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Analytic and Numeric Solutions of Discretized Constrained Optimal Control Problem with Vector and Matrix Coefficients

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Abstract

In this paper, the analytic and numeric solutions of general continuous linear quadratic optimal control problem are presented.The associated general Riccati differential equation is solved by numerical-analytical approach using variational iteration method. Numerical solutions of the constrained optimal control problem are obtained by shooting method and the conjugate gradient method(CGM) via quadratic programming of the discretized continuous optimal control problem. Our results show that both analytical and numerical solutions agree favourably.

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1 Introduction

Most of the algorithms for solving unconstrained optimal control problems are based on a class of descent methods which traditionally have been the principal methods for solving unconstrained minimization problems. Efficient, within this class, are steepest descent(SD), Flethcher-Reeves conjugate gradient algorithm (FRCG) [6], Polak- Rebiere method(PRGG) [10, 17] which had been classified as algorithms with no memory, and the Newton and quasi-Newton methods which update the hessian inverse of $f(\mathbf{X})$.In most applications, the conjugate gradient algorithm is more suitable when compared to other conjugate direction algorithms[8]. It totally outshines the steepest descent method, and compares more favourably with the Newton and quasi-Newton methods. For example, the Newton descent and Quasi-Newton descent are not suitable for minimising the Rayleigh quotient associated with a matrix, since an attempt to approximate the hessian at the minimum is a singular matrix [20]. Also when the dimension of the optimization variable is very large, most especially in optimal control, conjugate gradient method is preferred.

Most research works in the field of unconstrained optimization concentrate their efforts on algorithm with inaccurate or no line search. This is due to the fact that the line search part is time-consuming . Any adaptation of descent methods to solve real-life problems faces with

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the problem of determing α-the one dimensional step size accurately when the function under consideration is not quadratic. So many approaches had been given in [3, 17] for computing α to a tolerance, but still affect the quadratic termination of the conjugate gradient method and the optimal solution.

Optimal control problems governed by ordinary differential equations arise in a wide range of applications. Of a special interest is the linear quadratic optimal control problem (LQOCP), which had been greatly studied [2, 9, 14, 15, 17, 19] due to its interesting features and its wider applicabilities. Sargent[19] gave historical survey of optimal control and went on to review the different approaches to the numerical solution of optimal control problem.The function space algorithm for solving both continous and discrete linear quadratic optimal control problem was given by Polak[17, Ch2]. The determination of α calls for reinitialising of the algorithm, which in turn affect the quadratic termination of the conjugate gradient method.It proves to be successful computationally in the sense that it converges well enough, nevertheless, it involves an enormous variety of cumbersome calculations. The paper by Bersekas[2] tried to eliminate the problem of eigen-value associated with the hessian matrix associated with the problem and reinitialization of the algorithm. In his paper, the author examined the computational aspects of a certain class of discrete-time optimal control problems, in which the partial conjugate gradient algorithms operating in cycles of $s+1$ conjugate gradient steps($s \leq n$ state space dimension) was proposed and analysed. This modest research work incorporated the problem of determining the step size α accurately. The outstanding publication of Ibiejugba and Onumanyi [9] gave birth to extended conjugate gradient method(ECGM). In other to circumvent the numerical set-back in function space algorithm, the authors, constructed the control operator $\mathbf{A}$ which enables α to be determined accurately. Research works along this line are found in[13, 14, 15].

In this paper, we present the analytic solution and numerical solution by shooting method and conjugate gradient method(CGM) for the discretized constrained optimal control problem. We construct the hessian matrix which renders the optimal control problem amenable to the gradient methods. A similar approach was adopted in[15] but their presentation is purely mathematical abstraction. The construction resulted into large sparse quadratic programming problem using conjugate gradient method via penalty function method. The classical control parametrization method is flexible and efficient for a large class of optimal control problems, as discretized optimal control problems can be viewed as a Nonlinear programming problem with some special structure. Now, we consider a generalised problem to be solved analytically and numerically.

2 Analytical Solution

Consider the generalized optimal control problem:

$$\min \quad \mathbf{I}(\mathbf{x},\mathbf{u}) = \int_0^Z (\mathbf{x}^T(t)P\mathbf{x}(t) + \mathbf{u}^T(t)Q\mathbf{u}(t))\,dt \tag{2.1}$$

such that

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(\mathbf{t}), \quad \mathbf{x}(0) = \mathbf{x0}, \quad 0 \leq t \leq Z. \tag{2.2}$$

where $\mathbf{x}(t) \in I\!R^n$, $\mathbf{u}(t) \in I\!R^m$, $P_{n\times n}$, $Q_{m\times m}$ are symmetric positive definite and $A_{n\times n}$, $B_{n\times m}$ are not necessarily symmetric positive definite.

With appropiate conditions on the end points, adjoint variable $\mu(t) \in I\!R^n$ can be introduced by forming the required augmented functional from equation(2.1) and equation(2.2).

