Optimizing by Graphs

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Abstract

Here we first analyze the adequate graph-theoretical framework, and then we will shown some interesting results about these useful tools and its equivalence classes.

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1. Introduction

Topological Graph Theory is a branch of this mathematical discipline which studies the embedding of graphs in surfaces, but also analyze the graphs as topological spaces.

Such "embedding of graphs in surfaces" signifies that we want to drawthe graph on a surface without two edges intersecting. An example of such surfaces will be the sphere.

It will be also interesting to give some previous concepts, as that of *line graph, vertex-transitivity, edge-transitivity,* and so on.

Let G be a graph. Suppose that we denote by *V(G)* their set of nodes, and by *E(G)* their set of edges.

The so-called *line graph of G* will be the graph whose set of nodes is E (therefore, it coincides with the set of edges of G), and whose edges connect all pairs of E which have one common end (or extremity) in G.

Usually, it is abridged by *L(G).*

Hence, the *Line Graph of G* is another graph that represents the adjacencies between edges of G.

The Line Graph is sometimes called either *Adjoint Graph,* or *Interchange Graph,* but also *Edge Graph,* or *Derived Graph of G,* and so on.

A graph, *G,* is said to be *node-transitive* (or *vertex-transitive*), if for any two of its nodes, n_i and n_j, there is an automorphism which maps n_i to n_j.

A simple graph, *G,* is said to be *edge-transitive* (or *link-transitive*), if for any two of its edges, e and e', there is an automorphism which maps e into e'.

A simple graph, *G,* is said to be *symmetric,* when it is both, node-transitive and edge-transitive.

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But a simple graph, G, which is edge-transitive, but not node-transitive, is said *semi-symmetric.* Obviously, such a graph will be necessarily a bipartite graph.

A *clique* of a graph is its maximal complete subgraph.

Let Θ be a collection of elements into a finite set, S. The smallest subset $Y \subset S$ that meets every member of Θ is called their *hitting set,* or also their node cover.

2. Chordality

An auxiliar concept, which many times appears at Graph Theory problems, will be the *Chordality,* or character to be chordal a graph.

Let G be an undirected graph (UG). We says that G is *chordal,* if every cycle of length strictly greater than three ($l \in \{4, 5, 6, ...\}$) possesses a "chord".

This name ("chord") means an edge joining two non-consecutive nodes of the cycle.

Therefore, an UG will be *chordal,* if it does not contain an induced subgraph isomorphic to the n-cyclic graph, C_n, when $n > 3$.

And it also admits different, but equivalent, definitions, as

A graph is *chordal,* if it is the intersection graph of subtrees of a tree.

A graph is *chordal,* if it has an ordering such that for each node, the neighbours in front form a clique.

Recall that a *clique,* into a graph G, is a subset $C \subset V(G)$, such that every two members of C are adjacent.

The *chordality* result a *hereditary property,* because all the induced subgraphs of a chordal graph will be also chordal.

Chordal graphs are sometimes called either as *Triangulated Graphs,* or *Perfect Elimination Graphs.*

For instance, the interval graphs are chordal.

Any complete k-partite graph has maximum clique cardinality equal to k.

An *independent set* of a graph, G, is a subset of its nodes such that no two nodes in such subset may represent an edge of G.

For instance, in the case of the *utility graph,* departing from $K_{3,3}$, we have two independent sets, composed each one by three nodes colored either in black or in white, respectively (see the figure, please).

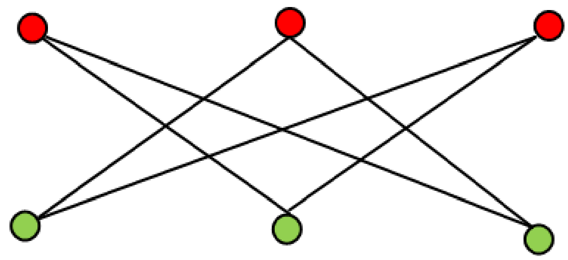
The name *utility graph* alluded here to the three-cottage problem. Such relatively known three-cottage problem asks about the planarity of the complete bipatite graph, $K_{3,3}$.

