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Edge Colouring of Cactus Graphs

Nasreen Khan†, Anita Pal‡ and Madhumangal Pal†

† Department of Applied Mathematics with Oceanology and Computer Programming,
Vidyasagar University, Midnapore-721 102, INDIA
e-mail: mmpalvu@gmail.com

‡ Department of Mathematics,
National Institute of Technology Durgapur,
Durgapur-713209, West Bengal, INDIA.
e-mail: anita.buie@gmail.com

Abstract. Edge colouring of an undirected graph $G = (V, E)$ is assigning a colour to each edge $e \in E$ so that any two edges having end-vertex in common have different colours. That is, the edge colouring problem asks for assigning colours from a minimum number of colours to edges of a graph such that no two edges with the same colour are incident to the same node. The minimum number of colours required for an edge colouring of G is denoted by $\chi'(G)$.

A cactus graph is a connected graph in which every block is either an edge or a cycle. In this paper, we colour the edges of a cactus graph with minimum number of colours.

Keywords: Graph colouring; edge colouring; cactus graph.

AMS Subject Classifications: 68Q22, 68Q25, 68R10.

1 Introduction

Cactus graph is a connected graph in which every block is either a cycle or an edge, in other words, no edge belongs to more than one cycle. Cactus graph have extensively studied and used as models for many real world problems. This graph is one of the most useful discrete mathematical structure for modelling problem arising in the real world. It has many applications

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in various fields like computer scheduling, radio communication system etc. Cactus graph have studied from both theoretical and algorithmic points of view. This graph is a subclass of planar graph and superclass of tree.

A proper colouring of $G = (V, E)$ is a map $c : E \rightarrow S$ (where S is the set of available colours) with $c(e) \neq c(f)$ for any adjacent edges e, f. The minimum number of colours needed to properly colour the edges of G, is called the chromatic index of G and denoted by $\chi'(G)$. The span of an edge colouring is the maximum colour number assigned to any edge of G. The edge colouring number of a graph G, denoted by $\chi'(G)$, is the least integer k such that all the edges of the graph are coloured by k-colours.

Let $\Delta = \Delta(G)$ denote the maximum degree of the vertices of the graph G. Obviously, $\chi'(G) \geq \Delta$ since all edges incident to the same vertex must be assigned different colours and the upper bound of $\chi'(G) \geq 2\Delta - 1$ also follow easily. Vizing [9, 10] and Gupta [4] independently proved that $\Delta+1$ colours suffice when G is a simple graph. Next Vizing proved the theorem that for any simple graph G, $\Delta \leq \chi'(G) \leq \Delta+1$ [9, 11]. Again we have from König's [5] theorem that every bipartite graph can be edge coloured with exactly Δ colours, i.e., $\chi'(G) = \Delta$. Shanon [8] proved that every graph can be edge coloured with at most $3\Delta/2$ colours, that is, $\chi'(G) \leq 3\Delta/2$. A few other upper bounds on $\chi'(G)$ have been known Andersen [1], Goldberg [3], Nishzeki and Kashiwagi [6].

Goldberg [2, 3] and Seymour's [7] conjecture is that if $\chi'(G) \geq \Delta + 2$, then $\chi'(G) = max_{H \subseteq G}[\frac{e(H)}{n(H)/2}]$.

Again if K_n be a complete graph of n vertices, then

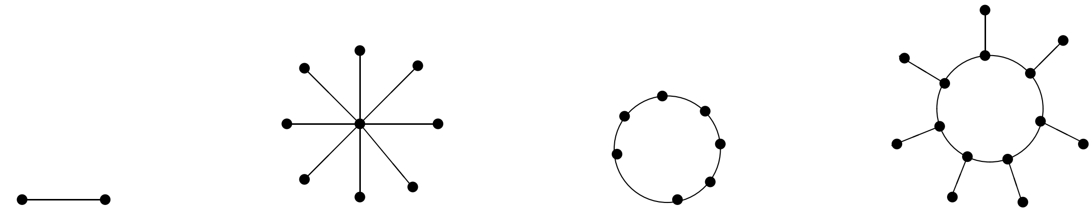
$$\chi'(K_n) = \begin{cases} n-1, & \text{if n is even,} \\ n, & \text{if n is odd.} \end{cases}$$

holds.

