*AMO-Advanced Modeling and Optimization, Volume 11, Number 4, 2009* *

# An Approach to Solve Two-Person Matrix Game via Entropy

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**Abstract**

This paper presents an Entropy Optimization Model on matrix game (two-person zero-sum game). The entropy function of the matrix game has been considered to the objective function and formulate the models. Obviously the formulated models are non-linear programming problem. For each player, this type of model is established and corresponding effecting solution is shown. Effectiveness is illustrated with the help of numerical example.

**Keywords:** Matrix Game, Entropy, Cross Entropy, Mathematical Programming.

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* **AMO-Advanced Modeling and Optimization. ISSN: 1841 - 4311**

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# 1 Introduction

Entropy optimization models have been successfully applied to practical problems in many scientific and engineering disciplines. Those disciplines include Information Theory ( [12], [15] ), Statistical Mechanics [13], Thermodynamics, Statistical Parameter Estimation and Inference, Economics, Business and Finance, Non-linear Spectral Analysis, Patter Recognition [20] , Urban and Regional planning, Queueing Theory and Linear Programming. Here we discuss about Entropy Optimization model in the area of two person zero-sum matrix game.

Game theory has a remarkable importance in both Operations Research and Systems Engineering due to its great applicability. Many real conflict problems can be modelled as games. However, the encountered conflict problems in economical, military and political fields become more and more complex and uncertain due to the existence of diversified factors. This situation will bring some difficulties in application of classical game theory. To remove this difficulties, we have been employed the entropy on two person zero-sum matrix game.

Many algorithms have been proposed for solving the constraint maximum entropy or minimum cross-entropy problem. Better known algorithms include Bregman's ( [3], [7] ) method, MART (Multiplicative Algebraic Reconstruction Technique), Newton's method ( [9], [8], [1] ), the Generalized Iterative Scaling Method ( GISM ) and the interior-point methods. Fang and Tsao [10] proposed an unconstrained dual method for optimizing entropy subject to linear constraints. Here we will derive the theory in detail and the K.K.T. conditions have played important role in developing solution for this model and provide numerical results without the help of above methodologies.

# 2 Mathematical Model

In a matrix game, assume that $s_1 = \{\alpha_1, \alpha_2, \dots \alpha_m\}$ and $s_2 = \{\beta_1, \beta_2, \dots \beta_n\}$ be the set of pure strategies for player I and player II, respectively. A payoff matrix of the player I and II are defined as follows:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \tag{1}$$

Let the mixed strategies for player I and II are

$$y = \{y_1, y_2, \dots y_m\}^T \tag{2}$$

and

$$z = \{z_1, z_2, \dots z_n\}^T \tag{3}$$

Then from our classical game theory, we can determine an optimal strategy $y$ of player I which is the solution of the following linear programming model.

**Model 1**

$$\min: \quad \frac{1}{v} = \sum_{i=1}^{m} Y_i \tag{4}$$

subject to

$$\sum_{i=1}^{m} a_{ij} Y_i \geq 1, \quad j = 1, 2, \dots, n \tag{5}$$

$$Y_i = \frac{y_i}{v} \geq 0, \quad i = 1, 2, \dots, m \tag{6}$$

Similarly, an optimal strategy $z$ of the player II is the solution of the following linear programming model.

**Model 2**

$$\max: \quad \frac{1}{w} = \sum_{j=1}^{n} Z_j \tag{7}$$

subject to

$$\sum_{i=1}^{n} a_{ij} Z_j \le 1, \quad i = 1, 2, \ldots, m \tag{8}$$

$$Z_i = \frac{z_i}{w} \ge 0, \quad j = 1, 2, \ldots, n \tag{9}$$

By the duality theorem of the linear programming, the maximum value of $v$ will be equal to the minimum value of $w$. This value represents the value of the matrix game. Again each player is interested in making moves which will be as surprising and as uncertain to the other player as possible. For this reason, the two players are involved in maximizing their entropies or involved in minimizing their cross-entropies. The mathematical form of entropies are as follows:

$$H_1 = -\sum_{i=1}^{m} Y_i \; ln(Y_i) \tag{10}$$

$$H_2 = -\sum_{j=1}^{n} Z_j \; ln(Z_j) \tag{11}$$

And the mathematical form of cross-entropies are as follows:

$$H_1^c = \sum_{i=1}^{m} Y_i \; ln(\frac{Y_i}{p_i^0}) \tag{12}$$

$$H_2^c = \sum_{j=1}^{n} Z_j \; ln(\frac{Z_j}{p_j^0}) \tag{13}$$

where $\{p_i^0, i = 1, 2, \ldots, m$ and $\{p_j^0, j = 1, 2, \ldots, n$ are the given priori distribution and, in the absence of it, they are follows uniform probability distribution.