Kuratowski (1930) proved that it is not so.

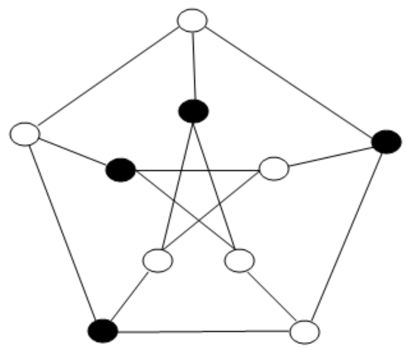
And therefore, the three-cottage problem has no solution.

Utility graph

From $K_{3,3}$



Petersen graph



Also can be noted in the partition into two classes of the family of nodes that belongs to the *Petersen graph,* as the second figure shown. It has ten nodes, fifteen edges, radius and diameter in both cases equal to two, chromatic number three, chromatic index four, and so on. It is strongly regular, 3-connected, it is 3-partite, a snark and cubic, with 120 automorphisms, being its automorphism group S_5, the symmetric group of order five.

Recall that a *snark* will be a connected, and bridgeless, cubic graph with chromatic number equal to four.

The Petersen graph is a little mathematical jewel, because it will be very useful for examples and counterexamples in many questions of graph theory.

In fact, it is drawn as a pentagon with a pentagram inside, with five rodes radiating from the nodes of the pentagram.

D. Knuth affirms that this graph is "a remarkable configuration that serves as a counterexample to many optimization predictions about what might be true for graphs in general".

The largest order n-graph which does not contain to K_p, the complete p-graph, is denominated the *Turán graph,* and it is denoted by $T_{n,\ p}$.

So, the well known *octahedron* is a cross polytope whose nodes and edges form a Turán graph; more exactly, it will be $T_{6,\ 3}$.

3. Planarity

A graph which can be embedded in the plane will be called a *planar graph.* It signifies that can be drawn on the plane, in such a way that its edges intersect only at their endpoints. Therefore, a non-planar graph is the one cannot be drawn in the plane without edge intersections.

When a planar graph is so drawn, we said either a *plane graph,* or a *planar embedding of the graph.*

If it can be drawn on the plane, also it can be drawn on the sphere, and vice versa.

Also it admits a generalization to graphs which can be drawn on a surface of a given genus.

According these terms, *planar graphs have graph genus zero,* because both, the plane and the sphere, are surfaces of such genus.

There exist a known result, due to the polish mathematician Kazimierz Kuratowski, which give us a characterization of planar graphs in terms of forbidden graphs:

A finite graph is planar if and only if it does not contain a subgraph that is a subdivision of K_5 *or* $K_{3,3}$.

A *subdivision* of a graph results from inserting nodes into edges.

So, for instance

$$\circ \to \circ \Rightarrow \circ - \circ \to \circ$$

in the particular case of a directed edge.

Recall that K_5 denotes the complete 5-graph.

And $K_{3,3}$ the complete bipartite graph on six nodes.

But this theorem may be expressed in some equivalent ways, as for instance, by:

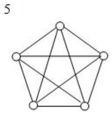
A finite graph is planar if and only if it does not contain a subgraph that is homeomorphic to K_5 *or* $K_{3,3}$.

The aforementioned problem, about the "embedding graphs into surfaces", is not only of theoretical interest, but it has many applications. For instance, in printing electronic circuits. In such cases, the purpose is to print (i.e. "embed") a circuit (i.e. "the graph") on a circuit board (i.e. "the surface") without two connections crossing each other, and resulting in an undesirable short circuit.

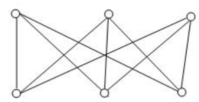
Hopcroft and Tarjan introduced a new method of testing the planarity of a graph in time linear to the number of edges. Because the efficient planarity testing is essential to graph drawing.

Also Chung et al. studied the problem of embedding a graph into a book, supposing situated for this their graph nodes on a line along the spine of such

K_5



$K_{3,3}$



book. And its edges are then drawn on separate pages, in such a way that edges that belongs to the same page do not cross between them. Such abstraction also has many practical applications, for instance, in layout problems that appears in the routing of multilayer printed circuit boards.