2 The edge colouring of induce sub-graphs of cactus graphs

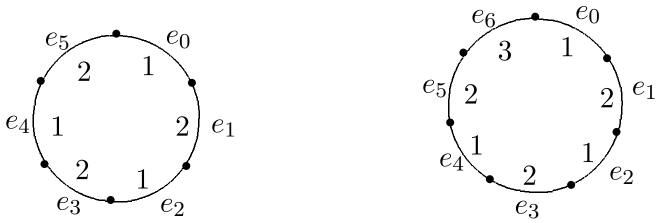
Let $G = (V, E)$ be a given graph and subset U of V the **induced subgraph** by U, denoted by $G[U]$, is the graph $G' = (U, E')$, where $E' = \{(u, v) : u, v \in U \text{ and } (u, v) \in E\}$.

The cactus graphs have many interested subgraphs, those are illustrated below. An edge, is nothing but P_2, so $\chi'(P_2) = 1$. The star graph $K_{1,\Delta}$ is a subgraph of cactus graph, therefore one can conclude the following result.



(a) An edge (b) A star ($K_{1,\Delta}$) (c) A cycle (C_n) (d) A sun (S_n)

Figure 1: Induce subgraphs of cactus graph.



(a) for $n = 2k$ (b) for $n = 2k+1$

Figure 2: Edge colouring of two cases of cycle

Lemma 1 *For any star graph $K_{1,\Delta}$, $\chi'(K_{1,\Delta}) = \Delta$, which is equal to $n-1$, where n is the number of vertices.*

Lemma 2 *For any cycle C_n of length n,* $\chi'(C_n) = \begin{cases} \Delta, & \textit{if } n \textit{ is even,} \\ \Delta + 1, & \textit{if } n \textit{ is odd.} \end{cases}$

Proof. Let C_n be a cycle of length n. We classify C_n into two groups, viz., C_{2k}, C_{2k+1}, i.e., one contains even number of edges and another contains odd number of edges.

Let $e_0, e_1, e_2, \ldots, e_{n-1}$ be the n number of edges of the cycle C_n. The edge colourings of that cycle are as follows.

Case I. Let $n = 2k$ (even), then the colour sequence of edges of C_{2k} are as

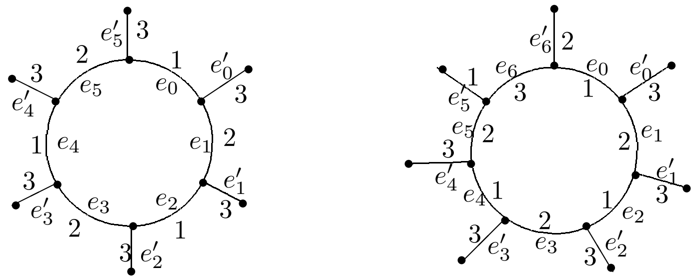
$$c(e_i) = \begin{cases} 1, & \text{if } i = 2k, \\ 2, & \text{if } i = 2k+1. \end{cases}$$

Case II. Let $n = 2k+1$ (odd).

In this case, the colouring for first $2k-1$ edges $e_0, e_1, e_2, \ldots, e_{2k-1} = e_{n-2}$ are same as in Case I. And for the last edge e_{n-1}, c is redefine as

$$c(e_{n-1}) = 3.$$

We know that for any cycle, $\Delta = 2$.



(a) for $n = 2k$ (b) for $n = 2k+1$

Figure 3: Edge colouring of two cases of Sun

Thus, from the above two cases (shown in Figure 2), it follows that

$$\chi'(C_n) = \begin{cases} 2 = \Delta, & \text{if } n \text{ is even,} \\ 3 = \Delta + 1, & \text{if } n \text{ is odd.} \end{cases}$$

□

Lemma 3 *For any sun S_{2n}, $\chi'(S_{2n}) = \Delta$ where Δ is the degree of S_{2n}.*

Proof. Let S_{2n} be constructed from C_n by adding an edge to each vertex. To colour this graph we consider two cases.

