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Sensitivity analysis for fuzzy linear programming problems with fuzzy variables

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Abstract

In this paper, we study how changes in the coefficients of objective function, the coefficients matrix and the right-hand-side vector of constraints of the fuzzy linear programming problems with the fuzzy order relation in the objective function and the constraints under ranking function affect the fuzzy optimal solution. We consider separate cases when changes occur in the data of the problem and derive bounds for parameter when the data are perturbed, while the fuzzy optimal solution is invariant. Finally, we obtain the optimal value function with fuzzy coefficients in each case and the results are described by some numerical examples.

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1 Introduction

The concept of decision making in fuzzy environment proposed by Bellman and Zadeh for the first time in [1]. The use of this concept in mathematical programming was proposed by Tanaka et al [9]. The first formulation of fuzzy linear programming (FLP) was given by Zimmermann [14]. Afterwards, many authors considered various types of FLP problems and proposed several approaches for solving these problems

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[2, 3, 5, 7]. One of these approaches for solving FLP problems is based on the concept of comparison of fuzzy numbers by use of ranking functions. In such methods authors define a crisp model which is equivalent to the FLP problem and then use optimal solution of the model as optimal solution of the FLP problem. The fuzzy dual problem was defined by the help of parametric linear program and was showed that the fuzzy primal and dual both have the same fuzzy solution under some suitable conditions by Verdegay [10]. The fuzzy variable linear programming problem has been explored by Zimmermann [15]. By using of certain linear ranking function for ordering trapezoidal fuzzy numbers, defined the dual of a fuzzy linear programming problem, then the duality results and complementary slackness have been given [8].

Sensitivity analysis is a basic tool for studying perturbations in optimization problems and it is one of the interesting researches in FLP problems. Sensitivity analysis in FLP was first considered by Hamacher et al [6], where a functional relation between changes of parameters of the right-hand-side and those of the optimal value of the primal objective function was derived for almost all conceivable cases. Fuller [4] showed that the solution to FLP problems with symmetrical triangular fuzzy numbers is stable with respect to small changes of centers of fuzzy numbers. Perturbations occur due to calculation errors or just to answer managerial questions " What if $\cdots$". Such questions propose after the simplex method and the related research area refers to as basis invariancy sensitivity analysis.

In addition, a lot of real-world problems have uncertainties in the data, coefficients and/or parameters, which form the fuzzy environment, because they are a mixture of measurements and perceptions, as described in [13]. In these cases, the values must be estimated by a decision maker that knows about the problem. This vagueness can be dealt with by stochastic process, approximate reasoning, chaos or fuzzy logic. In this context, the concept of fuzzy linear programming emerges when uncertain variables are used.

In this paper, we study basis invariancy sensitivity analysis for fuzzy linear programming problems with uncertain variables when there exists an fuzzy order relation under linear ranking function.

The paper is organized as follows: Section 2 states some basic concepts, namely, basic feasible solution, ranking function and fuzzy linear programming. Section 3 shows study of the sensitivity analysis for fuzzy linear programming with fuzzy order relation under ranking function, and obtaining lower and upper bounds for parameter when the problem data are perturbed. Any case is illustrated by an numerical example.

2 Preliminaries

Let R denote the set of all real numbers. In this paper, a fuzzy number will be a fuzzy set $\tilde{a} : R \longrightarrow [0, 1]$ with the following properties:

1. The membership function $\mu_{\tilde{a}}(x)$ is piecewise continuous,

2. $\tilde{a}$ is fuzzy convex; that is, $\mu_{\tilde{a}}(\lambda x + (1-\lambda)y) \geq \min\{\mu_{\tilde{a}}(x), \mu_{\tilde{a}}(y)\}$, $\forall x, y \in R$ and $\lambda \in [0,1]$,

3. There exist three intervals $[a,b],[b,c]$ and $[c,d]$ such that $\mu_{\tilde{a}}$ is increasing on $[a,b]$, equal to 1 on $[b,c]$, decreasing on $[c,d]$ and equal to 0 elsewhere.

The set of fuzzy numbers $\tilde{a} = (a^L, a^U, \alpha, \beta)_{LR}$, where $a^L \leq a^U, \alpha > 0, \beta > 0$ and $a^L, a^U, \alpha, \beta \in R$ will be denoted by $F(\mathbb{R})$. The arithmetic operations between fuzzy numbers, or fuzzy number and classical number is described as follows:

1. $x > 0, \quad x \in R; \quad x\tilde{a} = (xa^L, xa^U, x\alpha, x\beta),$

2. $x < 0, \quad x \in R; \quad x\tilde{a} = (xa^U, xa^L, -x\beta, -x\alpha),$

3. $\tilde{a} + \tilde{b} = (a^L + b^L, a^U + b^U, \alpha + \gamma, \beta + \theta).$

In the sequel we define a ranking function that represents the fuzzy number by a classical number.

