AMO - Advanced Modeling and Optimization, Volume 12, Number 2, 2010

Global Optimization Approach to the Solow Growth Theory

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Abstract

We consider the Solow growth model[Solow,1956] with nonconvex production functions and propose some global optimization methods and algorithms for solving the per capita consumption maximization problem.

Key words: Solow growth model, global optimization method, per capita consumption maximization problem.

1 Introduction

Let $f(K, L)$ be a concave, differentiable homogeneous production function, s the savings (saving rate), $0 \le s \le 1$, and μ depreciation rate of capital.

Assume that the labor grows at exponential rate η which means that

$$L' = L_0 e^{\eta t}. \tag{1.1}$$

Define k as per capita capital function:

$$k(t) = \frac{K(t)}{L(t)}.$$

*AMO - Advanced Modeling and Optimization. ISSN: 1841-4311**

Then

$$k'(t) = \left(\frac{K(t)}{L(t)}\right)' = \frac{K'L - L'K}{L^2} = \frac{1}{L}\left(K' - \frac{L'}{L}K\right) = \frac{1}{L}\left(K' - \eta K\right).$$

If we substitute $f(K, L)$ into this equation, we have

$$\begin{aligned} k' &= \frac{1}{L}\left(s(t)f(K, L) - \mu K - \eta K\right) \\ &= s(t)f\big(\frac{K}{L}, 1\big) - \mu\frac{K(t)}{L(t)} - \eta\frac{K(t)}{L(t)} \\ &= s(t)\varphi(k) - (\mu + \eta)k, \end{aligned}$$

where $\varphi(k)$ is the Solow per capita production function. Then per capita capital accumulation equation is

$$\begin{cases} k' = s(t)\varphi(k) - (\mu + \eta)k \\ k(0) = k_0. \end{cases} \tag{1.2}$$

Economic equilibria condition is:

$$k' = s(t)\varphi(k) - (\mu + \eta)k = 0. \tag{1.3}$$

Now we consider per capita consumption function

$$c(t) = \frac{C(t)}{L(t)} = \frac{(1 - s)f(K, L)}{L} = (1 - s)\varphi(k) \tag{1.4}$$

Assume that s is a constant function.
Let us consider the per capita consumption maximization problem subject to economic equilibria. That is

$$c = (1 - s)\varphi(k) \to \max, \tag{1.5}$$

$$s\varphi(k) - (\mu + \eta)k = 0. \tag{1.6}$$

Then problem (1.5)-(1.6) is equivalent to the following one dimensional problem:

$$\max_k \left[\varphi(k) - (\mu + \eta)k\right].$$

The solution k^* satisfies the equation

$$\varphi'(k^*) = (\mu + \eta). \tag{1.7}$$

So called the golden rule of level of accumulation is determined from (1.6) as:

$$s^* = \frac{\varphi'(k^*)k^*}{\varphi(k^*)}. \tag{1.8}$$

In order to further generalize the Solow growth model[Solow,1956] note that the main assumptions of the model were

- The production function is concave and homogeneous.
- Labor grows exponentially at rate η.
- The production f is autonomous, did not defend on time t.
- The saving function is constant.

2 Economic growth with logistic production functions

First, we consider problem (1.5)-(1.6) for the case when

$$\begin{cases} \varphi'(k) = \gamma\varphi(M - \varphi) \\ \varphi(0) = \varphi_0 \\ L = L_0 e^{\eta t}. \end{cases} \tag{2.1}$$

The solution of (2.1) is

$$\varphi(k) = \frac{\varphi_0 M}{\varphi_0 + (M - \varphi_0)^{-M\gamma k}}$$

Then problem (1.5)-(1.6) reduces to the problem:

$$\begin{cases} c = (1 - s)\varphi(k) \to \max, \\ s\varphi(k) = (\mu + \eta)k. \end{cases} \tag{2.2}$$

The problem (2.2) is equivalent to one variable maximization problem

$$c = \varphi(k) - (\mu + \eta)k \to \max,\ k > 0. \tag{2.3}$$

We find stationary points of the problem

$$c'(k) = \varphi'(k) - (\mu + \eta) = 0.$$

It can be easily calculated that

$$\varphi'(k) = \frac{M^2\gamma\varphi_0(M - \varphi_0)e^{-M\gamma k}}{\left(\varphi_0 + (M - \varphi_0)e^{-M\gamma k}\right)^2}.$$