Let e_0, e_1, e_2, …, e_{n-1} be the edges of C_n and e_i is adjacent to e_{i+1}. Here e_0 is adjacent to e_1 and e_{n-1}. To complete S_{2n}, we add an edge e'_i to each vertex v_i of C_n. We consider two cases.

Case I. Let $n = 2k$ (even). As Lemma 2, the colouring of C_n is given by

$$c(e_i) = \begin{cases} 1, & \text{if } i = 2k, \\ 2, & \text{if } i = 2k+1, \end{cases}$$

And the colouring of e'_i are assign as

$$c(e'_i) = 3, \text{ for } i = 0, 1, \ldots, n-1.$$

Case II. Let $n = 2k+1$ (odd). In this case the colouring of edges of the cycle C_n are same as in Case II given in Lemma 2.

Now we colour the other edges e'_i of S_{2n}. The colouring of first e'_0, e'_1, e'_2, …, e'_{n-3} edges are as

$$c(e'_i) = 3, \text{ for } i = 0, 1, \ldots, n-3.$$

And for two remaining edges e'_{n-1}, e'_{n-2} the values of c are

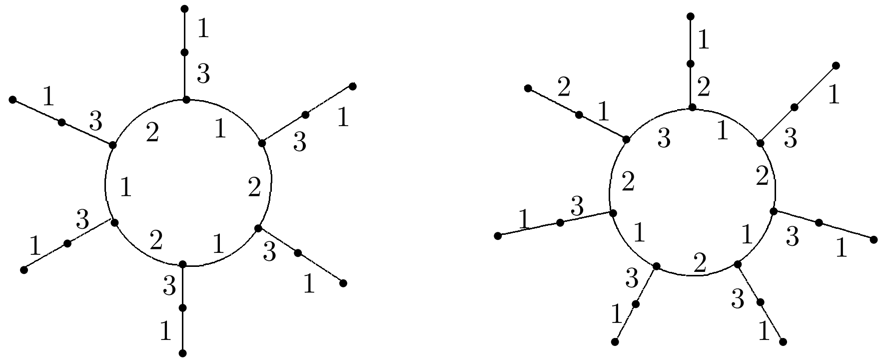


Figure 4: Illustration of Lemma 4

$$c(e'_{n-2}) = 1 \text{ and } c(e'_{n-1}) = 2.$$

Hence $\chi'(S_{2n}) = 3 = \Delta$. □

Corollary 1 *Let a graph G contains a cycle C_n of length n and one or more (but less than n) vertices contain edges. If Δ be the degree of G then $\chi'(G) = \Delta$.*

Lemma 4 *Let G be a graph obtained from S_{2n} by adding an edge to each of the pendent vertex of S_{2n}, then $\chi'(S_{2n}) = \lambda(G) = \Delta = 3$.*

Proof. Follows from Figure 4. □

Let us consider a graph which contains two cycles C_n and C_m of lengths n and m respectively and they have a common cutvertex v_0. Then we denote this graph as $C_n \bigcup_{v_0} C_m$ $(= G)$. The number of vertices and edges of G are $n+m$ and $n+m-1$ respectively. The colouring procedure is describe in the following lemma.

Lemma 5 *Let $G = C_n \bigcup_{v_0} C_m$ then $\chi'(C_n \bigcup_{v_0} C_m) = 4 = \Delta$, where Δ is the degree of the common vertex v_0.*

Proof. Let C_n and C_m be two cycles of G. Let e_0, e_1, e_2, …, e_{n-1} be the edges of C_n such that e_i is adjacent to e_{i+1}, $0 \leq i \leq n-2$ and e_0 is adjacent to e_{n-1}.

Again e'_0, e'_1, e'_2, …, e'_{m-1} be the edges of C_m such that e'_0 is adjacent to e'_1 and e'_{m-1}. Also, e'_i is adjacent to e'_{i+1}, $0 \leq i \leq m-2$.

Now we colour the edges of C_m by considering two cases, viz., $m = 2k$ (even) and $m = 2k+1$ (odd).

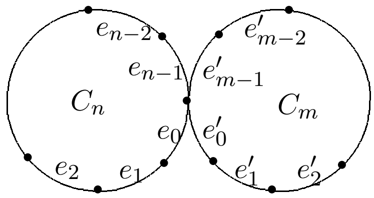


Figure 5: Illustration of Lemma 5

Case I. *If* $n = 2k$ *and* $m = 2k$.

