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A Boxed Optimization Reformulation for the Convex Second Order Cone Programming [1]

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Abstract: By using the the equivalent expression of the second order cone, we reformulate the convex second order cone programming as a boxed constrained optimization, which is a nonlinear programming with four nonnegative constraints. We give the conditions under which the stationary point of the reformulation problem solve the original problem.

Keywords: Convex second order cone programming, Arrow matrix, Stationary point.

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1. Introduction

The second order cone programming (SOCP) problem is a family of convex optimization problems more general than linear programming. SOCP problems include linear and convex quadratic programs as special cases. On the other hand, SOCP problems are special cases of semidefinite optimization (SDO) problems, and hence can be solved by using an algorithm for SDO problems.

In the last few years, the SOCP problem has received considerable attention from researchers because of its wide range of applications [1]. As mentioned by Kanzow, Ferenczi and Fukushima [7], the linear second-order cone program has been investigated in many previous works. For the study of many important applications and theoretical properties, see [1, 3]. Software for the solution of linear second-order cone programs is also available, see, for example, [10-13]. However, The treatment of the nonlinear second-order cone program is rather limited, see, for examples [3-5, 14-17]. These papers include kinds of solution methods (interior-point methods, smoothing methods, SQP-type methods, or methods based on unconstrained optimization) and certain theoretical properties or suitable reformulations of the second-order cone program.

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In this paper, we consider a boxed constrained optimization reformulation for the nonlinear convex second order cone programming problem, our reformulation is motivated by the method [2] for generalized second order cone complementarity. By using the arrow matrix, we reformulate the SOCP to a nonlinear programming with four nonnegative constraints. Furthermore, we give the conditions under which the stationary point of the reformulation problem solve the original problem.

2. The equivalent reformulation of SOCP

Given $f : \mathbb{R}^{n+1} \to \mathbb{R}$, we consider the following nonlinear convex second-order cone convex programming problem (SOCP), that is, finding $x \in \mathcal{K}$ such that

$$\begin{aligned} &\min f(x) \\ &s.t.\ x \succeq_{\mathcal{K}} 0 \end{aligned} \tag{1}$$

where $f(x)$ is a twice continuously differentiable convex function and $\mathcal{K}$ denotes the second-order cone:

$$\mathcal{K} = \{x \in \mathbb{R}^{n+1} | x_0^2 \geq \sum_{i=1}^{n} x_i^2\}.$$

If we define $A = diag(1, -1, -1, \ldots, -1)$, the second order cone can be expressed in matrix form as

$$\mathcal{K} = \{x \in \mathbb{R}^{n+1} | \frac{1}{2} x^T Ax \geq 0, x_0 \geq 0\}.$$

Then(1) can be written as

$$\begin{aligned} &\min f(x) \\ &s.t.\ \frac{1}{2} x^T Ax \geq 0, \\ &x_0 \geq 0. \end{aligned} \tag{2}$$

To solve (2), we consider introducing a merit function that embodies the KKT conditions. The KKT conditions of (2) are as follows:

$$\begin{aligned} &\nabla f(x) - \lambda Ax - \mu e_0 = 0, \\ &\frac{1}{2} x^T Ax - s = 0, \\ &x_0 - t = 0, \\ &\lambda s = 0,\ \lambda \geq 0,\ s \geq 0 \\ &\mu t = 0,\ \mu \geq 0,\ t \geq 0. \end{aligned} \tag{3}$$

where $\lambda, \mu\, s, t \in \mathbb{R}$ and $e_0 = (1, 0, \ldots, 0)^T \in \mathbb{R}^{n+1}$.

Let

$$F(x) = \frac{1}{2}[\|\nabla f(x) - \lambda Ax - \mu e_0\|^2 + (\frac{1}{2} x^T Ax - s)^2 + (x_0 - t)^2 + (\lambda s)^2 + (\mu t)^2]. \tag{4}$$

We can reformulate problem (2) as the following optimization problem:

$$\begin{aligned} &\min F(x) \\ &s.t. \lambda \geq 0,\ \mu \geq 0,\ s \geq 0, t \geq 0. \end{aligned} \tag{5}$$

The following result give the equivalent relation between the global minimizer of problem (2) and the KKT point of problem (2).

Theorem 1 If x^* is a solution of (2), then there exist

$$(\lambda^*, s^*, t^*, \mu^*) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+$$

such that $(x^*, \lambda^*, s^*, t^*, \mu^*)$ is a global minimizer of (5) with objective value zero. Conversely, if $(x^*, \lambda^*, s^*, t^*, \mu^*)$ is a global minimizer of (5) with objective value zero then $x^*$is the KKT point of (2).

Proof. First of all, if x^* is a solution of (2), we consider the possible two cases:

Case 1: $x^* = 0$, in this case, it is easily to get $\nabla f(x^*) = 0$ due to the convexity of $f(x)$, and therefore, let $\lambda^* \geq 0,\ \mu^* \geq 0, s^* = 0, t^* = 0$, we know that $(x^*, \lambda^*, s^*, t^*, \mu^*)$ is a global minimizer of (5) with objective value zero.