2.1 Ranking function

One of the ways for solving mathematical programming problems in a fuzzy environment is to compare fuzzy numbers. The comparison between fuzzy numbers is done by using a ranking function that attends some conditions described in [11]. An appropriate approach for ordering the elements of $F(\mathbb{R})$ is to define a ranking function $\mathcal{R} : F(\mathbb{R}) \to R$, which maps each fuzzy number into the real line, where a natural order exists. Some orders on $F(\mathbb{R})$ are defined as follows:

1. $\tilde{a} \leq^f \tilde{b}$ if and only if $\mathcal{R}(\tilde{a}) \leq \mathcal{R}(\tilde{b})$;

2. $\tilde{a} <^f \tilde{b}$ if and only if $\mathcal{R}(\tilde{a}) < \mathcal{R}(\tilde{b})$;

3. $\tilde{a} =^f \tilde{b}$ if and only if $\mathcal{R}(\tilde{a}) = \mathcal{R}(\tilde{b})$,

where $\tilde{a}$ and $\tilde{b}$ belong to $F(\mathbb{R})$, $\mathcal{R}$ is a ranking function, and the symbol " $\leq^f$ " represents the fuzzy fuzzy order relation.

We will restricted our attention to linear ranking functions; that is, a ranking function $\mathcal{R}$ such that

$$\mathcal{R}(k\tilde{a} + \tilde{b}) = k\mathcal{R}(\tilde{a}) + \mathcal{R}(\tilde{b}), \tag{1}$$

for any $\tilde{a}, \tilde{b} \in \mathcal{F}(\mathbb{R})$ and any $k \in R$.

According to this mind, we can choose a linear ranking function that satisfies Equation (1) as

$$\mathcal{R}(\tilde{a}) = c_L a^L + c_U a^U + c_\alpha \alpha + c_\beta \beta, \tag{2}$$

where c_L, c_U, c_α and c_β are arbitrary constants. A special form of the above ranking function was first proposed by Yager [12]:

$$\mathcal{R}(\tilde{a}) = \frac{a^L + a^U}{2} + \frac{\beta - \alpha}{4}. \tag{3}$$

Remark 1. For any fuzzy number $\tilde{a}$, the relation $\tilde{a} \geq^f \tilde{0}$ holds, if there exist $\epsilon \geq 0$ and $\xi \geq 0$ such that $\tilde{a} \geq^f (-\epsilon, \epsilon, \xi, \xi)$. In this way, we have $\mathcal{R}(\tilde{a}) \geq 0$ (we also assume $\tilde{a} =^f \tilde{0}$ if and only if $\mathcal{R}(\tilde{a}) = 0$). Hence, without loss of generality, we consider $\tilde{0} = (0, 0, 0, 0)$ as a trapezoidal fuzzy zero.

2.2 Fuzzy linear programming

Consider the primal problem in standard form

$$\begin{array}{rl} \min & \tilde{z} =^f c\tilde{x} \\ \text{s.t.} & A\tilde{x} =^f \tilde{b} \\ & \tilde{x} \geq^f \tilde{0}, \end{array} \qquad (FLP)$$

with dual

$$\begin{array}{rl} \max & \tilde{w} =^f y\tilde{b} \\ \text{s.t.} & yA \leq c, \end{array} \qquad (FLD)$$

where $\tilde{b} \in (\mathcal{F}(\mathbb{R}))^m, \mathbf{A} \in \mathbb{R}^{m\times n}, \mathbf{c}^t \in \mathbb{R}^n$ are data, $\tilde{x} \in (\mathcal{F}(\mathbb{R}))^n$ and $y \in \mathbb{R}^m$ are to be determined, and $\mathcal{R}$ is a linear ranking function as defined by (3).

Definition 2. A trapezoidal fuzzy vector $\tilde{x} \geq^f \tilde{0}$ is said to be a fuzzy feasible solution for (FLP) if $\tilde{x}$ satisfies the constraints $A\tilde{x} =^f \tilde{b}$.

Definition 3. A fuzzy feasible solution $\tilde{x}_*$ is called a fuzzy optimal solution for (FLP) if for all fuzzy feasible solutions $\tilde{x}$, we have $c^t\tilde{x}_* \leq^f c^t\tilde{x}$.

Definition 4. Let A be the coefficient matrix of the FLP problem with full row rank and B be a nonsingular sub-matrix $m \times m$ of A. Let $\{B_1, \ldots, B_m\} \subset \{1, \ldots, n\}$ denote the index set of the columns of matrix B. Let $N = \{1, 2, \ldots, n\} \setminus B$. In this case, vector $\tilde{x} =^f (\tilde{x}_B^T, \tilde{x}_N^T)^T =^f (B^{-1}\tilde{b}, \tilde{0})$ is called a basic solution. If $\tilde{x}_B \geq^f \tilde{0}$, then the fuzzy basic solution $\tilde{x}$ is called a fuzzy basic feasible solution ($FBFS$) and the corresponding fuzzy objective value is equal to $\tilde{z} =^f c_B\tilde{x}_B$, where $c_B = (c_{B_1}, \ldots, c_{B_m})$. Now corresponding to every index j, $1 \leq j \leq n$, define $z_j = c_By_j = c_BB^{-1}A_{.j}$, which $A_{.j}$ is jth column of A. Observe that for any basic index $j = B_i, 1 \leq i \leq m$, we have $B^{-1}A_{.j} = e_j$ where $e_j = (0, \ldots 0, 1, 0, \ldots, 0)^T$, since $Be_j = A_{.j}$ and so we have $z_j - c_j = 0$.

In the following state a fundamental theorem which plays an important role for sensitivity analysis in the FLP problem.

Theorem 5. *(Optimality conditions [8]) Assume the FLP problem with trapezoidal fuzzy variables is non-degenerate and B is a feasible basis. A fuzzy basic feasible solution* $\tilde{x}_B =^f B^{-1}\tilde{b}, \tilde{x}_N =^f \tilde{0}$ *is optimal to the FLP if and only if* $z_j = c_B B^{-1} A_{.j} \leq c_j$ *for all* $j, 1 \leq j \leq n$.