The colour sequence of C_n is same as in Case I given in Lemma 2. Now the colour sequence of C_m are as follows.

Here we colour the first edge e'_0 by 3 and last edge e'_{m-1} by 4, i.e.,

$$c(e'_0) = 3 \text{ and } c(e'_{m-1}) = 4.$$

Then the colour sequence of the remaining edges of C_m (considering C_m as C_n) are same as in Case I of Lemma 2.

If $n = 2k$ *and* $m = 2k + 1$.

Here also we colour the first and last edges e'_0 and e'_{m-1} by 3, 4 respectively. And the colour sequence of the remaining edges of C_m (considering C_m as C_n) are same as in Case II of Lemma 2.

Case II. *If* $n = 2k + 1$ *and* $m = 2k$.

The colour sequence of C_n are same as in Case II of Lemma 2. Now the colour sequence of edges of C_m are as follows.

Here we colour the first edge e'_0 by 2, i.e., $c(e'_0) = 2$. And the colouring of the remaining edges e'_i, $i = 1, 2, \ldots, m - 1$ are same as the above case [for $n = 2k$ and $m = 2k$].

If $n = 2k + 1$ *and* $m = 2k + 1$.

Here we colour the first edge e'_0 by 2. And the remaining edges will be coloured as in same manner given in above case I [for $n = 2k$ and $m = 2k + 1$].

Hence $\chi'(C_n \bigcup_{v_0} C_m) = 4 = \Delta$. □

Some times a cycle C_3 of length 3 is called a triangle. A triangle may be a subgraph of a cactus graph. Also, a triangle shape star, (i.e., all the triangles have a common cutvertex) be a subgraph of a cactus graph. Now, we consider a triangle shape star for edge colouring. Let $T_0, T_1, \ldots, T_{n-1}$ be the n triangles meet at a common cutvertex v_0 and we denote this graph by G_1, which is equivalent to $\bigcup\limits_{v_0} T_i$. The number of vertices and edges of G_1 are $2n + 1$ and $3n$

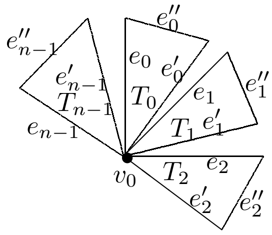


Figure 6: Illustration of Lemma 6

respectively. The labelling procedure is describe in the following lemma.

Lemma 6 *Let* $G_1 = \bigcup_{v_0} T_i$ *be a triangle shape star. Then* $\chi'(G_1) = \Delta$*, where* v_0 *is the common cutvertex of* T_i*'s and* Δ *is degree of* v_0*.*

Proof. Let us denote the n such triangles by $T_0, T_1, T_2, \ldots, T_{n-1}$ shown in Figure 6. Let e_i, e'_i, e''_i are the edges of T_i, $i = 0, 1 \ldots, n-1$.

The colouring of edges of T_i's as follows.

For $i = 0, 1, \ldots, n-1$, $c(e_i) = 2i+1$, $c(e'_i) = 2i+2$ and $c(e''_0) = 3$, $c(e''_i) = 1$, for $i = 1, 2, \ldots, n-1$.

Now, the minimum number of colouring of the edge e'_{n-1} of the triangle T_{n-1} is

$$c(e'_{n-1}) = 2(n-1) + 2 = 2n.$$

Here $\Delta = 2n$, therefore $\chi'(G_1) = 2n = \Delta$. □

Corollary 2 *If each vertex of star shape triangle (except cutvertex) has another edges then the value of* χ' *remains unchanged.*

Lemma 7 *Let a graph* G *contains* n *number of cycles of length* 3 *and* m *number of cycles of length* 4*. If they have a common cutvertex with degree* Δ *then* $\chi'(G) = \Delta$*.*

Proof. Let T_0, T_1, $\ldots$, T_{n-1} be the n number of cycles of length 3 and R_0, R_1, $\ldots$, R_{m-1} be the m number of cycles of length 4 given in Figure 7. Let v_0 be the cutvertex. Again let e_i, e'_i, e''_i be the edges of T_i and e^1_j, e^2_j, e^3_j, e^4_j be the edges of R_j.

