Comment on Combinatorial analysis by the Ihara zeta function of graphs

O. Shanker *

Abstract

In this short note we point out that a recent article in this journal incorrectly attributes some properties to Ihara zeta functions. The properties actually are attributable to another class of complex network zeta functions, which are used in computer science and mathematical physics applications.

In an interesting study [AGarrido2009] have presented many aspects of Ihara zeta functions of graphs. While the bulk of the analysis is valid, we note that the properties attributed to the Ihara zeta functions on pages 261 and 262, and the discussion of monotonicity, stability and Lipschitz Invariance, are not correct. These properties seem to be taken from a class of functions different from Ihara zeta functions, and the other class of functions are reported in [OShanker2007]. The class of functions for which these properties actually hold are now called complex network zeta functions [OShanker2008], to avoid confusion with Ihara graph zeta functions. They have been studied in the mathematical physics and computer science [OShanker2009, OShanker2010] literature.

To help clear up the confusion, we briefly present here the definition of the complex network zeta function, with particular reference to the properites presented in [AGarrido2009] which actually hold for the complex network zeta functions and not for Ihara graph zeta functions. Let us denote by r_{ij} the distance from node i to node j of a complex network (the length of the shortest path connecting the node i to node j). r_{ij} is ∞ if there is no path from node i to node j. This definition of distance satisfies the triangle inequality, and hence the nodes of the complex network form a metric space.

*Mountain View, CA 94041, U. S. A. Email: oshanker@gmail.com
***AMO - Advanced Modeling and Optimization. ISSN: 1841-4311**

The complex network zeta function $\zeta_G(\alpha)$ is defined as

$$\zeta_G(\alpha) := \limsup_{node\ i} \sum_{j \neq i} r_{ij}^{-\alpha}. \tag{1}$$

Originally the definiton was given as an average over all nodes (average over i), but the formulation in terms of $\limsup$ given in Eq. 1 makes it smoother to apply for formally infinite graphs. The definition Eq. 1 can be expressed as a weighted sum over the node distances. When the exponent α tends to infinity, the sum in Eq. 1 gets contributions only from the nearest neighbours of a node. The other terms tend to zero. Thus, $\zeta_G(\alpha)$ tends to the average vertex degree for the complex network. When α is zero the sum in Eq. 1 gets a contribution of one from each node. This means that $\zeta_G(\alpha)$ is $N-1$, where N is the graph size, measured by the number of nodes. Hence $\zeta_G(\alpha)$ tends to infinity as the system size increases.

Furthermore, $\zeta_G(\alpha)$ is a decreasing function of α,

$$\zeta_G(\alpha_1) > \zeta_G(\alpha_2), \tag{2}$$

if $\alpha_1 < \alpha_2$. Thus, if it is finite for any value of α, it will remain finite for all higher values of α. If it is infinite for some value of α, it will remain infinite for all lower values of α. Thus, there is at most one value of α, $\alpha_{transition}$, at which $\zeta_G(\alpha)$ transitions from being infinite to being finite. This is reminiscient of the behaviour of Hausdorff dimension [Falconer2003]. In fact, we can define the complex network dimension as the value of the exponent α at which $\zeta_G(\alpha)$ transitions from being infinite to being finite, i.e.,

$$d_{zeta-function} := \alpha_{transition}. \tag{3}$$

If $\zeta_G(\alpha)$ remains infinite in the large system limit for all values of α, we define the graph dimension to be infinite. For regular discrete d-dimensional lattices $\mathbf{Z}^d$ with distance defined using the L^1 norm the complex network zeta function can be explicitly evaluated [OShanker2007]. It is a sum of shifted Riemann zeta functions. One finds that the transition occurs at $\alpha = d$, as is reasonable for a good definition of the complex network dimension.

