}, 1)\| = 0.$$

Hence, $\tilde{u}$ is a solution of $\mathrm{VI}(F, \Omega)$. Set $u^* = \tilde{u}$ in (27), we again have

$$\|u^{k+1} - \tilde{u}\| \leq \|u^k - \tilde{u}\|,$$

and the whole sequence $\{u^k\}$ converges to $\tilde{u}$, a solution of $\mathrm{VI}(F, \Omega)$. This completes the proof.

4 Numerical experiment

In this section, we give some preliminary computational results to test the ability of the proposed Algorithm 3.1. All codes are written in MATLAB 7.1 and run on a PIV 2.0 GHz personal computer.

The example used here is taken from the test problems of Zhang and Han[5], which constraint set S and the mapping f are taken, respectively, as

$$S = \{x \in R^5_+ | \sum_{i=1}^{5} x_i = 10\}.$$

and

$$f(x) = Mx + \rho C(x) + q.$$

where M is an $R^{5\times 5}$ asymmetric positive matrix and $C_i(x) = \arctan(x_i - 2), i = 1, 2, \cdots, 5$. The parameter ρ is used to vary the degree of asymmetry and nonlinearity, and the data of example are illustrate as follows:

$$M = \begin{pmatrix} 0.726 & -0.949 & 0.266 & -1.193 & -0.504 \\ 1.645 & 0.678 & 0.333 & -0.217 & -1.443 \\ -1.016 & -0.225 & 0.769 & 0.943 & 1.007 \\ 1.063 & 0.587 & -1.144 & 0.550 & -0.548 \\ -0.256 & 1.453 & -1.073 & 0.509 & 1.026 \end{pmatrix}$$

and

$$q = (5.308,\ 0.008\ , -0.938,\ 1.024,\ -1.312)'$$

In this experiment, we take the stopping criterion $\varepsilon = 10^{-6}$. For Algorithm 3.1, we take $y^0 = 5$ as the initial point and $\beta_0 = 0.6, \mu = 0.85$, $\gamma_1 = \gamma_2 = 1.4$, $v = 0.25$ and $\delta = 0.8$ for $\rho = 10$ and $\rho = 20$. For the method in [5], denoted by Zhang and Han's method, we take $\beta_k \equiv 0.06$, $\delta = 1.35$ when $\rho = 10$ and $\beta_k \equiv 0.05$, $\delta = 1.35$ when $\rho = 20$. The results for $\rho = 5$ and $\rho = 10$ are listed in Tables 1 and 2. In these tables, $'$IT$'$ denote the number of iterations, and $'$CPU$'$ denotes the cputime in seconds.

The results in Tables 1,2 indicate that the proposed Algorithm 3.1 is efficient. Though its CPU time is almost the same as Zhang and Han's method, its iterative number is smaller than the latter. As this descent method only requires function evaluations per iteration, it is attractive from a computational point of view.

Table 1: Numerical results for $\rho = 10$

Starting point	Algorithm	IT	CPU
(25, 0, 0, 0, 0)	Algorithm 3.1	97	0.01
	Zhang and Han's method	119	0.01
(10, 0, 0, 0, 0)	Algorithm 3.1	86	0.01
	Zhang and Han's method	99	0.01
(10, 0, 10, 0, 10)	Algorithm 3.1	81	0.01
	Zhang and Han's method	108	0.01
(0, 2.5, 2.5, 2.5, 2.5)	Algorithm 3.1	89	0.01
	Zhang and Han's method	109	0.02

Table 2: Numerical results for $\rho = 20$

Starting point	Algorithm	IT	CPU
(25, 0, 0, 0, 0)	Algorithm 3.1	110	0.01
	Zhang and Han's method	116	0.02
(10, 0, 0, 0, 0)	Algorithm 3.1	99	0.01
	Zhang and Han's method	173	0.02
(0, 0, 0, 0, 0)	Algorithm 3.1	108	0.02
	Zhang and Han's method	173	0.02
(2.5, 0, 2.5, 0, 2.5)	Algorithm 3.1	98	0.01
	Zhang and Han's method	170	0.02

References

[1] Gabay D, *Applications of the method of multipliers to variational inequalities.* In Fortin M, Glowinski R. eds, Augmented Lagrangian Methods: Applications to the Solution of Boundary-Valued Problems. North-Holland, Amsterdam, 299-331, 1983.

[2] Gabay D, Mercier B, A dual algorithm for the solution of nonlinear variational problems via finite-element approximations, *Computers and Mathematics with Applications*, vol. 2, no. 1, pp. 17-40, 1976.

[3] M.Sun, J.Xu, Z.Shi, A new projection-type alternating direction method for variational inequality problems, *Chinese Journal of Engineering Mathematics*, vol.23, no. 6, pp. 1101-1104, 2006.

[4] D.Han, A modified alternating direction method for variational inequality problems, *Applied Mathematics and Optimization*, vol. 45, pp. 63-74, 2002.

[5] W.Zhang, D.Han, A new alternating direction method for co-coercive variational inequality problems, *Computers and Mathematics with Applications*, vol. 57, no. 7, pp. 1168-1178, 2009.

[6] B.C.Eaves, Computing stationary points, *Mathematical Programming Study*. vol. 7, pp. 1-14, 1978.

[7] M.Sun, A New Projection-type Alternating Direction Method For Monotone Variational Inequality Problems, *Journal of the Operations Research Society of Japan*. vol. 52, no. 1, pp. 1-10, 2009.