3 Sensitivity analysis

Consider the primal problem (FLP). Suppose that the simplex method produced an optimal basis B. We shall describe how to make use of the optimality conditions in order to find new optimal solution if some of the problem data change without resolving the problem from scratch. In particular the following variations in the primal problem will be considered:

- Change in the cost vector c,
- Change in the right hand side vector $\tilde{b}$,
- Change in the constraint matrix A,
- Addition of a new activity (trapezoidal fuzzy variable),
- Addition of a new constraint.

3.1 Change in the cost vector c

Given an optimal basic feasible solution, suppose that the cost coefficient of the fuzzy variable $\tilde{x}_k$ is changed from c_k to $c_k^{'}$, that $c_k^{'} := c_k + \lambda \triangle c_k$. The effect of this change on the final tableau will occur in the cost row. We are going to determine the λ that make the old solution be still optimal. Consider the following two separation cases:

Case 1: $\tilde{x}_k$ is a non-basic variable.

In this case c_B is not affected, and hence $z_j := c_B B^{-1} A_{.j}$ is not changed for any j. Thus $z_k - c_k$ is replaced by $z_k - c_k^{'}$. Now, to preserve optimality, we must have

$$z_k - c_k^{'} = c_B B^{-1} A_{.k} - c_k - \lambda \triangle c_k \leq 0,$$

this implies

$$\lambda \begin{cases} \geq \dfrac{z_k - c_k}{\triangle c_k}, & \text{if } \triangle c_k > 0 \\ \leq \dfrac{z_k - c_k}{\triangle c_k}, & \text{if } \triangle c_k < 0 \end{cases} \tag{4}$$

Hence for any change in c_k, satisfying (4), the current optimal solution remains optimal and the value of the objective function also does not change since $\tilde{x}_k =^f \tilde{0}$.

Case 2: $\tilde{x}_t$ is a basic variable, say $\tilde{x}_t := \tilde{x}_{B_k}$.

Let c_{B_k} be replaced by $c'_{B_k} := c_{B_k} + \lambda\triangle c_{B_k}$. In this case the evaluations of $z_j := c_B B^{-1} A_{.j}$ for all non-basic variables are affected by any change in c_k and we should have

$$\begin{aligned} z'_j - c_j &= c'_B B^{-1} A_{.j} - c_j = (c_{B_1}, \ldots, \underset{\underset{kth}{\uparrow}}{c'_{B_k}}, \ldots, c_{B_m}) B^{-1} A_{.j} - c_j \\ &= c_B B^{-1} A_{.j} - c_j + (0, \ldots, \lambda\triangle c_{B_k}, \ldots, 0) B^{-1} A_{.j} \\ &= c_B B^{-1} A_{.j} - c_j + \lambda\triangle c_{B_k} \sum_{i=1}^{m} \beta_{ki} A_{ij} \leq 0, \quad j \in N \end{aligned}$$

where $B^{-1} = (\beta_{ij})$. This implies that

$$\lambda \begin{cases} \geq \dfrac{c_j - z_j}{\triangle c_{B_k} \sum_{i=1}^{m} \beta_{ki} A_{ij}}, & \text{if } \triangle c_{B_k} \sum_{i=1}^{m} \beta_{ki} A_{ij} < 0 \\ \leq \dfrac{c_j - z_j}{\triangle c_{B_k} \sum_{i=1}^{m} \beta_{ki} A_{ij}}, & \text{if } \triangle c_{B_k} \sum_{i=1}^{m} \beta_{ki} A_{ij} > 0 \end{cases} \tag{5}$$

Hence

$$\begin{aligned} \max_{j \in N} \left\{ \frac{c_j - z_j}{\triangle c_{B_k} \sum_{i=1}^{m} \beta_{ki} A_{ij}} : \triangle c_{B_k} \sum_{i=1}^{m} \beta_{ki} A_{ij} < 0 \right\} \leq \lambda \leq \\ \min_{j \in N} \left\{ \frac{c_j - z_j}{\triangle c_{B_k} \sum_{i=1}^{m} \beta_{ki} A_{ij}} : \triangle c_{B_k} \sum_{i=1}^{m} \beta_{ki} A_{ij} > 0 \right\}. \end{aligned} \tag{6}$$

Thus if (6) is satisfied, changes in c_k will not affect the original optimal basis or the value of the optimal solution. The only change will occur in the optimal value of the objective function $\tilde{z}$, and the new optimal value will be equal to

$$\begin{aligned} \tilde{z}'_* =^f c'_B B^{-1} \tilde{b} &=^f c_B B^{-1} \tilde{b} + (0, \ldots, \lambda\triangle c_{B_k}, \ldots, 0) B^{-1} \tilde{b} \\ &=^f \tilde{z}_* + \lambda\triangle c_{B_k} \sum_{i=1}^{m} \beta_{ki} \tilde{b}_i, \end{aligned}$$

where is a fuzzy linear function with respect to λ.

