Asymmetry and Complex Networks

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Abstract

Asymmetry -and *Symmetry*- and their corresponding and dual degrees are not only fundamental concepts, but also very promising tools on many mathematical fields. For instance, it will be a cornerstone of Modern Science.

If we assume that a Complex System, or its representation, by Complex Networks or Graphs may have a modifiable degree of symmetry, then it is possible to simplify the equations that describe them.

For these reasons, there are a tenacious search for a unified description by the subjacent notion that a valid (and therefore, desirable) theory would be the more symmetrical possible, by new results based on fuzzy measures.

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1 Introduction

Usually we may distinguish four structural models [3, 9, 12, 19, 22, 23, 26] when we describes Complex Systems by Complex Networks, i. e. using Graph Theory. So, we can mentionate

- *Regular Networks,*

- *Random Networks,*

- *Small-World Networks,*

and

- *Scale-free Networks*

But also it is possible to introduce some new versions, according to the new measures of Symmetry/Asymmetry Level Measures.

We introduce here some of the more necessary tools for such new advances.

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1.1 Analysis of Complex Network features

In Regular Networks, each node is connected to all other nodes. I.e. they are fully connected.

Because of such a type of structure, they have the lowest path length (L), and the lowest diameter (D), being

$$L = D = 1$$

Also they have the highest clustering coefficient *(C).* So, it holds

$$C = 1$$

Furthermore, with the highest possible number of edges, given by

$$Card\ (E) = n\ (n\text{-}1)/2 \sim n^2$$

As related to *Random Graphs (RGs),* we can say that each pair of nodes is connected with probability *p.*

They have a low average path length, according to

$$L \approx (\ln\ n) \,/\, \ln \langle k \rangle \backsimeq \ln\ n,\ for\ n \gg 1$$

It is because the total network may be covered in <k> steps, from which

$$n \propto \langle k \rangle^L$$

Moreover, they possess a low clustering coefficient, when the graph is sparse. Thus,

$$C = p = \langle k \rangle \,/ n \ll 1$$

given that the probability of each pair of neighboring nodes to be connected is precisely equal to *p.*

The Small-World effect is observed on a network when it has a low average path length. I.e.

$$L << n, \text{ for } n >> 1$$

Recall the now famous "six degrees of separation", also called the

“small-world phenomenon”

The subjacent idea is that two arbitrarily selected people may be connected by only six degrees of separation (in average, and it is not much larger than this value). Therefore, the diameter of the corresponding graph is not much larger than six. For instance, on social connections. So, the Small-World property will be interpreted as that despite its large size (of the corresponding graph), the shortest path between two nodes is small, as e.g. on WWW, or on the Internet.

Self-similarity on network indicates that it is approximately similar to any part of itself, and therefore, it is *fractal.* In many cases, the real networks possess all these properties, i.e. they are *Fractal, Small-World,* and *Scale-Free.*

Fractal dimensions describe self-similarity of diverse phenomena: images, temporal signals,... Such fractal dimension gives us an indication of how completely a fractal appears to fill the space, as one zooms down to finer and finer scales. It is, so, a statistical measure.

The most important of such measures are

- *Rényi dimension,*

- *Hausdorff dimension,*

and

- *Packing dimension.*

Fuzzy set approach also may produce some consistent models [16, 17].

In the case of the Watts-Strogatz Small-World model, proposed in 1998, it represents a hybrid case between a Random Graph and a Regular Lattice [9, 24, 26].

So, Small-World models share with Random Graphs some common features, such as

- the Poisson or Binomial degree distribution, near to Uniform type;
- network size: it does not grow;
- each node has approximately the same number of edges.

Therefore, it shows a homogeneous nature. Because their ease of implementation, the more usual procedures to compute such measures will be

correlation dimension

and

box counting

WS-models show the low average path length typical of Random Graphs,

$$L \sim \ln\ n, \text{ for } n >> 1$$

And also such models give us the usual high clustering coefficient of Regular Lattices, being

$$C \approx 0.75, \text{ for } k >> 1$$

In consequence, WS-models have a small-world structure, being well clustered.

The Random Graphs coincide on the small-world structure, but they are poorly clustered.

This model (WS) has a peak degree distribution, of Poisson type.

With reference to the last model [3, 9, 12, 16], called Scale-Free Network, this appears when the degree distribution follows a Power-Law. I.e.

$$P(k) \propto k^{-\gamma}$$

In such a case, there exist a small number of highly connected nodes, called *Hubs,*
which are the tail of the distribution.

On the other hand, the great majority of the sets of their nodes have few connections, representing the head of such distribution.

Such a model was introduced by Albert-Laszló Barabási and Réka Albert, in 1999. Some of their essential *features* are

- *non-homogeneous nature,* in the sense that some (few) nodes have many edges from them, and the remaining nodes only have very few edges, or links.