Case I. *When m=1.*

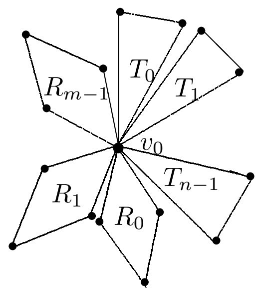


Figure 7: A graph contains n number of C_3's and m number of C_4's

When G contains n number of C_3 and one C_4, then the colouring of edges of R_0 are (according to the previous lemma) as

$$c(e_0^k) = \begin{cases} 2n+1, & \text{when } k=1; \\ 1, & \text{when } k=2; \\ 2, & \text{when } k=3; \\ 2n+2, & \text{for } k=4. \end{cases}$$

Here $\Delta = 2n+2$, therefore $c(e_0^4) = \Delta$.

Case II. *When m=2.*

i.e., G contains n number of C_3's and two C_4's. The colour of edges of R_2 are assign as per following rule.

$$c(e_1^k) = \begin{cases} 2n+3, & \text{when } k=1; \\ 1, & \text{when } k=2; \\ 2, & \text{when } k=3; \\ 2n+4, & \text{for } k=4. \end{cases}$$

Here $\Delta = 2n+4$, therefore $c(e_1^4) = \Delta$.

In general, we colour the edges of R_m as

$$c(e_m^k) = \begin{cases} 2n+2m-1, & \text{when } k=1; \\ 1, & \text{when } k=2; \\ 2, & \text{when } k=3; \\ 2n+2m, & \text{for } k=4. \end{cases}$$

Here $\Delta = 2n+2m$, therefore $c(e_m^4) = \Delta$.

Hence $\chi'(G) = \Delta$. □

Corollary 3 *If a graph G contains finite number of cycles of any length with a common cutvertex, then $\chi'(G) = \Delta$, where Δ is the degree of cutvertex.*

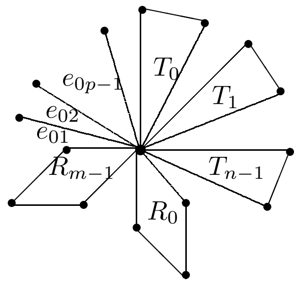


Figure 8: Illustration of Lemma 8

Lemma 8 *Let G be a graph which contains finite number of cycles of any length and finite number of edges. If they have a common cutvertex with degree Δ then $\chi'(G) = \Delta$.*

Proof. First we prove that if G contains n number of cycles of length 3, m number of cycles of length 4 and p number of edges then $\chi'(G) = \Delta$.

Case I. *When p=1.*

i.e, G contains n number of C_3's, m number of C_4's and one edge. Let e_{01} be that edge. Here we follow the previous lemma. Then the colouring of that edge is $c(e_{01}) = 2n + 2m + 1$.

Here $\Delta = 2n + 2m + 1$, therefore $c(e_{01}) = \Delta$.

Case II. *When p=2.*

i.e, G contains n number of C_3's, m number of C_4's and two edges. Let e_{02} be the second edge. Then $c(e_{02}) = 2n + 2m + 2$.

Here $\Delta = 2n + 2m + 2$, therefore $c(e_{02}) = \Delta$.

Then by mathematical induction we prove that when G contains n number of C_3's, m number of C_4's and p number of edges then $\chi'(G) = \Delta$.

Therefore, the general form of this lemma can be proved using the mathematical induction.

□

Corollary 4 *When the end vertices of the edges of the graph G have another edges and each vertex of the cycles have another edges and each vertex of the cycle have another edges, then the value of χ' remains unchanged.*

Lemma 9 *Let G be a graph contains a cycle of any length and each vertex of the cycle has another cycle of any length. Then if Δ is the degree of G, then $\chi'(G) = \Delta$.*

Proof. At first we prove that if G contains a cycle of any length and each vertex of the cycle have another cycles of length 3 then $\chi'(G) = \Delta$.