The Hamiltonian function is given as

$$\mathbf{H}(\mathbf{x},\mathbf{u},\mu,\dot{\mathbf{x}}) = \mathbf{x}^T(t)P\mathbf{x}(t) + \mathbf{u}^T(t)Q\mathbf{u}(t) + \mu^T(t)(\dot{\mathbf{x}} - A\mathbf{x}(t) - B\mathbf{u}(\mathbf{t})) \tag{2.3}$$

From the knowlegde of calculus of variation[18], we form the necessary conditions for optimal control problem using the Euler-Lagrangian(E-L) equations for $\mathbf{H}$ regarded as function of the four vector variables $(\mathbf{x},\mathbf{u},\mu,\dot{\mathbf{x}})$. Thus, the $E-L$ system can be written

$$\frac{d}{dt}\left[\frac{\partial \mathbf{H}}{\partial \dot{\mathbf{x}}}\right] = \frac{\partial \mathbf{H}}{\partial \mathbf{x}} \tag{2.4}$$

$$\frac{d}{dt}\left[\frac{\partial \mathbf{H}}{\partial \dot{\mathbf{u}}}\right] = \frac{\partial \mathbf{H}}{\partial \mathbf{u}} \tag{2.5}$$

$$\frac{d}{dt}\left[\frac{\partial \mathbf{H}}{\partial \dot{\mu}}\right] = \frac{\partial \mathbf{H}}{\partial \mu} \tag{2.6}$$

which gives

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(\mathbf{t}) \tag{2.7}$$

$$\dot{\mu} = 2P\mathbf{x} - A^T\mu \tag{2.8}$$

$$\mathbf{u} = \frac{Q^{-1}B^T\mu}{2} \tag{2.9}$$

substituting $\mathbf{u}$ in equation(2.9) into equation(2.7), we have the following dynamical equations in matrix form as

$$\begin{pmatrix} \dot{\mathbf{x}} \\ \cdots \\ \dot{\mu} \end{pmatrix} = \underbrace{\begin{pmatrix} A & \vdots & \frac{BQ^{-1}B^T}{2} \\ \cdots & & \cdots \\ 2P & \vdots & -A^T \end{pmatrix}}_{\mathbf{M}} \begin{pmatrix} \mathbf{x} \\ \cdots \\ \mu \end{pmatrix} \tag{2.10}$$

The solution of differential system(2.10) exists and is unique[12].The general solution of which is given as

$$\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ \vdots \\ x_n(t) \\ \mu_1(t) \\ \mu_2(t) \\ \mu_3(t) \\ \vdots \\ \mu_n(t) \end{pmatrix} = e^{t\mathbf{M}} \begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \\ \vdots \\ x_n(0) \\ \mu_1(0) \\ \mu_2(0) \\ \mu_3(0) \\ \vdots \\ \mu_n(t) \end{pmatrix} \tag{2.11}$$

Since we know $\mathbf{x}(0)$, the task is to choose $\mu(0)$ so that the tranversality condition $\mu(Z) = 0$ is satisfied.Assuming a linear function of the form:

$$\mu(t) = C(t)\mathbf{x}(t) \tag{2.12}$$

where $C(t)$ is an $n \times n$ symmetric, negative semidefinite matrix with element varying over time, leads to the matrix **Riccati Equation** for $C(t)$[17]:

$$\dot{C} = 2P - A^T C - CA - \frac{CBQ^{-1}B^T C}{2} \tag{2.13}$$

since $\mu(Z) = 0$, the terminal condition is $C(Z) = 0$. So we have obtained a non-linear system of first-order ODEs in $C(\cdot)$ with a terminal boundary condition. Hence $\mu(0)$ can be obtained by solving equation(2.13).

3 Numerical Solutions

In order to make equations(2.1)-(2.2) amenable to conjugate gradient method, we shall replace the constrained optimal control problem by appropiate discretized optimal control problem. Breaking the interval $[0, Z]$ into s equal intervals with knots $t_0 < t_1 < t_2 \cdots < t_s$ and Δt_j (say $\Delta t_j = 0.01$) and $t_j = j\Delta t_j$. If these intervals are small enough, we can assume that in any interval $[j-1, j]$, the values $\mathbf{x}(t)$ and $\mathbf{u}(t)$ can be approximated by zero order spline $\mathbf{x}_j$ and $\mathbf{u}_j$ respectively. Our objective function (2.1) is then approximated by;

$$min \;\; I = \sum_{j=0}^{s} \mathbf{x}_j^T \mathbf{N} \mathbf{x}_j + \mathbf{u}_j^T \mathbf{T} \mathbf{u}_j \tag{3.1}$$

where $\mathbf{N} = \mathbf{P}\Delta t_j$ and $\mathbf{T} = \mathbf{Q}\Delta t_j$, and the differential equation(2.2) by

$$\dot{\mathbf{x}}_j = \mathbf{A}\mathbf{x}_j + \mathbf{B}\mathbf{u}_j \tag{3.2}$$

Furthermore, we shall use finite difference approximation to write

$$\mathbf{x}_{j+1} = \mathbf{x}_j + \dot{\mathbf{x}}_j \Delta t_j \tag{3.3}$$

Thus, the resulting discretized optimal control problem is:

$$min \;\; I = \sum_{j=0}^{s} \mathbf{x}_j^T \mathbf{N} \mathbf{x}_j + \mathbf{u}_j^T \mathbf{T} \mathbf{u}_j \tag{3.4}$$

subject to

$$\mathbf{x}_{j+1} = \mathbf{C}\mathbf{x}_j + \mathbf{D}\mathbf{u}_j, \tag{3.5}$$

where $\mathbf{C} = (I_{n\times n} + \mathbf{A})\Delta t_j$, $\mathbf{D} = \mathbf{B}\Delta t_j$, $j = 0, 1, \cdots, s-1$ and $\mathbf{x}_0$ given.