- as related to the network size, *it continuously grows;*

and

- regarding to the connectivity, it obeys a *Power-Law distribution.*

Many massive graphs, such as the WWW graph, share certain characteristics, described as such aforementioned Power-Law.

Bollobás and Riordan [9, 10] consider a Random Graph process in which nodes are added to the graph one at a time, and joined to a fixed number of earlier nodes, chosen with probability proportional to their degree.

After n steps, the resulting graph has diameter approximately

$$log\ n$$

This affirmation is true for $n = 1$.
But for $n \geq 2$, the diameter value would be asymptotically

$$(logn) / \log(\log\ n)$$

Another very interesting mechanism is the so-called

Preferential Attachement process
(PA by acronym)

It will be any class of processes in which some quantity is distributed among a number of sets (for instance, objects or individuals), according to how much they already have, so that intuitively *"the rich get richer"*

(the more interrelated get more new connections than those who are not).

The principal scientific interest in PA is that it may produce interesting power law distributions.

Analytic solutions for PA mechanism were showed by

Dogorotsev et al. (2000)

and then, by Krapivsky et al., working on an independent way.

But it was Bela Bollobás who proved this rigorously.

A very notable example of Scale-Free Network may be the World Wide Web. As we know [1, 3, 14, 15], it is a collection of many possibly very different subnetworks.

Related to the Web graph characteristics, we notice the *Scale Invariance* as being very important [21, 26].

Another interesting feature is the possibility to obtain a measurement of the World-Wide Web (its diameter, i.e. the shortest distance between any pair of nodes into the system), or at least a bound, either a mean value, ...

The WWW representation is made by a very large digraph, whose nodes are documents, and whose edges are links (URLs), pointing from one document to another.

Réka Albert et al. found that the average of the shortest path between two nodes will be

$$\langle d \rangle = 0.35 + 2.06 \ \log N$$

where N is the number of nodes in the Random Graph considered.

This shows that the Web is a Small-World network.

In particular, if we take

$$N = 8 \ x \ 10^8$$

we will obtain

$$\langle d_{Web} \rangle = 18.59$$

This important result means that two randomly chosen nodes (documents), on the graph which represent the Web, are only on average nineteen clicks (or steps into the Web graph) from each other.

For a given value of the number of nodes, N, the distribution associated to d is of Gaussian type.

It will be also very remarkable the logarithmic dependence of such diameter on the value of N. In this sense, R. Albert et al. indicate that the future evaluation of $\langle d \rangle$, with the increasing of the Web, would change from 19 to only 21.

1.2 Fuzzy Asymmetry

Also it will be possible to introduce some new asymmetry and symmetry level measures as by [16, 17].

Let *(E, d)* be a fuzzy metric space.

Note. Our results may also be applied to some different space classes.

We proceed to define both new fuzzy measures.

Such functions might be defined as some of the type $\{L_i\}_{i \in \{s,a\}}$,with $i \in \{a, s\}$ where s denotes symmetry, and a denotes asymmetry.

Suppose that from here we denote by *c(A)* the cardinal of a fuzzy set, *A.*

We denote by *H (A)* its entropy measure, and by *Sp (A)* its corresponding specificity measure.

Theorem 1. Let *(E, d)* be a fuzzy metric space, with A as a subset of *E,* and let H and *Sp* be both fuzzy measures defined on *(E, d).*

Then, the function operating on A as

$$L_s(A) = Sp(A)\left(\tfrac{1-c(A)}{1+c(A)}\right) + \tfrac{1}{1+H(A)}$$

will be also a fuzzy measure.

This measure is called *Symmetry Level Function.*

Theorem 2. Let *(E, d)* be a fuzzy metric space, being A any subset of *E,* and let H and *Sp* be both precedent fuzzy measures defined on *(E, d).*

Then, the function

$$L_a(A) = 1 - \left\{Sp(A)\left(\tfrac{1-c(A)}{1+c(A)}\right) + \tfrac{1}{1+H(A)}\right\}$$

This measure is called *Asymmetry Level Function.*

Corollary 1. In the precedent hypotheses, the Symmetry Level Function is a Normal Fuzzy Measure.

Corollary 2. Also the Asymmetry Level Function will be a Normal Fuzzy Measure.

Recall that the values of a fuzzy measure, Sp, decrease as the size of the considered set increases. And that theRange of the Specificity Measure, Sp, will be [0, 1].

2 Conclusion

So, we hope to have achieved our initial purpose, that of attempting to provide a comprehensive vision on the principal aspects, and essential properties, of Complex Networks, from a new Mathematical Analysis point of view, and in particular show new promising results about the Functions of Symmetry/Asymmetry Levels.

2.1 References

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