If we substitute $z = e^{-M\gamma k}$ and $A = M^2\gamma\varphi_0(M-\varphi_0)$ then

$$\frac{Az}{\left(\varphi_0 + (M-\varphi_0)z\right)^2} = \mu + \eta.$$

This reduces to the following quadratic equation

$$(M-\varphi_0)^2(\mu+\eta)z^2 + \left[2\varphi_0(M-\varphi_0)(\mu+\eta) - A\right]z + (\mu+\eta)\varphi_0^2 = 0.$$

Denote by D:

$$D = \left[2\varphi_0(M-\varphi_0)(\mu+\eta) - A\right]^2 - 4(\mu+\eta)^2(M-\varphi_0)^2\varphi_0^2. \quad (2.4)$$

The solution z is found as

$$z^* = -\frac{\left[2\varphi_0(M-\varphi_0)(\mu+\eta) - A\right] + \sqrt{D}}{2(\mu+\eta)(M-\varphi_0)^2}. \quad (2.5)$$

Then

$$k^* = -\frac{1}{M\gamma} ln z^*.$$

It can be checked that $c''(k^*) = \varphi''(k^*) < 0$. This means that z^* is a global solution to the problem.

Lemma 2.1 $0 < z^* < 1$.

The proof is obvious consequence of (2.4) an (2.5).

Problem (2.3) has been solved numerically for different parameters of $\varphi_0, M, \mu, \eta, \gamma$. We present the numerical results in the following table 1.

Table 1.

φ_0	M	μ	η	γ	Solutions		
					k^*	c^*	s^*
1.5	10	0.01	0.023	0.2	4.06913	9.849	0.01345
0.3	5	0.2	0.014	0.5	2.7135	4.332	0.1182
4	13	0.043	0.032	0.7	0.89	12.924	0.005185

Finally, we consider the case when both functions $\varphi(k)$ and $L(t)$ are logistic. It means that

$$\begin{cases} \varphi'(k) = \gamma\varphi(M-\varphi) \\ \varphi(0) = \varphi_0 \\ L'(t) = \sigma L(T-L) \\ L(0) = L_0. \end{cases} \quad (2.6)$$

Capital accumulation equation is

$$K'(t) = sf(K, L) - \mu K(t).$$

Per capita capital function is

$$k(t) = \frac{K(t)}{L(t)}.$$

Then

$$\begin{aligned} k' &= \left(\frac{K(t)}{L(t)}\right)' = \frac{K'L - L'K}{L^2} \\ &= \frac{K'}{L} - \left(\frac{L'}{L}\right)\left(\frac{K}{L}\right) \\ &= s\varphi(k) - \mu k - \sigma(T - L)k. \end{aligned}$$

The stability condition is

$$s\varphi(k) = \big(\mu + \sigma(T - L)\big)k. \tag{2.7}$$

Then problem (1.5)-(1.6) reduces to one variable parametric maximization problem:

$$c(k) = \varphi(k) - \big(\mu k - \sigma(T - L)\big)k \to \max_{k>0},\ t \in [0, +\infty[. \tag{2.8}$$

We solve the above problem numerically on $[0, 1]$ for the given parameters of $\varphi_0, M, \mu, \gamma\ L_0, T, \sigma$ and $t = 0; t = 0.5; t = 1$. The numerical results are given in the following tables.

Table 2. $t = 0$

φ_0	M	μ	γ	L_0	T	σ	Solutions		
							k^*	c^*	s^*
1	7	0.01	0.2	2	5	0.1	3.6992	5.6243	0.1694
2	12	0.2	0.13	3	8	0.21	2.6709	7.7980	0.2998
5	15	0.043	0.5	4	9	0.42	0.6153	13.390	0.0896

Table 3. $t = 0.5$

φ_0	M	μ	γ	L_0	T	σ	Solutions		
							k^*	c^*	s^*
1	7	0.01	0.2	2	5	0.1	3.7783	5.7388	0.1554
2	12	0.2	0.13	3	8	0.21	2.9082	8.7641	0.2306
5	15	0.043	0.5	4	9	0.42	0.7794	14.4122	0.0336