By parameter optimization[7], the discretized problem becomes a large sparse quadratic programming problem. We give a matrix representation

$$I(\mathbf{z}) = \mathbf{z}^T \mathbf{M} \mathbf{z} + \mathbf{c} \tag{3.6}$$

subject to

$$\mathbf{V}\mathbf{z} = \mathbf{k} \tag{3.7}$$

and

$$\mathbf{z}^T = (\mathbf{x}_1^T, \mathbf{x}_2^T, \cdots, \mathbf{x}_s^T, \mathbf{u}_0^T, \mathbf{u}_1^T, \cdots, \mathbf{u}_s^T) \tag{3.8}$$

where $\mathbf{M}$ is a block diagonal matrix of order $(n+m)s+m$, with entries given by:

$$[\mathbf{M}]_{ii} = \mathbf{N}, \quad i = 1, 2, \cdots, s$$

and

$$[\mathbf{M}]_{ii} = \mathbf{T}, \quad i = s+1, s+2, \cdots, 2s+1$$

where i^{th} element correspond to i^{th} block and $\mathbf{c} = \mathbf{x}_0\mathbf{N}\mathbf{x}_0$. The matrix $\mathbf{V}$ is block matrix of order $ns \times (n+m)s+m$ with the representation

$$\mathbf{V} = \begin{pmatrix} \mathbf{E} & \vdots & \mathbf{F} & \vdots & \mathbf{0} \end{pmatrix} \tag{3.9}$$

where $\mathbf{E}$ is an $ns \times ns$ block bidiagonal matrix with principal block diagonal elements $[\mathbf{E}]_{ii} = I_{n\times n}$ and lower block principal diagonal elements $[\mathbf{E}]_{ij} = -\mathbf{C}, \forall i,j$ *block* such that $i = j+1$. $[\mathbf{F}]$ is $ns \times ms$ block diagonal matrix with block diagonal elements $[\mathbf{F}]_{ii} = -\mathbf{D}$ and $[\mathbf{0}]$ is an $ns \times m$ zero matrix. The column vector $[\mathbf{k}]$ is of order $ns \times 1$ with entries given by:$[\mathbf{k}]_{1-n,1} = A\mathbf{x}_0$ and$[\mathbf{k}]_{i1} = 0, \; i = n+1, n+2, \cdots, ns$.

Using proposition 2.8 of [5], the quadratic programming(QP) problem(3.6)-(3.7) is equivalent to the solution of the saddle point system of linear equations

$$\begin{pmatrix} \mathbf{M} & \mathbf{V}^T \\ \mathbf{V} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{z} \\ \lambda \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{k} \end{pmatrix} \tag{3.10}$$

where $\lambda \in I\!R^{ns}$ is the Lagrange multipliers. If $\mathbf{V}$ is a full row rank matrix, we can solve equation(3.10) effectively by the Gaussian elimination with suitable pivoting strategy, or by a symmetric factorization which takes into account that equation(3.10) is indefinite. Alternatively, we can use MINRES, a Krylov space method which generates the iterates minimizing the Euclidean of the residual in the Krylov space. The performance of the MINRES depend on the spectrum of the KKT system(3.10), similarly as the performance of the conjugate gradient method.

The unconstrained minimization problem by penalty function method is

$$\min L_\rho(\mathbf{z}) = \mathbf{z}^T\mathbf{M}\mathbf{z} + \mathbf{c} + \rho \langle \mathbf{V}\mathbf{z} - \mathbf{k}, \mathbf{V}\mathbf{z} - \mathbf{k} \rangle \tag{3.11}$$

on expansion, we have

$$\min L_\rho(\mathbf{z}) = \mathbf{z}^T\mathbf{A}_\rho\mathbf{z} + \mathbf{B}\mathbf{z} + \mathbf{C} \tag{3.12}$$

Equation(3.12) is the quadratic form representation for the unconstrained minimization problem, where $L_\rho(\mathbf{z})$ is penalized Lagrangian,ρ is penalty parameter, the penalized matrix $\mathbf{A}_\rho = \left[\mathbf{M} + \rho\mathbf{V}^T\mathbf{V}\right]$, $\mathbf{B} = -2\rho\mathbf{k}^T\mathbf{k}\mathbf{V}$ and $\mathbf{C} = \rho\mathbf{k}^T\mathbf{k} + \mathbf{c}$.

Proposition 3.1. *Consider the continuous optimal control problem(2.1)-(2.2) and the associated discretized optimal control problem(3.4)-(3.5), the matrix* $\mathbf{M}$ *defined in (3.6) is positive symmetric definite and well-conditioned.*

Proof. The positive symmetric definiteness is immediate,since $\mathbf{M}$ is a block diagonal matrix with positive symmetric block diagonal elements.Thus for any $\mathbf{z} \in I\!R^{(n+m)s+m}$,$\mathbf{z}^T\mathbf{M}\mathbf{z} > 0$ and $\mathbf{M} = \mathbf{M}^T$.
Let $b \in R^{(n+m)s+m}$,then the system of linear equations

$$\mathbf{M}\mathbf{z} = \mathbf{b} \tag{3.13}$$

is stable with respect to perturbation of the entries of $\mathbf{M}$, $\mathbf{z}$ and $\mathbf{b}$. The condition number $K(\mathbf{M})$

is small depending on ratio $\lambda_{\max}/\lambda_{\min}$,where $\lambda_{\max}$ and $\lambda_{\min}$ are the maximum and minimum eigenvalues of $\mathbf{M}$.Hence $\mathbf{M}$ is well-conditioned. □

The property of a problem of being well-conditioned is independent of the numerical method that is being used to solve it. Since we have established the positive symmtric definiteness of $\mathbf{M}$, we state the following lemmas.