Example 1. *Consider the following fuzzy linear programming problem .*

$$\begin{array}{ll} \min & \tilde{z} =^f -\tilde{x}_1 - \tilde{x}_2 - 2\tilde{x}_3 \\ s.t: & \tilde{x}_1 + \tilde{x}_2 + 2\tilde{x}_3 + \tilde{x}_4 =^f (5,8,2,5) \\ & \tilde{x}_1 - \tilde{x}_2 \qquad + \tilde{x}_5 \ =^f (6,10,2,6) \\ & \tilde{x}_1 + \tilde{x}_2 + \tilde{x}_3 + \tilde{x}_6 =^f (1,6,7,8) \\ & \tilde{x}_1,\ \tilde{x}_2,\ \tilde{x}_3,\ \tilde{x}_4,\ \tilde{x}_5,\ \tilde{x}_6 \geq^f \tilde{0}, \end{array}$$

The final simplex tableau is given as follows

Basis	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{x}_3$	$\tilde{x}_4$	$\tilde{x}_5$	$\tilde{x}_6$	*R.H.S.*	$\mathcal{R}$
$\tilde{z}$	0	0	0	-1	0	0	$(-8,-5,5,2)$	$-\frac{29}{4}$
$\tilde{x}_3$	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	0	0	$(\frac{5}{2},4,1,\frac{5}{2})$	$\frac{29}{8}$
$\tilde{x}_5$	1	-1	0	0	1	0	$(6,10,2,6)$	9
$\tilde{x}_6$	$\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	1	$(-3,\frac{7}{2},\frac{19}{2},9)$	$\frac{1}{8}$

The optimal solution is $\tilde{x}_1^* =^f (0,0,0,0), \tilde{x}_2^* =^f (0,0,0,0), \tilde{x}_3^* =^f (\frac{5}{2},4,1,\frac{5}{2})$ and $\tilde{z}_* =^f (-8,-5,5,2)$.

Here the matrix of the optimal basis is $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ and its inverse $\mathbf{B}^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{pmatrix}$.

If $c_3 = -2 \to c_3' = -2 + 3\lambda$, then by using (6) we get

$$\lambda \leq 0,$$

and the optimal value function in this region is as follows

$$\tilde{z}(\lambda) =^f (-8,-5,5,2) + (\frac{15}{2}, 12, 3, \frac{15}{2})\lambda.$$

3.2 Change in the requirement vector $\tilde{b}$

Since optimality condition, $z_j - c_j \leq 0, \forall j \in N$, does not depend on the requirement vector, any change in the requirement vector does not affect the optimality condition. It however affects the values of the basic variables and hence the value of the objective function. Thus if the magnitude of the change in the requirement vector be such that it preserves the feasibility of the optimal basis, then the original optimal basis remains optimal.

Let the requirement vector $\tilde{b}$ be replaced by $\tilde{b}' :=^f \tilde{b} + \lambda \triangle \tilde{b}$, where $\triangle \tilde{b}$ is a constant fuzzy vector. Then $B^{-1}\tilde{b}$ will be replaced by $B^{-1}\tilde{b}'$. The new right hand side can be calculated without explicitly evaluating $B^{-1}\tilde{b}'$. This is evident by noting that

$$B^{-1}\tilde{b}' =^f B^{-1}\tilde{b} + \lambda B^{-1}\triangle\tilde{b}. \tag{7}$$

For maintaining the feasibility, we must have

$$B^{-1}\tilde{b} + \lambda B^{-1}\triangle\tilde{b} \geq^f \tilde{0},$$

which this is equivalent to

$$\mathcal{R}(B^{-1}\tilde{b}) + \lambda\mathcal{R}(B^{-1}\triangle\tilde{b}) = B^{-1}\mathcal{R}(\tilde{b}) + \lambda B^{-1}\mathcal{R}(\triangle\tilde{b}) \geq 0,$$

where $\mathcal{R}(\tilde{b}) = (\mathcal{R}(\tilde{b}_1), \ldots, \mathcal{R}(\tilde{b}_m))^t$ and $\mathcal{R}(\triangle\tilde{b}) = (\mathcal{R}(\triangle\tilde{b}_1), \ldots, \mathcal{R}(\triangle\tilde{b}_m))^t$, or

$$\sum_{i=1}^{m} \beta_{hi}\mathcal{R}(\tilde{b}_i) + \lambda \sum_{i=1}^{m} \beta_{hi}\mathcal{R}(\triangle\tilde{b}_i) \geq 0, \quad h = 1, 2, \ldots, m.$$

The last relation implies that

$$\lambda \begin{cases} \geq -\dfrac{\sum_{i=1}^{m} \beta_{hi}\mathcal{R}(\tilde{b}_i)}{\sum_{i=1}^{m} \beta_{hi}\mathcal{R}(\triangle\tilde{b}_i)}, & \text{if } \sum_{i=1}^{m} \beta_{hi}\mathcal{R}(\triangle\tilde{b}_i) > 0 \\ \leq -\dfrac{\sum_{i=1}^{m} \beta_{hi}\mathcal{R}(\tilde{b}_i)}{\sum_{i=1}^{m} \beta_{hi}\mathcal{R}(\triangle\tilde{b}_i)}, & \text{if } \sum_{i=1}^{m} \beta_{hi}\mathcal{R}(\triangle\tilde{b}_i) < 0 \end{cases} \tag{8}$$