Case 2: $x^* \neq 0$, in this case, notice that $x_0^* > 0$, thus there is at most one active constraint at x^*. If $\frac{1}{2}x^{*T}Ax^* > 0$, both constraints are nonactive and the gradient of the objective must be zero at x^*, that is,$\nabla f(x^*) = 0$. If $\frac{1}{2}x^{*T}Ax^* = 0$, the gradient of the unique active constraints is $Ax^* = (x_0^*, -x_1^* \ldots, -x_n^*)^T \neq 0$, forming a linearly independent set,implying that constraint qualifications hold at x^*. In the both cases there exist Lagrange multipliers $\lambda^* \geq 0, \mu^* = 0$ (for $x_0^* > 0$) such that $\nabla f(x^*) - \lambda^* Ax^* - \mu^* e_0 = 0$ and $\lambda^*(\frac{1}{2}x^{*T}Ax^*) = 0,\ \mu^* x_0 = 0$. Let $s^* = \frac{1}{2}x^{*T}Ax^*,\ t^* = x_0^*$, by using the last equation of (3), we conclude that $s^* \geq 0,\ t^* \geq 0$. Thus the objective value $F(x^*) = \frac{1}{2}[\|\nabla f(x^*) - \lambda^* Ax^* - \mu^* e_0\|^2 + (\frac{1}{2}x^{*T}Ax^* - s^*)^2 + ({x_0}^* - t^*)^2 + (\lambda^* s^*)^2 + (\mu^* t^*)^2] = 0$. Therefore, given a solution of (2), we obtain an optimal solution of (5) with objective value zero.

Conversely, if $(x^*, \lambda^*, s^*, t^*, \mu^*)$ is a global minimizer of (5) with objective value zero, we have that

$F(x^*) = \frac{1}{2}[\|\nabla f(x^*) - \lambda^* Ax^* - \mu^* e_0\|^2 + (\frac{1}{2}x^{*T}Ax^* - s^*)^2 + ({x_0}^* - t^*)^2 + (\lambda^* s^*)^2 + (\mu^* t^*)^2] = 0$, which means

$$\begin{aligned}
&\nabla f(x^*) - \lambda^* Ax^* - \mu^* e_0 = 0, \\
&\frac{1}{2}x^{*T}Ax^* - s^* = 0, \\
&x_0^* - t^* = 0, \\
&\lambda^* s^* = 0,\ \lambda^* \geq 0,\ s^* \geq 0, \\
&\mu^* t^* = 0,\ \mu^* \geq 0,\ t \geq 0.
\end{aligned} \tag{6}$$

So x^* is the KKT point of (2).

The following Theorem shows that under certain condition, the stationary point of (5) is a solution of (2).

Theorem 2 Let $(x^*, \lambda^*, \mu^*, s^*, t^*)$ be a stationary point of (5), define

$$H_f = \nabla^2_{xx} f(x^*) - \lambda A$$

if $H_f$is positive definite then x^* is a solution of (2).

Proof. Let

$$\begin{aligned} l_1 &= \nabla f(x^*) - \lambda^* A x^* - \mu^* e_0, \\ l_2 &= \frac{1}{2} x^{*T} A x^* - s^*, \\ l_3 &= x_0^* - t^*, \end{aligned} \tag{7}$$

and

$$L(x, \lambda, \mu, s, t) = F(x) - \theta_1 \lambda - \theta_2 \mu - \theta_3 s - \theta_4 t.$$

The first order necessary optimality conditions (KKT) of (5) can be written as

$$H_f^T l_1 + l_2 (Ax^*) + l_3 e_0 = 0, \tag{8}$$

$$-l_1^T (Ax^*) + (\lambda^* s^*) s^* - \theta_1 = 0, \tag{9}$$

$$-l_1^T e_0 + (t^* \mu^*) t^* - \theta_2 = 0, \tag{10}$$

$$-l_2 + (\lambda^* s^*) \lambda^* - \theta_3 = 0, \tag{11}$$

$$-l_3 + (t^* \mu^*) \mu^* - \theta_4 = 0. \tag{12}$$

From (9) and (11) we get

$$l_1^T (Ax^*) l_2 = (\lambda^* s^*)^3 + \theta_1 \theta_3, \tag{13}$$

(10) and (12) imply

$$l_1^T e_0 l_3 = (t^* \mu^*)^3 + \theta_2 \theta_4. \tag{14}$$

Premultiplying (8) by l_1 we obtain

$$0 = l_1^T H_f^T l_1 + l_2 l_1^T (Ax^*) + l_3 l_1^T e_0 = l_1^T H_f l_1 + (\lambda^* s^*)^3 + \theta_1 \theta_3 + (t^* \mu^*)^3 + \theta_2 \theta_4, \tag{15}$$

where

$$\begin{aligned} (\lambda^* s^*)^3 + \theta_1 \theta_3 \geq 0, \\ (t^* \mu^*)^3 + \theta_2 \theta_4 \geq 0. \end{aligned}$$

By (15) and the above two inequalities and the assumption that H_f is positive definite, we have

$$l_1 = 0, \quad \lambda^* s^* = 0, \quad t^* s^* = 0. \tag{16}$$

Therefore, using (8) and (16), $l_2(Ax^*) + l_3 e_0 = 0$ and $x_i^* l_2 = 0$, for $i = 1, \ldots, n$. If for some $k = 1, \ldots, n$ $x_k^* \neq 0$, then $l_2 = 0$, and $l_3 = 0$, which implies $F(x^*) = 0$.

If $x_i^* = 0$ for $i = 1, \ldots, n$, then

(i) If $s^* > 0$, (16) and (11) imply $\theta_3 = 0, l_2 = -\theta_3 = 0$ and $l_3 = 0$, then $F(x^*) = 0$.

(ii) If $s^* = 0$, by (16) and the definition of l_2 we obtain

$$0 \geq -\theta_3 = l_2 = \frac{1}{2} x_0^2 \geq 0 \Longrightarrow l_2 = -\theta_3 = 0, \tag{17}$$

and again $l_3 = 0$ and $F(x^*) = 0$. And we get the desired result.

3. Conclusion

In this paper, we reformulate the nonlinear convex second order cone programming problem as a nonlinear programming problem with four nonnegative constraints. Furthermore, we give the conditions under which the stationary point of the reformulation problem solve the original problem. How to design an algorithm to solve the SOCP based on this reformulation deserves further study, we leave it as the future work.

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