Lemma 1 (Dostiàl[5])**.** *Let* $\mathbf{M} \in I\!R^{((n+m)s+m)\times((n+m)s+m)}$ *be a symmetric positive definite matrix, let* $\mathbf{V} \in I\!R^{(ns)\times((n+m)s+m)}$,$\rho > 0$*, and let* $Ker\mathbf{M} \cap Ker\mathbf{V} = \mathbf{0}$*. Then the penalized matrix* $\mathbf{A}_\rho$ *is positive definite.*

Lemma 2 (Dostiàl[5])**.** *Let* $\mathbf{M} \in I\!R^{((n+m)s+m)\times((n+m)s+m)}$ *be a symmetric positive definite matrix, let* $\mathbf{V} \in I\!R^{(ns)\times((n+m)s+m)}$,$\mu > 0$ *such that*

$$\mathbf{z}^T\mathbf{M}\mathbf{z} \geq \mu \|\mathbf{z}\|, \mathbf{z} \in Ker\mathbf{V}$$

Then $\mathbf{A}_\rho$ *is positive definite for sufficiently large* ρ

The lemmas ensure the sufficient condition for $\mathbf{z}^* \in I\!R^{((n+m)s+m)}$ to a be local minimum point. We solve the unconstrained minimization equation(3.12) by conjugate gradient algorithm in the inner loop and enforce the feasibility condition in the outer loop as stated in the following algorithm.

Algorithm 1 Conjugate Gradient Algorithm for Constrained Optimal Control Problem

Step 1. Select a $\mathbf{z}_{0,0} \in I\!R^{(n+m)s+m}$,$c > 0$and $\rho_0 > 0$. Set $k = 0$.
Step 2. Set $i = 0$ and set $p_0 = -g_0 = -\Delta L_\rho(\mathbf{z}_{0,0})$
Step 3. Compute $\alpha_i = \frac{p_i^T p_i}{p_i^T \mathbf{A}_\rho p_i}$
Step 4. Set $\mathbf{z}_{0,i+1} = \mathbf{z}_{0,i} + \alpha_i p_i$
Step 5. Compute $\Delta L_\rho(\mathbf{z}_{i+1}$
Step 6. If $\Delta L_\rho(\mathbf{z}_{0,i+1}) = 0$ and $\mathbf{V}\mathbf{z}_{0,i+1} = \mathbf{k}$ stop;else go to step 7.
Step 7. If $\Delta L_\rho(\mathbf{z}_{0,i+1}) \neq 0$, set

$$g_{i+1} = \Delta L_\rho(\mathbf{z}_{0,i+1}),$$
$$p_{i+1} = -g_{i+1} + \gamma_i p_i, \text{ with } \gamma_i = \frac{g_{i+1}^T g_{i+1}}{g_i^T g_i}$$

Step 8. Set $i = i + 1$, and go to step 3.
Step 9. Else if $\mathbf{V}\mathbf{z}_{0,i+1} \neq \mathbf{k}$, set $\rho_{k+1} = c\rho_k$; set $k = k + 1$ and go to step 2.

4 Hypothetical Examples

In this section, we demonstrate the reliability of our approach to discretized optimal control problem to other methods. We compared our result with the solutions obtained by shooting method for the dynamical equation(2.11).All computations in the following examples were performed in the MATLAB environment, Version 7.6.0324 Release(2008a) running on a Microsoft Windows $Vista^{TM}$ Home Premium operating system with an Intel(R)Pentium(R) Dual processor running at 1.87GHz.

Example 1. *Consider the following constrained optimal control problem*

$$\min \quad I(x,u) = \int_0^1 (x_1^2 + x_1x_2 + x_2^2 + 2u_1^2 + 2u_1u_2 + u_2^2)\,dt \tag{4.1}$$
such that

$$\begin{aligned} \dot{x}_1 &= x_1 - x_2 + 2u_1 + u_2 \\ \dot{x}_2 &= x_1 + x_2 - u_2 \end{aligned} \tag{4.2}$$

where $\mathbf{x}(0) = \mathbf{1}_{2\times 1}$, $P = \begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix}$, $Q = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ *and* $B = \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix}$.
By eq(2.10) $\boldsymbol{M}$ *is given as :*

$$\mathbf{M} = \begin{pmatrix} 1 & -1 & \vdots & 1 & 0 \\ 1 & 1 & & 0 & 1 \\ \cdots & & \cdots & & \cdots \\ 2 & 1 & & -1 & -1 \\ 1 & 2 & \vdots & 1 & -1 \end{pmatrix} \tag{4.3}$$

By equation(2.13), we have the resulted non-linear, coupled system of ordinary differential equations;

$$\dot{c}_{11} = 2 - 2c_{11} - c_{12} - c_{21} - c_{12}c_{21} - c_{11}^2 \tag{4.4}$$
$$\dot{c}_{12} = 1 - 2c_{12} + c_{11} - c_{22} - c_{11}c_{12} - c_{12}c_{22} \tag{4.5}$$
$$\dot{c}_{21} = 1 - 2c_{21} + c_{11} - c_{22} - c_{11}c_{21} - c_{21}c_{22} \tag{4.6}$$
$$\dot{c}_{22} = 2 - 2c_{22} + c_{12} + c_{21} - c_{12}c_{21} - c_{22}^2 \tag{4.7}$$