Thus the range for λ for which the optimal basis remains optimal is

$$\max_{1\leq h\leq m} \Big\{ -\frac{\sum_{i=1}^{m} \beta_{hi}\mathcal{R}(\tilde{b}_i)}{\sum_{i=1}^{m} \beta_{hi}\mathcal{R}(\triangle\tilde{b}_i)} : \sum_{i=1}^{m} \beta_{hi}\mathcal{R}(\triangle\tilde{b}_i) > 0 \Big\} \leq \lambda \leq$$

$$\min_{1\leq h\leq m} \Big\{ -\frac{\sum_{i=1}^{m} \beta_{hi}\mathcal{R}(\tilde{b}_i)}{\sum_{i=1}^{m} \beta_{hi}\mathcal{R}(\triangle\tilde{b}_i)} : \sum_{i=1}^{m} \beta_{hi}\mathcal{R}(\triangle\tilde{b}_i) < 0 \Big\}. \tag{9}$$

The new solution of the problem is given by (7) and the value of the objective function is a fuzzy linear function with respect to λ:

$$\tilde{z}'_* =^f c_B B^{-1}(\tilde{b} + \lambda\triangle\tilde{b}) =^f c_B B^{-1}\tilde{b} + \lambda c_B B^{-1}\triangle\tilde{b} =^f \tilde{z}_* + \lambda c_B B^{-1}\triangle\tilde{b}.$$

Example 2. *Consider Example 1. Let* $\triangle\tilde{b} =^f ((4,2,1,3),(3,2,5,2),(1,2,3,1))^t$ *be perturbing direction, then by using (9) we get*

$$\max\{-\frac{29}{14}, -\frac{36}{7}\} \leq \lambda \leq \min\{\frac{1}{6}\}.$$

Therefore, the stability range of optimal solution is

$$-\frac{29}{14} \leq \lambda \leq \frac{1}{6},$$

and the optimal value function in this region is as follows

$$\tilde{z}(\lambda) =^f (-8,-5,5,2) + (-2,-4,3,1)\lambda.$$

3.3 Change in the constraint matrix

We now discuss the effect of changing some of the entries of the constraint matrix A. Two cases are possible, namely, changes involving non-basic columns and changes involving basic columns.

Case 1: Change in the non-basic columns

Suppose that some of the non-basic columns $A_{.j}, j \in N_1 \subseteq N$ are replaced to $A'_{.j} := A_{.j} + \lambda\triangle A_{.j}, j \in N_1$, that $\triangle A_{.j}s$ are the perturbation vectors. Then the new updated columns are

$$\begin{pmatrix} c_B B^{-1}A'_{.j} - c_j \\ B^{-1}A'_{.j} \end{pmatrix}, j \in N_1.$$

It is clear that the feasibility condition is not distributed. To preserve the optimality we must have,

$$\begin{aligned} z'_j - c_j &= c_B B^{-1}A'_{.j} - c_j \\ &= c_B B^{-1}(A_{.j} + \lambda\triangle A_{.j}) - c_j \\ &= (z_j - c_j) + \lambda c_B B^{-1}\triangle A_{.j} \\ &\leq 0, \qquad j \in N_1. \end{aligned}$$

This implies that

$$\lambda \begin{cases} \geq \dfrac{c_j - z_j}{c_B B^{-1}\triangle A_{.j}}, & \text{if } c_B B^{-1}\triangle A_{.j} < 0, j \in N_1 \\ \leq \dfrac{c_j - z_j}{c_B B^{-1}\triangle A_{.j}}, & \text{if } c_B B^{-1}\triangle A_{.j} > 0, j \in N_1 \end{cases} \tag{10}$$

Thus the range for λ for which the optimal basis remains optimal is

$$\max_{j\in N_1}\left\{\frac{c_j-z_j}{c_BB^{-1}\triangle A_{.j}}:c_BB^{-1}\triangle A_{.j}<0\right\}\leq\lambda\leq \min_{j\in N_1}\left\{\frac{c_j-z_j}{c_BB^{-1}\triangle A_{.j}}:c_BB^{-1}\triangle A_{.j}>0\right\}. \tag{11}$$

Example 3. *Consider Example 1. If* $A'_{.1}=A_{.1}+\lambda\triangle A_{.1}$ and $A'_{.4}=A_{.4}+\lambda\triangle A_{.4}$ *where* $\triangle A_{.1}=(1,2,-1)^t$ and $\triangle A_{.4}=(-3,1,\frac{2}{3})$ *then by using (11) we have*

$$0\leq\lambda\leq\frac{1}{3}.$$

Case 2: Change in the basic column

In this part, our goal is to determine the lower and upper bounds for λ which guarantee that the replacement $A_{.k}$ by $A'_{.k}:=A_{.k}+\lambda\triangle A_{.k}, k\in B$, does not affect the optimal basis, and the original optimal solution $\tilde{x}_*$ remains feasible and optimal. By taking this replacement, the optimal basis B will be replaced with $\overline{B}:=B+\lambda\triangle A_{.k}e_k^t$ where e_j is a unit vector. The inverse matrix $\overline{B}$ is

$$\begin{aligned}\overline{B}^{-1}&=B^{-1}-\lambda\frac{B^{-1}\triangle A_{.k}e_k^tB^{-1}}{1+\lambda e_k^tB^{-1}\triangle A_{.k}}\\&=B^{-1}-\lambda\frac{B^{-1}\triangle A_{.k}\beta_{k.}}{1+\lambda\sum_{i=1}^m\beta_{ki}\triangle A_{ik}},\qquad 1+\lambda\sum_{i=1}^m\beta_{ki}\triangle A_{ik}\neq0,\end{aligned}\tag{12}$$