A relevant property of the Petersen graph is their non-planarity, with its consequences. It is because we can prove that it contains both, K_5 and $K_{3,3}$, as minors.

Two different perspectives of the Petersen graph will be seeing it either as the complement of the line graph of K_5, or also as the Kneser graph $KG_{5,2}$.

The Petersen graph can be drawn on a torus without edge crossings. Therefore, it has orientable genus equal to one.

If we pass to non-orientable surfaces, the simplest on which can be embedded without crossings (the Petersen graph) will be the projective plane.

4. Colorability

Graph Coloring is an assignment of labels to elements of a graph, subject to certain constraints. Usually, and by tradition, such labels are called "colors". It proceeds of attempts to mathematization of problems as the known of "four colors", and other related with them.

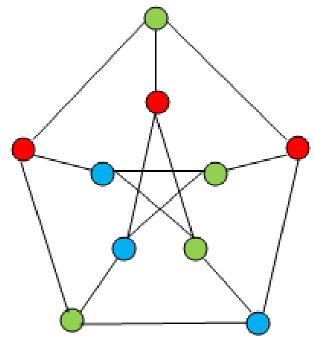
So, it will be interpreted as a way of coloring the nodes of a graph such that no two adjacent nodes share the same color. But it is properly called a *node coloring.*

If we substitute the term "node" by "edge", we have the *edge coloring.*

And if such substitution is by "face" or region, then we have described the *face coloring.*

The more typical representation of such "colors" are the first non-negative integers; so, it will be 1, 2, 3..., instead of Red, Blue, etc., only possible when we need only few labels.

A proper node coloring of the Petersen graph, using three colors, the minimum posible nr.



The finite collection of such labels, for instance, $\{1, 2, 3, ...\}$, is called the *"color set"*.

But the nature of the coloring problem not depends indeed on the nature of such labels. It only depends on the number of such "colors".

Graph coloring posseses many applications, and it is actually an active field of research.

A coloring that use, at most, k colors is called a proper *k-coloring.*

We said *Chromatic Number* of the graph G, denoted by $\chi(G)$, the smallest number of colors needed to color such graph.

If G will be a graph which admits a proper k-coloring, it is *k-colorable.*

When its chromatic number is exactly *k,* that is, $\chi(G) = k$, it will said *k-chromatic.*

A subset of nodes associated to the same color is called a *color class.*

The *chromatic polynomial* counts the number of ways a graph can be colored using no more than a given number of colors.

So, such polynomial will be a function that counts the number of k-colorings of G.

It contains at least as much information as does the chromatic number, about the colorabiity of G.

It is the smallest positive integer that is not a root of the chromatic polynomial, as

$$\chi(G) = \min\{k : P(G,\ k) > 0\}$$

For instance, the chromatic polynomial of the complete n-graph, K_n, would be

$$P(G,\ t) = t\,(t-1)\,(t-2)\,...\,(t-(n-1))$$

And for the cycle n-graph, C_n, we have

$$P(G,\ t) = (t-1)^n + (-1)^n\,(t-1)$$

5. Perfectibility

A graph is *perfect,* if for every induced subgraph, the maximum size of a clique (the largest clique of such subgraph) coincides with its chromatic number. That is, if $\chi\,(G)$ is equal to the maximum size of a complete subgraph.

We may remember that the clique number ever provides a lower bound for the chromatic. In the particular case of perfect graphs, this inequality reveals an equality.

It will be very interesting, because in many applications, some problems that are until now intractable, can be solved in the class of perfect graphs.

We made to mention some principal families of graphs that are perfect.

So, for instance,

- *interval graphs*

- *chordal graphs*

- *bipartite graphs*

-*distance-hereditary graphs*

- *permutation graphs*

- *wheel graphs*

-*comparability graphs*

and so on.

The analysis of Perfect Graphs proceeds from Claude Berge, when working on a problem of Information Theory, exactly the Shannon´s capacity of a graph.

In 1961, he proposed two famous conjectures about such graphs. The second conjecture implies the first, being so called *"strong"* and *"weak"* conjectures, respectively.