Solving equations(4.4), (4.5), (4.6) and (4.7) by means of Backward-sweep Runge-Kutta method, we obtained;

$$\mathbf{C}(0) = \begin{pmatrix} -\frac{26127}{10000} & -\frac{2809}{10000} \\ -\frac{2809}{10000} & -\frac{19187}{10000} \end{pmatrix} \tag{4.8}$$

with this the analytical solution for ***Riccati differential equation*** *can be obtained by He's variational iteration method[1] as follows;*

$$c_{11,n+1}(t) = c_{11,n}(t) + \int_0^t \lambda_{11}(s)\left[\frac{dc_{11}}{ds} - 2 + 2\tilde{c}_{11} + \tilde{c}_{12} + \tilde{c}_{21} + \tilde{c}_{12}\tilde{c}_{21} + \tilde{c}_{11}^2\right] ds \tag{4.9}$$
$$c_{12,n+1}(t) = c_{12,n}(t) + \int_0^t \lambda_{12}(s)\left[\frac{dc_{12}}{ds} - 1 + 2\tilde{c}_{12} - \tilde{c}_{11} + \tilde{c}_{22} + \tilde{c}_{11}\tilde{c}_{12} + \tilde{c}_{12}\tilde{c}_{22}\right] ds \tag{4.10}$$
$$c_{21,n+1}(t) = c_{21,n}(t) + \int_0^t \lambda_{21}(s)\left[\frac{dc_{12}}{ds} - 1 + 2\tilde{c}_{21} - \tilde{c}_{11} + \tilde{c}_{22} + \tilde{c}_{11}\tilde{c}_{21} + \tilde{c}_{21}\tilde{c}_{22}\right] ds \tag{4.11}$$
$$c_{22,n+1}(t) = c_{22,n}(t) + \int_0^t \lambda_{22}(s)\left[\frac{dc_{11}}{ds} - 2 + 2\tilde{c}_{22} - \tilde{c}_{12} - \tilde{c}_{21} + \tilde{c}_{12}\tilde{c}_{21} + \tilde{c}_{22}^2\right] ds \tag{4.12}$$
$$\tag{4.13}$$

where $\tilde{c}_{ij,n}$ *are considered as restricted variation i.e.* $\delta c_{ij,n} = 0$ *and* $t \in [0,1]$*. Its stationary* *conditions can be obtained as:*

$$\begin{aligned}
1+\lambda_{11}(t) &= 0, \quad \acute{\lambda}_{11}(t)\,|_{s=t}=0, && (4.14)\\
1+\lambda_{12}(t) &= 0, \quad \acute{\lambda}_{12}(t)\,|_{s=t}=0, && (4.15)\\
1+\lambda_{21}(t) &= 0, \quad \acute{\lambda}_{21}(t)\,|_{s=t}=0, && (4.16)\\
1+\lambda_{22}(t) &= 0, \quad \acute{\lambda}_{22}(t)\,|_{s=t}=0. && (4.17)
\end{aligned}$$

Thus, the Lagrange multipliers are $\lambda_{11}(s),\lambda_{12}(s),\lambda_{21}(s)$ *and* $\lambda_{22}(s)=-1$ *[1].*
We can take the linearized solution $c_{11}=\frac{1}{2}(2-Ae^{-2t})$, $c_{12}=\frac{1}{2}(1-Be^{-2t})$, $c_{21}=\frac{1}{2}(1-Ce^{-2t})$ *and* $c_{22}=\frac{1}{2}(2-De^{-2t})$ *as the initial approximation, the condition* $\mathbf{C}(0)$ *gives us* $A=-\frac{36117}{10000}$,$B=C=-\frac{7809}{10000}$ *and* $D=-\frac{29187}{10000}$*. Then we get:*

$$\begin{aligned}
c_{11,1}(t) &= 1-t-\frac{36117}{10000}e^{-2t}+\frac{54780183}{40000000}-\frac{95661}{20000}e^{-2t}-\frac{136541817}{40000000}e^{-4t}+\frac{5}{8}\ln(e^{-2t})\\
c_{12,1}(t) &= \frac{1}{2}-\frac{7809}{10000}e^{-2t}+\frac{39605133}{50000000}-\frac{2067}{1000}e^{-2t}+\frac{63744867}{50000000}e^{-4t}+\frac{1}{2}\ln(e^{-2t})\\
c_{21,1}(t) &= \frac{1}{2}-\frac{7809}{10000}e^{-2t}+\frac{39605133}{50000000}-\frac{2067}{1000}e^{-2t}+\frac{63744867}{50000000}e^{-4t}+\frac{1}{2}\ln(e^{-2t})\\
c_{22,1}(t) &= 1+t-\frac{29187}{10000}e^{-2t}+\frac{1968771}{8000000}+\frac{10113}{4000}e^{-2t}+\frac{18257229}{8000000}e^{-4t}+\frac{5}{8}\ln(e^{-2t})
\end{aligned}$$