Table 4. $t = 1$

φ_0	M	μ	γ	L_0	T	σ	Solutions		
							k^*	c^*	s^*
1	7	0.01	0.2	2	5	0.1	3.8679	5.8580	0.1408
2	12	0.2	0.13	3	8	0.21	3.1960	9.6917	0.1648
5	15	0.043	0.5	4	9	0.42	0.9767	14.8359	0.0096

3 Global Optimization approach to the Growth theory

Assume that the production function $f(K, L)$ is differentiable and homogenous, and satisfies the Lipschitz condition with respect to K with the constant $M > 0$. It means that

$$|f(K', L) - f(K'', L)| \leq M\|K' - K''\|, \; \forall L > 0.$$

Then it is clear that the function $\varphi(k)$ is nonconvex and satisfies the Lipschitz condition with the constant M.

The problem (1.5)-(1.6) becomes a nonconvex optimization problem and can be solved by global optimization techniques. Consider the above problem on a sufficiently large interval $[0, b]$.

$$F(k) = -\varphi(k) + (\mu + \eta)k \to \min, \; k \in [0, b]. \tag{3.1}$$

Problem (3.1) belongs to a class of global optimization and its solution can be found by the method of piecewise linear function [Horst,1995].

Now we present the algorithm of the method. Introduce the function $P(k_1, y)$ for any fixed $y \in [0, b]$ in the following way:

$$P(k_1, y) = F(y) - M|k_1 - y|, \; k_1 \in [0, b].$$

Choose an arbitrary point $k_1^0 \in [0, b]$ and set $y^0 = k_1^0$. Define piecewise linear function $S_1(k_1) = P(k_1, y^0)$.

Let y^1 be the solution of the problem:

$$S_1(k_1) \to \min, \; k_1 \in [0, b].$$

Find a point k_1^1 such that

$$F(k_1^1) = \min\left\{F(k_1^0); F(y^1)\right\}, \; k_1^1 = k_1^0 \vee y^1.$$

Construct piecewise linear function $S_2(k_1)$ by

$$S_2(k_1) = \max\left\{S_1(k_1); P(k_1, y^1)\right\}.$$

Let y^2 be the solution of the problem:

$$S_2(k_1) \to \min,\ k_1 \in [0, b].$$

Define k_1^2 so as to satisfy

$$F(k_1^2) = \min\left\{F(k_1^1), F(y^2)\right\},\ k_1^2 = k_1^1 \vee y^2.$$

Analogously, we can write the iteration process for the t-th step as follows:

$$F(k_1^t) = \min\left\{F(k_1^{t-1}); F(y^t)\right\}),\ k_1^t = k_1^{t-1} \vee y^t,$$

where $y^t : S_t(k_1) \to \min,\ k_1 \in [0, b]$,

$$S_t(k_1) = \max\left\{S_{t-1}(k_1); P(k_1, y^{t-1})\right\}.$$

Consequently, the method of piecewise linear functions generates two sequences of points y^t and k_1^t. It is well known [Vasiliev,1996] that sequences of points k_1^t and y^t produced by the above algorithm converges to the global minimizer for the problem, i.e.,

$$\lim_{t\to\infty} F(k_1^t) = \lim_{t\to\infty} S_t(k_1^t) = \min_{k\in[0,b]} F(k).$$

For example, $\eta = 0.4, \mu = 1.6, b = 5.4$ and

$$\begin{aligned}\varphi(k) = &- 0.0787k^7 + 1.5459k^6 - 12.2406k^5 + 49.8829k^4 \\ &- 109.764k^3 + 122.579k^2 - 51.4616k.\end{aligned}$$

Then problem (1.5)-(1.6) reduces to the problem

$$c^* = \max_{[0,5.4]}\left\{\varphi(k) - 2k\right\}.$$

If we solve the problem by the above algorithm, then solutions will be:

$$k^* = 5.06; c^* = 4.3277.$$

4 Conclusion

We showed that when the production functions of the Solow growth model are nonconvex, then the per capita consumption maximization problem can be reduced to one variable global maximization problem. We have proposed global optimization methods for solving the above problem. Some numerical examples are provided.

Acknowledgements

This paper was done during the first author staying at the Institute of Mathematics of the Humboldt-University Berlin under DFG fellowship from June 20, 2009 to September 20, 2009.

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