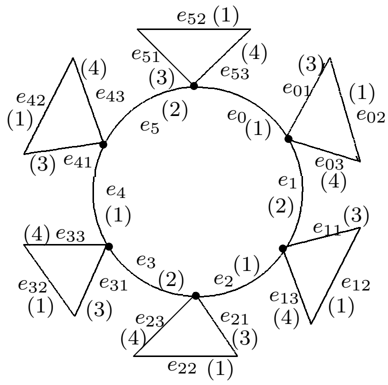


Figure 9: For $n = 2k$

Let $e_0, e_1, e_2, \ldots, e_{n-1}$ be the edges of C_n and $e_{01}, e_{02}, e_{03}; e_{11}, e_{12}, e_{13}; \ldots; e_{n-1,1}, e_{n-1,2}, e_{n-1,3}$ be the edges of all C_3's. That is, here the number of cutvertices is n.

Case I. Let $n = 2k$, i.e., even.

In this case, the colour sequence of the edges e_0, e_1, e_2, ..., e_{n-1} is assign according as Lemma 2 (for $n = 2k$) and the colour of all C_3's are as follows.

The colour of the edges of main cycle are either 1 or 2. Here e_{i1} and e_{i3} of C_3's are adjacent to e_i and e_{i+1}. The colours of e_{i1} and e_{i3} should not be 1 or 2 but we may assign its colour by 3 or 4. The edges e_{i2} are not adjacent to e_i and e_{i+1}. So we assign its colour by 1 or 2. Thus the colour sequence of all C_3's are as follows:

for $i = 1, 2, 3,$

$$c(e_{ij}) = \begin{cases} 3, & \text{for } j = 1; \\ 1, & \text{for } j = 2; \\ 4, & \text{for } j = 3. \end{cases}$$

Case II. Let $n = 2k + 1$, i.e., odd.

In this case, the colour sequence of the edges e_0, e_1, e_2, ..., e_{n-1} of main cycle C_n are same as in Case II of Lemma 2 (for $n = 2k$).

Now, the colour sequence of the edges $e_{01}, e_{02}, e_{03};\ e_{11}, e_{12}, e_{13};\ \ldots\ ;\ e_{n-3,1}, e_{n-3,2}, e_{n-3,3}$ of C_3's are same as in Case I. And we colour the remaining two C_3's as

$$c(e_{n-2,j}) = \begin{cases} 1, & \text{for } j = 1; \\ 2, & \text{for } j = 2; \\ 4, & \text{for } j = 3. \end{cases} \quad \text{and} \quad c(e_{n-1,j}) = \begin{cases} 2, & \text{for } j = 1; \\ 1, & \text{for } j = 2; \\ 4, & \text{for } j = 3; \end{cases}$$

respectively.

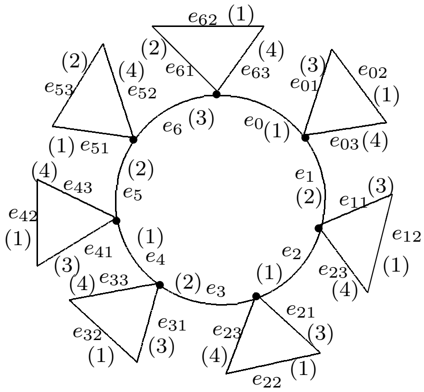


Figure 10: For $n = 2k + 1$

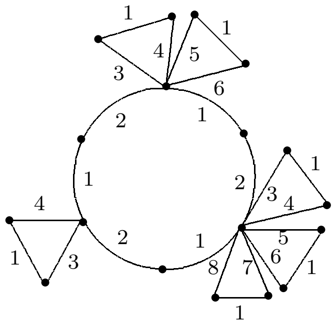


Figure 11: Illustration of Corollary 6.

Thus from the above two cases, it follows that $\chi'(G) = 4 = \Delta$. Similarly, when the cut vertices of G contain cycles of lengths four, five or more, then, $\chi'(G) = \Delta$.

So the general form of lemma can be proved using mathematical induction. □

Corollary 5 *If the vertices of all C_3's contain one, two or more edges then χ' value remains unchanged.*

Corollary 6 *If each vertex of a cycle of any length contains two, three or more cycles of any length, then the value of χ' value remains unchanged.*

From Figure 11 we see that the degree of the graph is 8 and the value of χ' is 8, i.e., $\chi'(G) = \Delta$. Now, we label another important subclass of cactus graph called caterpillar graph.