In same manner,the rest of the components of the iteration formulae can be obtained using MATLAB package. Hence by equation(2.12), $\mu(0)^T=(\frac{-1814}{625}\ \frac{-21947}{10000})^T$*. Thus, the solution is given as;*

$$\begin{aligned}
x_1(t) &= \left(\frac{233}{1000}e^{(\frac{1071}{625})t}+\frac{9767}{10000}e^{-(\frac{1071}{625})t}\right)cos\left(\left(\frac{9677}{10000}\right)t\right)\\
&\quad -\left(\frac{723}{5000}e^{(\frac{1071}{625})t}+\frac{5833}{5000}e^{-(\frac{1071}{625})t}\right)sin\left(\left(\frac{9677}{10000}\right)t\right) && (4.18)\\
x_2(t) &= \left(\frac{7}{40}e^{(\frac{1071}{625})t}+\frac{33}{40}e^{-(\frac{1071}{625})t}\right)cos\left(\left(\frac{9677}{10000}\right)t\right)\\
&\quad +\left(\frac{261}{5000}e^{(\frac{1071}{625})t}+\frac{8979}{10000}e^{-(\frac{1071}{625})t}\right)sin\left(\left(\frac{9677}{10000}\right)t\right) && (4.19)\\
\mu_1(t) &= \left(\frac{259}{5000}e^{(\frac{1071}{625})t}-\frac{14771}{5000}e^{-(\frac{1071}{625})t}\right)cos\left(\left(\frac{9677}{10000}\right)t\right)\\
&\quad -\left(\frac{367}{5000}e^{(\frac{1071}{625})t}-\frac{31177}{10000}e^{-(\frac{1071}{625})t}\right)sin\left(\left(\frac{9677}{10000}\right)t\right) && (4.20)\\
\mu_2(t) &= \left(\frac{381}{2500}e^{(\frac{1071}{625})t}-\frac{23471}{10000}e^{-(\frac{1071}{625})t}\right)cos\left(\left(\frac{9677}{10000}\right)t\right)\\
&\quad +\left(\frac{1}{80}e^{(\frac{1071}{625})t}-\frac{1293}{625}e^{-(\frac{1071}{625})t}\right)sin\left(\left(\frac{9677}{10000}\right)t\right) && (4.21)
\end{aligned}$$

using equation(2.9), the control variables are given as

$$\begin{aligned} u_1(t) &= \left(\frac{1021}{10000}e^{\left(\frac{1071}{625}\right)t} - \frac{26507}{10000}e^{-\left(\frac{1071}{625}\right)t}\right) cos\left(\left(\frac{9677}{10000}\right)t\right) + \\ &\quad \left(-\frac{61}{2000}e^{\left(\frac{1071}{625}\right)t} + \frac{1049}{2000}e^{-\left(\frac{1071}{625}\right)t}\right) sin\left(\left(\frac{9677}{10000}\right)t\right) \quad (4.22) \\ u_2(t) &= \left(-\frac{381}{2500}e^{\left(\frac{1071}{625}\right)t} + \frac{23471}{10000}e^{-\left(\frac{1071}{625}\right)t}\right) cos\left(\left(\frac{9677}{10000}\right)t\right) \\ &\quad + \left(-\frac{1}{80}e^{\left(\frac{1071}{625}\right)t} + \frac{1293}{625}e^{-\left(\frac{1071}{625}\right)t}\right) sin\left(\left(\frac{9677}{10000}\right)t\right) \quad (4.23) \end{aligned}$$

The analytic objective value is $I = 2.5466$. *The objective value obtained by shooting method[4] is* $I = 2.5837$, *why the objective value by conjugate gradient method(CGM) is* $I = 2.5820$. *The graphs below show the agreement of the numerical methods with the analytical solution. The CGM solution agrees more favourably compared with the shooting method and most importantly at the boundary conditions.*

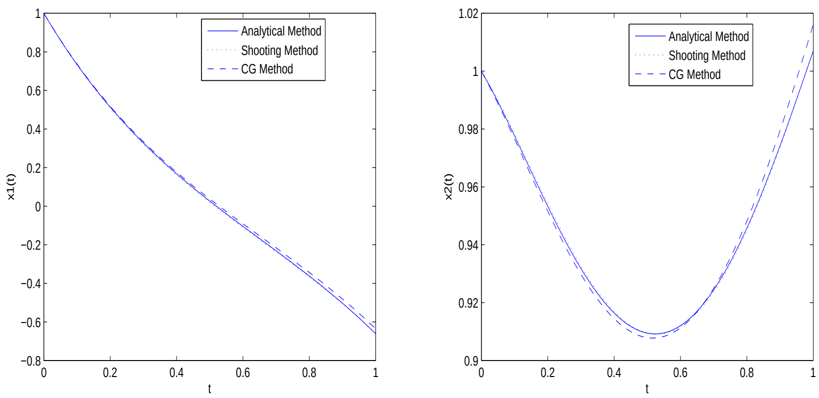

Figure 1: The graphs of state variables $x_1(t)$ and $x_2(t)$ against time for example(1)

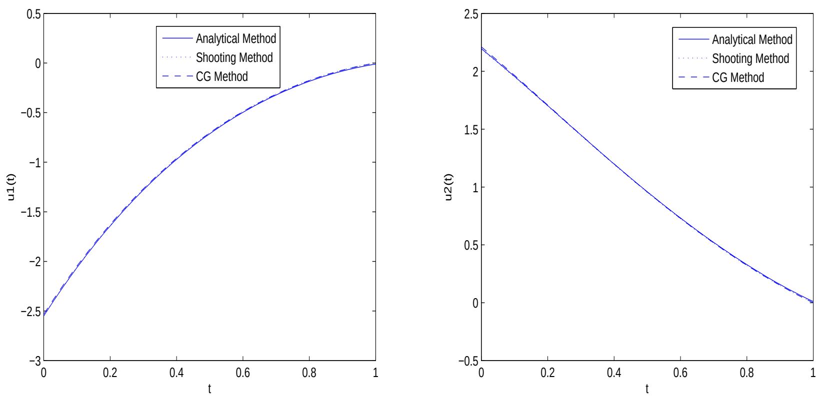

Figure 2: The graphs of control variables $u_1(t)$ and $u_2(t)$ against time for example(1)