by the Sherman-Morrison formulas, where $B^{-1}=(\beta_{ij})$ and $\beta_{k.}$ is the k-th row B^{-1}. This change of the basis matrix will affect the feasibility of vector $\tilde{x}_*$. However, it may affect the optimality condition and the optimal value of the objective function $\tilde{z}$. Therefore

$$\begin{aligned}\tilde{x}_{\overline{B}}&=^f\overline{B}^{-1}\tilde{b}\\&=^f\Big(B^{-1}-\lambda\frac{B^{-1}\triangle A_{.k}\beta_{k.}}{1+\lambda\sum_{i=1}^m\beta_{ki}\triangle A_{ik}}\Big)\tilde{b}\\&=^f\tilde{x}_B-\lambda\frac{B^{-1}\triangle A_{.k}\beta_{k.}\tilde{b}}{1+\lambda\sum_{i=1}^m\beta_{ki}\triangle A_{ik}}.\end{aligned}\tag{13}$$

Now the i-th component of $\tilde{x}_{\overline{B}}$ is given by

$$(\tilde{x}_{\overline{B}})_i=^f\sum_{j=1}^m\beta_{ij}\tilde{b}_j-\lambda\frac{\sum_{j=1}^m\beta_{ij}\triangle A_{jk}\sum_{j'=1}^m\beta_{kj'}\tilde{b}_{j'}}{1+\lambda\sum_{i'=1}^m\beta_{ki'}\triangle A_{i'k}},\qquad i=1,2,\ldots,m. \tag{14}$$

This new basic solution $\tilde{x}_{\overline{B}}$ will be feasible if

$$\sum_{j=1}^{m}\beta_{ij}\tilde{b}_j-\lambda\frac{\sum_{j=1}^{m}\beta_{ij}\triangle A_{jk}\sum_{j'=1}^{m}\beta_{kj'}\tilde{b}_{j'}}{1+\lambda\sum_{i'=1}^{m}\beta_{ki'}\triangle A_{i'k}}\geq^f\tilde{0},\qquad i=1,2,\ldots,m. \tag{15}$$

Without loss of generality, assume that $1+\lambda\sum_{i'=1}^{m}\beta_{ki'}\triangle A_{i'k}>0$ and this implies

$$\lambda\begin{cases}>\dfrac{-1}{\sum_{i'=1}^{m}\beta_{ki'}\triangle A_{i'k}}, & \text{if } \sum_{i'=1}^{m}\beta_{ki'}\triangle A_{i'k}>0\\[2ex] <\dfrac{-1}{\sum_{i'=1}^{m}\beta_{ki'}\triangle A_{i'k}}, & \text{if } \sum_{i'=1}^{m}\beta_{ki'}\triangle A_{i'k}<0\end{cases} \tag{16}$$

Due to (16), the relation (15) satisfies if

$$\lambda\Big(\sum_{j=1}^{m}\beta_{ij}\tilde{b}_j\sum_{i'=1}^{m}\beta_{ki'}\triangle A_{i'k}-\sum_{j=1}^{m}\beta_{ij}\triangle A_{jk}\sum_{j'=1}^{m}\beta_{kj'}\tilde{b}_{j'}\Big)\geq^f-\sum_{j=1}^{m}\beta_{ij}\tilde{b}_j.$$

Hence for maintaining feasibility, we must have

$$\max_{1\leq i\leq m}\Big\{\frac{-\sum_{j=1}^{m}\beta_{ij}\mathcal{R}(\tilde{b}_j)}{H_i}\ :\ H_i>0\Big\}\leq\lambda\leq\min_{1\leq i\leq m}\Big\{\frac{-\sum_{j=1}^{m}\beta_{ij}\mathcal{R}(\tilde{b}_j)}{H_i}\ :\ H_i<0\Big\}, \tag{17}$$

where $H_i=\sum_{j=1}^{m}\beta_{ij}\mathcal{R}(\tilde{b}_j)\sum_{i'=1}^{m}\beta_{ki'}\triangle A_{i'k}-\sum_{j=1}^{m}\beta_{ij}\triangle A_{jk}\sum_{j'=1}^{m}\beta_{kj'}\mathcal{R}(\tilde{b}_{j'}),\ i=1,2,\ldots,m.$
Now, to preserve optimality, we must have

$$z_j'-c_j=c_{\overline{B}}\overline{B}^{-1}A_{.j}-c_j=c_B\Big(B^{-1}-\lambda\frac{B^{-1}\triangle A_{.k}\beta_{k.}}{1+\lambda\sum_{i'=1}^{m}\beta_{ki'}\triangle A_{i'k}}\Big)A_{.j}-c_j$$

$$=z_j-c_j-\lambda\frac{\sum_{i=1}^{m}\sum_{j'=1}^{m}\sum_{i'=1}^{m}c_{B_i}\beta_{ij'}\triangle A_{j'k}\beta_{ki'}A_{i'j}}{1+\lambda\sum_{i'=1}^{m}\beta_{ki'}\triangle A_{i'k}}\leq 0,\qquad j\in N. \tag{18}$$