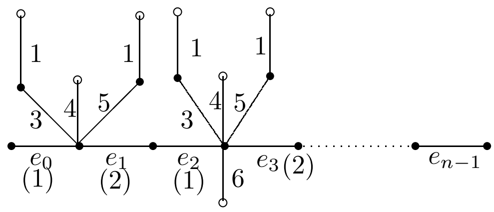


Figure 12: Here $\Delta = 6$ and so $\chi'(G) = 6 = \Delta$

3 Edge Colouring of Caterpillar Graph

Definition 1 *A caterpillar C is a tree where all vertices of degree ≥ 3 lie on a path, called the backbone of C. The hairlength of a caterpillar graph C is the maximum distance of a non-backbone vertex to the backbone.*

Lemma 10 *If G be a caterpillar graph and Δ be its degree, then $\chi'(G) = \Delta$.*

Proof. Let P_n be a path of caterpillar graph, where P_n contains n number of edges e_0, e_1, e_2, ..., e_{n-1}. At first we colour the edges of the caterpillar graph and then colour the other edges of that graph.

Now, we colour the edges of P_n as follows:

$$c(e_i) = \begin{cases} 1, & \text{for } i = k; \\ 2, & \text{for } i = k+1; \end{cases}$$

where $k = 0, 1, 2, \ldots, n-1$.

We may not assign same colours as path P_n of the other edges of the caterpillar graph, as they are adjacent. So, we may colour the other edges by 3, 4, and so on (minimum number of colouring depends on the degree of that graph).

The edge colouring of the caterpillar graph is shown in Figure 12.

So, we see from Figure 12 that when the degree of the graph increases the minimum number of colouring increases and is exactly equal to its maximum degree of of vertex. Here from Figure 12 we see that $\Delta = 6$ therefore $\chi'(G) = \Delta$.

Hence the proof. □

4 Edge Colouring of Lobster

Another subclass of cactus graph is called lobster graph. The definition of lobster graph is given below.

Definition 2 *A lobster is a tree having a path (of maximum length) from which every vertex has distance at most k, where k is an integer.*

The maximum distance of the vertex from the path is called the diameter of the lobster graph. There are many types of lobsters given in literature like diameter 2, diameter 4, diameter 5, etc.

Lemma 11 *Let G be a lobster graph. If Δ be the maximum degree of the lobster graph, then $\chi'(G) = \Delta$.*

Proof. Let P_n be the path (of length n) of the lobster graph. Let e_0, e_1, ..., e_{n-1} be the edges of P_n. First we colour the edges of P_n as follows:

$$c(e_i) = \begin{cases} 1, & \text{for } i = k; \\ 2, & \text{for } i = k+1; \end{cases}$$

for $k = 0, 1, 2, \ldots, n-1$.

Then we colour the other edges of that graph. Let l_i be the other edges where $i = 0, 1, 2, \ldots$. The colours of the edges of P_n are either 1 or 2. Here l_i's are adjacent to e_i and e_{i+1}, $0 \leq i \leq n-2$. So, we may not colour the edges l_i's by 1 or 2. We may colour the edges by 3, 4 or more (it depends on the degree of each vertex).

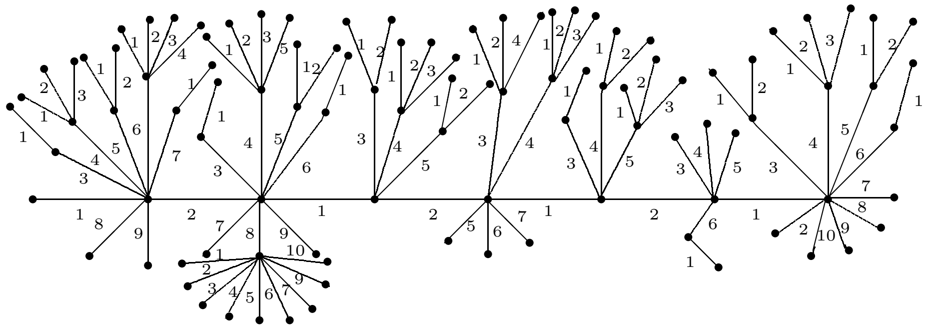


Figure 13: A diameter 2 lobster graph

Form Figure 13 we see that maximum degree of the two diameter lobster graph is 10, i.e., $\Delta = 10$. Also we see that maximum colour number is 10 which is equal to the degree of vertex of 2-diameter lobster graph.