Example 2. *Consider the constrained optimal control problem*

$$\begin{array}{l} \min \quad I(x,u) = \int\limits_0^1 (2x_1^2 + x_1x_2 + x_2^2 + u_1^2 + \tfrac{1}{2}u_1u_2 + u_2^2)\,dt \\ \textit{such that} \end{array} \tag{4.24}$$

$$\begin{array}{rcl} \dot{x}_1 &=& 2x_1 - x_2 + u_1 + u_2 \\ \dot{x}_2 &=& x_1 - x_2 - u_1 \end{array} \tag{4.25}$$

where $\mathbf{x}(0) = \mathbf{1}_{2\times 1}$, $P = \begin{pmatrix} 2 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}$, $Q = \begin{pmatrix} 1 & \frac{1}{4} \\ \frac{1}{4} & 1 \end{pmatrix}$, $A = \begin{pmatrix} 2 & -1 \\ 1 & -1 \end{pmatrix}$ *and* $B = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$. *By eq(2.10)* $\boldsymbol{M}$ *is given as :*

$$\mathbf{M} = \begin{pmatrix} 2 & -1 & \vdots & \frac{4}{5} & -\frac{2}{5} \\ 1 & -1 & & -\frac{2}{5} & \frac{8}{15} \\ \cdots & & \cdots & & \cdots \\ 4 & 1 & & -2 & -1 \\ 1 & 2 & \vdots & 1 & 1 \end{pmatrix} \tag{4.26}$$

By equation(2.13), we have the resulted non-linear, coupled system of ordinary differential equations;

$$\dot{c}_{11} = 4 - 4c_{11} - c_{12} - c_{21} + \frac{2}{5}c_{11}c_{12} + \frac{2}{5}c_{11}c_{21} - \frac{8}{15}c_{12}c_{21} - \frac{4}{5}c_{11}^2 \tag{4.27}$$

$$\dot{c}_{12} = 1 - c_{12} + c_{11} - c_{22} - \frac{4}{5}c_{11}c_{12} + \frac{2}{5}c_{11}c_{22} - \frac{8}{15}c_{12}c_{22} + \frac{2}{5}c_{12}^2 \tag{4.28}$$

$$\dot{c}_{21} = 1 - c_{21} + c_{11} - c_{22} - \frac{4}{5}c_{11}c_{21} + \frac{2}{5}c_{11}c_{22} - \frac{8}{15}c_{21}c_{22} + \frac{2}{5}c_{21}^2 \tag{4.29}$$

$$\tag{4.30}$$

$$\dot{c}_{22} = 2 + 2c_{22} + c_{12} + c_{21} - \frac{4}{5}c_{12}c_{21} + \frac{2}{5}c_{12}c_{22} + \frac{2}{5}c_{21}c_{22} - \frac{8}{15}c_{22}^2 \tag{4.31}$$

$$\tag{4.32}$$

Solving equations(4.27), (4.28), (4.29) and (4.31) as in example 1,we obtained

$$\mathbf{C}(0) = \begin{pmatrix} -\frac{52385}{10000} & \frac{3979}{10000} \\ \frac{3979}{10000} & -\frac{8521}{10000} \end{pmatrix} \tag{4.33}$$

Hence by equation(2.12), $\mu(0)^T = (-\frac{24241}{5000} \quad -\frac{113}{250})^T$*. Thus, the solution is given as;*

$$\begin{aligned} x_1(t) &= \left(\frac{301}{10000}e^{\left(\frac{20483}{10000}\right)t} + \frac{9699}{10000}e^{-\left(\frac{20483}{10000}\right)t}\right) cos\left(\left(\frac{9809}{10000}\right)t\right) \\ &\quad -\left(\frac{571}{10000}e^{\left(\frac{20483}{10000}\right)t} + \frac{183}{250}e^{-\left(\frac{20483}{10000}\right)t}\right) sin\left(\left(\frac{9809}{10000}\right)t\right) \end{aligned} \tag{4.34}$$

$$\begin{aligned} x_2(t) &= \left(\frac{277}{10000}e^{\left(\frac{20483}{10000}\right)t} + \frac{4861}{5000}e^{-\left(\frac{20483}{10000}\right)tt}\right) cos\left(\left(\frac{9809}{10000}\right)t\right) \\ &\quad -\left(\frac{23}{5000}e^{\left(\frac{20483}{10000}\right)t} - \frac{37071}{10000}e^{-\left(\frac{20483}{10000}\right)t}\right) sin\left(\left(\frac{9809}{10000}\right)t\right) \end{aligned} \tag{4.35}$$

$$\begin{aligned} \mu_1(t) &= \left(\frac{27}{1250}e^{\left(\frac{20483}{10000}\right)t} - \frac{24349}{5000}e^{-\left(\frac{20483}{10000}\right)t}\right) cos\left(\left(\frac{9809}{10000}\right)t\right) \\ &\quad -\left(\frac{503}{10000}e^{\left(\frac{20483}{10000}\right)t} - \frac{2639}{500}e^{-\left(\frac{20483}{10000}\right)t}\right) sin\left(\left(\frac{9809}{10000}\right)t\right) \end{aligned} \tag{4.36}$$