Since by (16), $1 + \lambda \sum_{i'=1}^{m} \beta_{ki'} \triangle A_{i'k} > 0$, (18) reduces to

$$\lambda\Big((z_j - c_j)\sum_{i'=1}^{m} \beta_{ki'} \triangle A_{i'k} - \sum_{i=1}^{m}\sum_{j'=1}^{m}\sum_{i'=1}^{m} c_{B_i}\beta_{ij'}\triangle A_{j'k}\beta_{ki'}A_{i'j}\Big) \leq c_j - z_j. \qquad (19)$$

Hence in order to maintain the optimality of the new solution, λ must satisfies

$$\max_{j\in N}\Big\{\frac{c_j - z_j}{M_j} : \ M_j < 0\Big\} \leq \lambda \leq \min_{j\in N}\Big\{\frac{c_j - z_j}{M_j} : \ M_j > 0\Big\}, \qquad (20)$$

where $M_j = (z_j - c_j)\sum_{i'=1}^{m} \beta_{ki'} \triangle A_{i'k} - \sum_{i=1}^{m}\sum_{j'=1}^{m}\sum_{i'=1}^{m} c_{B_i}\beta_{ij'}\triangle A_{j'k}\beta_{ki'}A_{i'j}, \ j \in N$.

Therefore, we have proved the following theorem:

Theorem 6. *If λ satisfies (16), (17) and (20) then $\tilde{x}_*$ is an optimal solution of the perturbed problem.*

In the stability region of the Theorem 6, the optimal value function is a fuzzy linear fractional function as follows

$$\tilde{z}(\lambda) =^f \tilde{z}_* - \lambda\frac{\sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{j'=1}^{m} c_{B_i}\beta_{ij}\triangle A_{jk}\beta_{kj'}\tilde{b}_{j'}}{1 + \lambda\sum_{i'=1}^{m}\beta_{ki'}\triangle A_{i'k}}.$$

Example 4. *Consider Example 1. If $A'_{.3} = A_{.3} + \lambda\triangle A_{.3}$, where $\triangle A_{.3} = (-\frac{1}{4}, 0, -1)^t$, then by using the Theorem 6 we obtain the following interval for λ*

$$0 \leq \lambda < 8,$$

and the optimal value function is a fuzzy linear fractional function as follows:

$$\tilde{z}(\lambda) =^f (-8, -5, 5, 2) - \frac{\lambda}{8 - \lambda}(5, 8, 2, 5).$$

3.4 Adding a new activity

Suppose that a new activity $\tilde{x}_{n+1}$ with unit cost c_{n+1} and consumption column a_{n+1} is considered for possible production. Without resolving the problem, we can easily determine wether producing $\tilde{x}_{n+1}$ is worthwhile or not.

It is obvious that the original optimal solution is feasible to the modified problem. It also remains optimal if $z_{n+1} - c_{n+1} \leq 0$. In this ways $\tilde{x}^*_{n+1} =^f \tilde{0}$. If however, $z_{n+1} - c_{n+1} > 0$, then $\tilde{x}_{n+1}$ is introduced into the basis and the primal simplex method may be applied to find an optimal solution to the modified problem.

3.5 Adding a new constraint

Suppose that a new constraint is added to the problem after an optimal solution has already been obtained. If the optimal solution to the original problem satisfies the new constraint, it is obvious that it is also an optimal solution to the modified problem. If it does not satisfy the new constraint, a new optimal solution has to be found.

Suppose that B is the optimal basis before adding constraint $a^{m+1}\tilde{x} \leq^f \tilde{b}_{m+1}$. The corresponding tableau is shown below:

$$\tilde{z} + (c_B B^{-1} N - c_N)\tilde{x}_N =^f c_B B^{-1}\tilde{b}$$
$$\tilde{x}_B + B^{-1} N \tilde{x}_N =^f B^{-1}\tilde{b}. \tag{21}$$

The constraint $a^{m+1}\tilde{x} \leq^f \tilde{b}_{m+1}$ is rewritten as $a_B^{m+1}\tilde{x}_B + a_N^{m+1}\tilde{x}_N + \tilde{x}_{n+1} =^f \tilde{b}_{m+1}$, where a^{m+1} is decomposed into $(a_B^{m+1} \;\; a_N^{m+1})$ and $\tilde{x}_{n+1}$ is a nonnegative slack variable. Multiplying equation (21) by a_B^{m+1} and subtracting from the new constraint gives the following system:

$$\tilde{z} + (c_B B^{-1} N - c_N)\tilde{x}_N =^f c_B B^{-1}\tilde{b}$$
$$\tilde{x}_B + B^{-1} N \tilde{x}_N =^f B^{-1}\tilde{b}$$
$$(a_N^{m+1} - a_B^{m+1} B^{-1} N)\tilde{x}_N + \tilde{x}_{n+1} =^f \tilde{b}_{m+1} - a_B^{m+1} B^{-1}\tilde{b}.$$

These equations give us a basic solution of the new problem. The only possible violation of optimality of the new problem is the sign of $\tilde{b}_{m+1} - a_B^{m+1} B^{-1}\tilde{b}$, if $\tilde{b}_{m+1} - a_B^{m+1} B^{-1}\tilde{b} \geq^f \tilde{0}$, then the current solution is optimal. Otherwise, if $\tilde{b}_{m+1} - a_B^{m+1} B^{-1}\tilde{b} <^f \tilde{0}$ then the dual simplex method [8] is used to restore feasibility.