Let us define the distance for regular discrete d-dimensional lattices $\mathbf{Z}^d$ using the L^1 norm:

$$\|\vec{n}\|_1 = \|n_1\| + \cdots + \|n_d\|. \tag{4}$$

We are viewing the lattice as a graph, with the lattice points as the nodes and the links being to the closest neighbours along the coordinate axes. Because of the homogenity of the lattice points, the definition Eq. 1 can be expressed as a Dirichlet series expression:

$$\zeta_G(\alpha) = \sum_r S(r)/r^\alpha, \tag{5}$$

where the graph surface function, $S(r)$, is defined as the number of nodes which are exactly at a distance r from a given node, averaged over all nodes of the network. For a one-dimensional regular lattice there are two nearest neighbours, two next-nearest neighbours, etc. Thus, the graph surface function $S_1(r)$ is exactly two for all values of r. The complex network zeta function $\zeta_G(\alpha)$ is equal to $2\zeta(\alpha)$, where $\zeta(\alpha)$ is the usual Riemann zeta function. The $S_d(r)$ satisfy the recursion relation

$$S_{d+1}(r) = 2 + S_d(r) + 2\sum_{i=1}^{r-1} S_d(i). \tag{6}$$

This result follows by choosing a given axis of the lattice and summing over cross-sections for the allowed range of distances along the chosen axis. Asymptotically, $S_d(r) \to 2^d r^{d-1}/\Gamma(d)$ as $r \to \infty$. $r \to \infty$ corresponds to $\alpha \to \alpha_{transition}$. Thus, $\zeta_G(\alpha) \to 2^d\zeta(\alpha - d + 1)/\Gamma(d)$ as $\alpha \to \alpha_{transition}$. $\zeta_G(\alpha) \to 2d$ as $\alpha \to \infty$. Table 1 gives $\zeta_G(\alpha)$ as a function of the lattice dimension d. It follows from the recursion relation 6 that $S_d(r)$ is a polynomial of order $d-1$ in r, with only even or odd terms present. $\zeta_G(\alpha)$ is the sum of different Riemann zeta functions. The poles of $\zeta_G(\alpha)$ occur for $\alpha = d$ and for all positive integer values of α which are less than d and differ from d by an even number.

Table 1: Graph zeta function for different regular lattices.

Dimension d	$\zeta_G(\alpha)$
1	$2\zeta(\alpha)$
2	$4\zeta(\alpha-1)$
3	$4\zeta(\alpha-2)+2\zeta(\alpha)$
4	$\frac{8}{3}\zeta(\alpha-3)+\frac{16}{3}\zeta(\alpha-1)$
5	$\frac{4}{3}\zeta(\alpha-4)+\frac{20}{3}\zeta(\alpha-2))+2\zeta(\alpha)$
$\alpha \to \alpha_{transition}$	$O(2^d\zeta(\alpha-d+1)/\Gamma(d))$

In applying Eq. 6 to calculate the surface function for higher values of d, we need an expression for the sum of positive integers raised to a given power k. The following result is useful:

$$\sum_{i=1}^{r} i^k = \frac{r^{k+1}}{(k+1)} + \frac{r^k}{2} + \sum_{j=1}^{(k+1)>2j} \frac{(-1)^{j+1}2\zeta(2j)k!r^{k+1-2j}}{(2\pi)^{2j}(k+1-2j)!}. \tag{7}$$

Another formula which can be used recursively is

$$\sum_{k=1}^{n} \binom{n+1}{k} \sum_{i=1}^{r} i^k = (r+1)((r+1)^n - 1). \tag{8}$$

References

[1] K. Falconer. *Fractal Geometry: Mathematical Foundations and Applications.* Wiley, second edition, 2003.

[2] Angel Garrido, (2009) Combinatorial analysis by the Ihara zeta function of graphs, *Advanced Modeling and Optimization* vol.**11**, pp.253-278

[3] O. Shanker, (2007) Zeta Function and Dimension of Complex Network, *Mod. Phys. Lett.* vol.**B21**, pp.639-644

[4] O. Shanker, (2008) Sharp dimension transition in a shortcut model, *J. Phys. A* vol.**41(28)**, doi: 10.1088/1751-8113/41/28/285001

[5] O. Shanker, (2009) Defining the Dimension of a Complex Network and Zeta Functions, *International Journal of Computer Research* vol.**16**, 293-312

[6] O. Shanker, (2010) Complex network dimension and path counts, *Theoretical Computer Science* vol.**411**, pp.2454-2458