$$\begin{aligned} \mu_2(t) &= \left(\frac{1101}{10000}e^{\left(\frac{20483}{10000}\right)t} - \frac{5621}{10000}e^{-\left(\frac{20483}{10000}\right)t}\right) cos\left(\left(\frac{9809}{10000}\right)t\right) \\ &\quad -\left(\frac{83}{10000}e^{\left(\frac{20483}{10000}\right)t} + \frac{9357}{2500}e^{-\left(\frac{20483}{10000}\right)t}\right) sin\left(\left(\frac{9809}{10000}\right)t\right) \end{aligned} \tag{4.37}$$

using equation(2.9), the control variables are given as

$$\begin{aligned} u_1(t) &= \left(\frac{501}{10000}e^{\left(\frac{20483}{10000}\right)t} - \frac{8241}{5000}e^{-\left(\frac{20483}{10000}\right)t}\right) cos\left(\left(\frac{9809}{10000}\right)t\right) + \\ &- \left(\frac{157}{10000}e^{\left(\frac{20483}{10000}\right)t}\frac{2567}{625}e^{-\left(\frac{20483}{10000}\right)t}\right) sin\left(\left(\frac{9809}{10000}\right)t\right) \end{aligned} \tag{4.38}$$

$$\begin{aligned} u_2(t) &= \left(\frac{233}{10000}e^{\left(\frac{20483}{10000}\right)t} - \frac{5057}{2500}e^{\left(\frac{20483}{10000}\right)t}\right) cos\left(\left(\frac{9809}{10000}\right)t\right) \\ &\quad -\left(\frac{53}{2500}e^{\left(\frac{20483}{10000}\right)t} - \frac{16123}{10000}e^{-\left(\frac{20483}{10000}\right)t}\right) sin\left(\left(\frac{9809}{10000}\right)t\right) \end{aligned} \tag{4.39}$$

The analytic objective value is $I = 2.6460$*. The objective value obtained by shooting method is* $I = 2.6965$*, why the objective value by conjugate gradient method(CGM) is* $I = 2.6946$*. The graphs below shows the solutions of the numerical methods with the analytical solution. The CGM solution agrees more favourably compared with the shooting method and most importantly at the boundary conditions as in example 1 above.*

5 Conclusion

The results obtained by the discretized continuous algorithm via quadratic programming for solving constrained optimal control problems is inevitable in real-world problems, where they are becoming too complex to allow analytical solution. The algorithm converges in four cycles and is well suited for a certain class of discretized constrained optimal control problems. The conjugate gradient method(CGM) competes more favourably than the shooting method since its objective values 2.5820 and 2.6946 are much closer to the analytic solutions 2.5466 and 2.6460 than the shooting method's objective values 2.5837 and 2.6965 repectively, more importantly at the boundary points of the system variables. It demonstrates superiority over other method

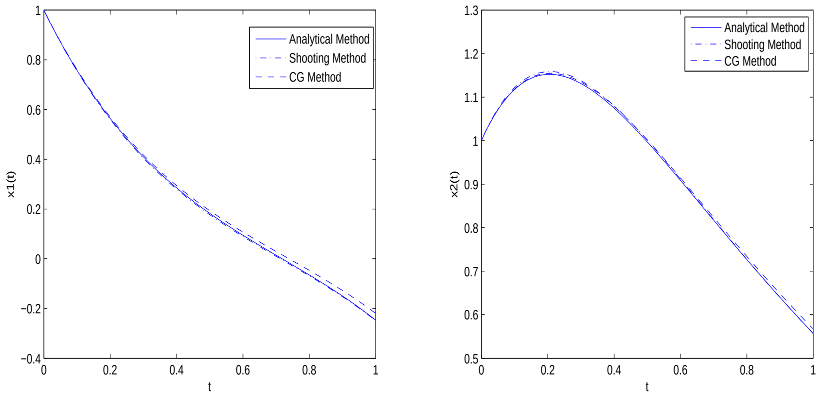

Figure 3: The graphs of state variables $x_1(t)$ and $x_2(t)$ against time for example(2)

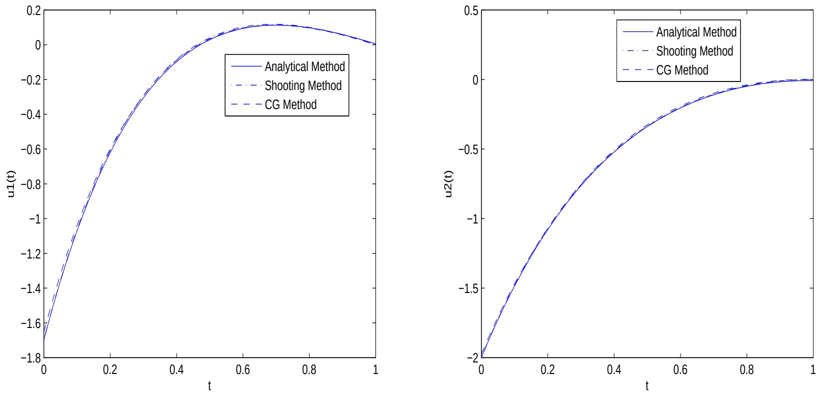

Figure 4: The graphs of control variables $u_1(t)$ and $u_2(t)$ against time for example(2)

such as function space algorithm[17] and partial conjugate gradient method[2] in the sense that step length is computed in finite arithmetic and both the gradient and hessian are easily computed. Also, it is preferable to other numerical approaches that inculcate conjugate gradient method[9],[13], and [14] as tool for determining the local optimal point. The algorithm is flexible and it can be adopted for optimal control problems constrained by non-linear differential equation.

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