Example 5. *Consider the following fuzzy linear programming problem*

$$\begin{array}{ll} \min & \tilde{z} =^f 6\tilde{x}_1 + 10\tilde{x}_2 \\ s.t: & -2\tilde{x}_1 - 5\tilde{x}_2 + \tilde{x}_3 =^f (-8, -5, 5, 2) \\ & -3\tilde{x}_1 - 4\tilde{x}_2 + \tilde{x}_4 =^f (-10, -6, 6, 2) \\ & \tilde{x}_1,\ \tilde{x}_2,\ \tilde{x}_3,\ \tilde{x}_4 \geq^f \tilde{0}, \end{array}$$

The final simplex table is given as follows

Final

$Basis$	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{x}_3$	$\tilde{x}_4$	$R.H.S.$	$\mathcal{R}$
$\tilde{z}$	0	0	$-\frac{6}{7}$	$-\frac{10}{7}$	$(-\frac{62}{7}, \frac{300}{7}, \frac{360}{7}, \frac{418}{7})$	$\frac{267}{14}$
$\tilde{x}_2$	0	1	$-\frac{3}{7}$	$\frac{2}{7}$	$(-\frac{5}{7}, \frac{12}{7}, \frac{18}{7}, \frac{19}{7})$	$\frac{15}{28}$
$\tilde{x}_1$	1	0	$\frac{4}{7}$	$-\frac{5}{7}$	$(-\frac{2}{7}, \frac{30}{7}, \frac{30}{7}, \frac{38}{7})$	$\frac{16}{7}$

The optimal solution is $\tilde{x}_1^* =^f (-\frac{2}{7}, \frac{30}{7}, \frac{30}{7}, \frac{38}{7}), \tilde{x}_2^* =^f (-\frac{5}{7}, \frac{12}{7}, \frac{18}{7}, \frac{19}{7})$ and the optimal value is equal to $\tilde{z} =^f (-\frac{62}{7}, \frac{300}{7}, \frac{360}{7}, \frac{418}{7})$. Suppose that the constraint$-\frac{1}{4}\tilde{x}_1 \geq^f (-2, -1, 3, 7)$ is added to the problem, then

$$a_N^3 - a_B^3 B^{-1} N = [-\frac{1}{7}, \frac{5}{28}],$$

$$\tilde{b}_3 - a_B^3 B^{-1} \tilde{b} =^f (-\frac{1}{14}, \frac{29}{14}, \frac{117}{14}, \frac{57}{14}).$$

So we have the following tableaus:

$Basis$	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{x}_3$	$\tilde{x}_4$	$\tilde{x}_5$	$R.H.S.$	$\mathcal{R}$
$\tilde{z}$	0	1	$-\frac{6}{7}$	$-\frac{10}{7}$	0	$(-\frac{62}{7}, \frac{300}{7}, \frac{360}{7}, \frac{418}{7})$	$\frac{267}{14}$
$\tilde{x}_2$	0	1	$-\frac{3}{7}$	$\frac{2}{7}$	0	$(-\frac{5}{7}, \frac{12}{7}, \frac{18}{7}, \frac{19}{7})$	$\frac{15}{28}$
$\tilde{x}_1$	1	0	$\frac{4}{7}$	$-\frac{5}{7}$	0	$(-\frac{2}{7}, \frac{30}{7}, \frac{30}{7}, \frac{38}{7})$	$\frac{16}{7}$
$\tilde{x}_5$	0	0	$-\frac{1}{7}$	$\frac{5}{28}$	1	$(-\frac{1}{14}, \frac{29}{14}, \frac{117}{14}, \frac{57}{14})$	$-\frac{1}{14}$

Basis	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{x}_3$	$\tilde{x}_4$	$\tilde{x}_5$	*R.H.S.*	$\mathcal{R}$
$\tilde{z}$	0	0	0	$-\frac{35}{14}$	-6	$(-\frac{149}{7}, \frac{303}{7}, \frac{531}{7}, \frac{769}{7})$	$\frac{273}{14}$
$\tilde{x}_2$	0	1	0	$-\frac{1}{4}$	-3	$(-\frac{97}{14}, \frac{27}{14}, \frac{207}{14}, \frac{389}{14})$	$\frac{3}{4}$
$\tilde{x}_1$	1	0	0	0	4	$(-\frac{4}{7}, \frac{88}{7}, \frac{264}{7}, \frac{152}{7})$	2
$\tilde{x}_3$	0	0	1	$-\frac{5}{4}$	-7	$(-\frac{29}{2}, \frac{1}{2}, \frac{57}{2}, \frac{117}{2})$	$\frac{1}{2}$

Therefore, the new optimal solution is $\tilde{x}_1^* =^f (-\frac{4}{7}, \frac{88}{7}, \frac{264}{7}, \frac{152}{7})$, $\tilde{x}_2^* =^f (-\frac{97}{14}, \frac{27}{14}, \frac{207}{14}, \frac{389}{14})$, $\tilde{x}_3^* =^f (-\frac{29}{2}, \frac{1}{2}, \frac{57}{2}, \frac{117}{2})$ and the optimal value is equal to $\tilde{z}^* =^f (-\frac{149}{7}, \frac{303}{7}, \frac{531}{7}, \frac{769}{7})$.

4 Conclusion

The fuzzy linear programming problems with fuzzy variables are proposed by using ranking function in this paper. We then addressed the basis invariancy sensitivity analysis under ranking function, and obtained lower and upper bounds for parameter. Finally, we showed that the optimal value function is a fuzzy linear or a fuzzy linear fractional function.

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