They will be expressed by

(First Conjecture) *The complement of each perfect graph is also a perfect graph.*

(Second Conjecture) *A graph is perfect* if and only if *it is a Berge graph.*

That is, it has no induced subgraph isomorphism to an odd cycle of length at least five, or the complement of such an odd cycle.

The first of such conjectures was proved by Lászlo Lovász, in 1972.

And the second, by Chudnovsky, Robertson, Seymour and Thomas, in 2005.

Recall the subsequent concepts,

The *complement* of a graph G, denoted by $\overline{G}$, has the same node set as G, and distinct nodes are adjacent in $\overline{G}$ just when they are not adjacent in G.

A *hole* of G is an induced subgraph of G which is a cycle of length $l \geq 4$.

An *antihole* of G is an induced subgraph of G whose complement is a hole in $\overline{G}$.

A graph, G, is *Berge,* if every hole and and antihole of G has even length. So, a graph that does not contain neither odd holes nor odd antiholes is called a Berge graph.

We may modulate the strictness of such features of perfection, by these subtle descriptions:

G is a *trivially perfect graph* when in every induced subgraph, the size of the largest independent set equals the number of maximal cliques.

G is a *strongly perfect graph* when every induced subgraph has an independent set intersecting all its maximal cliques.

G is a *very strongly perfect graph* when in every induced subgraph, every node belongs to an independent set meeting all maximal cliques.

6. Computational Complexity

There are very hard optimization problems, as

- independent set
- clique
- colouring
- clique cover

Departing of the idea of graph product, we have that there is a clique of size greater or equal to k in G if and only if there is a clique of size greater or equal to k^2, in GxG.

In certain graphs such hard problems may be considered of "reasonable" dificulty.

So, taking

$$\alpha(G),\ \omega(G),\ \chi(G),\ and\ \theta(G)$$

relative to independent set, clique, coloring and clique cover, and being $\overline{G}$ the complement graph of G, it holds

$$\begin{aligned}\alpha(G) &= \omega\left(\overline{G}\right)\\ \chi(G) &= \theta\left(\overline{G}\right)\\ \omega(G) &\leq \chi(G)\\ \alpha(G) &\leq \theta(G)\end{aligned}$$

It will be interesting to analyze when such precedent inequalities transforms to the subsequent equalities

$$\begin{aligned}\omega(G) &= \chi(G)\\ \alpha(G) &= \theta(G)\end{aligned}$$

It is very related with the "degrees of perfectness", because we may define that

A graph is *perfect,* if for every induced subgraph, $H \subset G$, we have

$$\omega(H) = \chi(H)$$

A graph is *co-perfect,* if for every induced subgraph, $H \subset G$, we have

$$\alpha(H) = \theta(H)$$

To the question on what graphs are perfect, the answer is the Bipartite Graphs. The same question, but now about the co-perfect, it is the same: the Bipartite Graphs. And if we want to know cases of non acomplishment of such conditions, the answer will be the odd cycles.

In 1984, Grötschel, Lovász, and Scrijver showed that the weighted versions of the stable set problem, the clique problem, the coloring problem, and the clique covering problem are solvable in polynomial time, when we consider perfect graphs.

Lovász also proved, in 1983, that the problem of recognizing Berge graphs is $co-NP$. But until the proof of the Strong Perfect Graph Theorem was known, whether or not it is also in the complexity class denoted by P. Chudnovsky et al. discover a polynomial time algorith for this purpose.

Finding the hitting set is an NP-complete problem. Also the problem of finding the size of a clique, for a given graph, is an NP-complete problem. And the chromatic number problem is one of the famous Karp (1972) relation of 21 NP-complete problems.

Note that one of the major applications of Graph Coloring resides at the register allocation, in compilers.

Conclusion

So, we will work with the support on a more powerful analytical framework, improving our theoretical basis, being coherent with the precedent results.

The problems also may be translated to coloring of graphs, intervening features such as chordality, planarity, perfectness of graphs, and so on.

My current objective of research is to made such adequated translation, until to search a new way to solve the control problem of the asymptotic behaviour, when we consider the aforementioned (in our precedent papers) ratio.

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