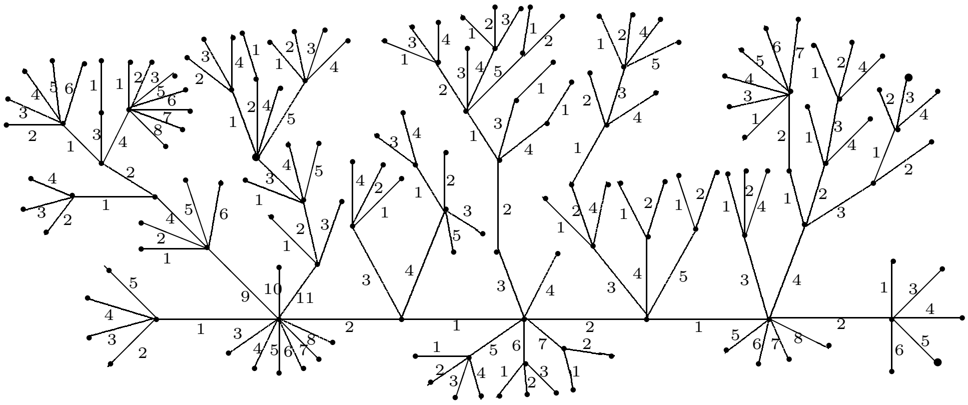


Figure 14: A diameter 5 lobster graph

Form Figure 14 we see that maximum degree of the 5 diameter lobster graph is 11, i.e., $\Delta = 11$. Also we see that maximum colour number is 11 which is equal to the maximum degree of the vertex of 5-diameter lobster graph.

From the above results we say that in general, for any lobster graph G, $\chi'(G) = \Delta$, where Δ is maximum degree of the vertex. $\square$

The edge colouring of all subgraphs of cactus graphs and their combinations are discussed in the previous lemmas. From these results we conclude that the value of χ' of any cactus graph can not be more than $\Delta + 1$. Hence we state the following theorem. Figure 15 illustrate the edge colouring of a cactus graph.

Theorem 1 *If Δ is the degree of a cactus graph G, then $\Delta \leq \chi'(G) \leq \Delta + 1$.*

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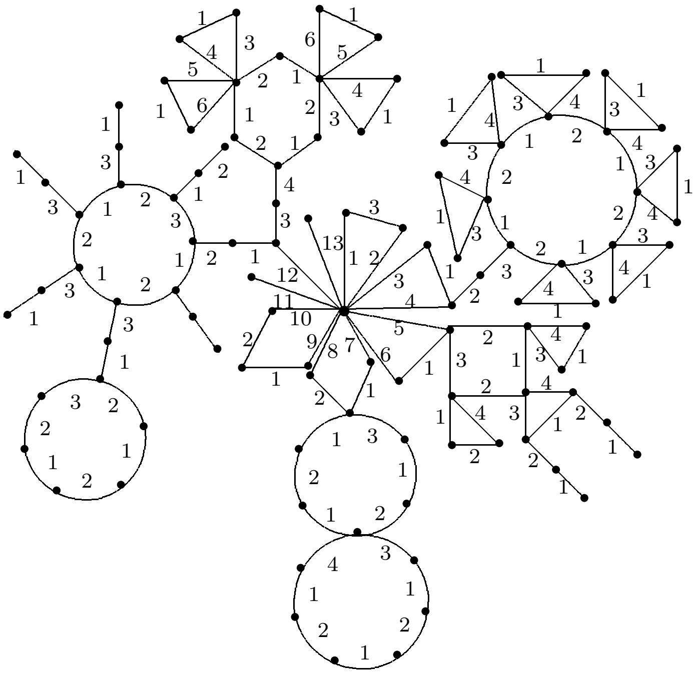

Figure 15: Illustration of Theorem 1.

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