## 2.1 Entropy Optimization Models

Let us first established the entropy optimization model for maximization type by considering following principle.
‘ Out of all possible distributions that are consistent with moment constraint, choose one that has the maximum entropy’. This principle was proposed by Janes [?] and has been known as Principle of Maximum Entropy or Janes’ Maximum Entropy principle. From this point of view, we formulated a new mathematical model namely Entropy Optimization Model on two person zero-sum matrix game in which the entropy function of the matrix game has been considered to the objective function. The entropy optimization model for player II is represented as follows.
**Model 3**

$$\max: \quad H_2 = -\sum_{j=1}^{n} Z_j \; ln(Z_j) \tag{14}$$

subject to

$$\sum_{i=1}^{n} a_{ij} Z_j \leq 1, \quad i = 1, 2, \ldots, m \tag{15}$$

$$Z_i = \frac{z_i}{w} \geq 0, \quad j = 1, 2, \ldots, n \tag{16}$$

Similarly we established the entropy optimization model for the minimization type by considering following principle.
‘Out of all probability distributions satisfying the given moment constraints, choose the distribution that minimizes the cross-entropy with respect to the given priori distribution and, in the absence of it, choose the distribution that minimizes the cross-entropy with respect to the uniform distribution’. This principle was proposed by Kullback-Leibler [15] and has been known as Principle of Minimum Cross Entropy or Kullback-Leibler’s

Minimum Cross Entropy principle. From this point of view, we formulated a new mathematical model namely Entropy Optimization Model on two person zero-sum game in which the entropy function of the matrix game has been considered to the objective function. The Entropy Optimization Model for player I is represented as follows.

**Model 4**

$$\min: \quad H_1^c = \sum_{i=1}^{m} Y_i \; ln(\frac{Y_i}{p_i^0}) \tag{17}$$

subject to

$$\sum_{i=1}^{m} a_{ij} Y_i \geq 1, \quad j = 1, 2, \ldots, n \tag{18}$$

$$Y_i = \frac{y_i}{v} \geq 0, \quad i = 1, 2, \ldots, m \tag{19}$$

where $\{p_i^0, i = 1, 2, \ldots, m$ is the given priori distribution and, in the absence of it, it is uniform probability distribution.

# 3 Solution Procedure

**Model 3** is non-linear programming model with concave objective function and convex feasible region so the the K.K.T. conditions must gives optimum solution, which is maximum. For this purpose we construct the Lagrangian of this model is as follows.

$$L(Z, R, S) = -\sum_{j=1}^{n} Z_j \; ln(Z_j) - \sum_{i=1}^{m} r_i (\sum_{j=1}^{n} a_{ij} Z_j - 1 + s_i^2) \tag{20}$$

where $R = (r_1, r_2, \ldots, r_m)$ is Lagrange multipliers and $S = (s_1, s_2, \ldots, s_m)$associated with constraints (15). The optimality condition gives

$$Z_j = \exp(-1 - \sum_{i=1}^{m} a_{ij} r_i), \quad j = 1, 2, \ldots, n \tag{21}$$

$$\sum_{i=1}^{n} a_{ij} Z_j \leq 1, \quad i = 1, 2, \ldots, m \tag{22}$$

$$r_i(\sum_{i=1}^{n} a_{ij} Z_j - 1) = 0, \quad i = 1, 2, \ldots, m \tag{23}$$

$$r_i \geq 0, \quad i = 1, 2, \ldots, m \tag{24}$$

$$Z_j \geq 0, \quad j = 1, 2, \ldots, n \tag{25}$$

Hence the objective function (14) is maximum for those values of $Z$ and $R$ which satisfy the relation (21) - (25). After that we determine the value of the game namely $v$ and the optimal strategy $z$ for player II. But this relations (21)-(25) give implicit relation so it is not easy to determine $Z$. There are various methods available to determine the solution but to solve equation (21)-(25)we use Lingo package to obtain the solution by considering the following model.

**Model 5**

$$\max: \quad L(Z, R, S) = -\sum_{j=1}^{n} Z_j \; ln(Z_j) - \sum_{i=1}^{m} r_i(\sum_{j=1}^{n} a_{ij} Z_j - 1 + s_i^2) \tag{26}$$

subject to

$$Z_j = \exp(-1 - \sum_{i=1}^{m} a_{ij} r_i), \quad j = 1, 2, \ldots, n \tag{27}$$

$$\sum_{i=1}^{n} a_{ij} Z_j \leq 1, \quad i = 1, 2, \ldots, m \tag{28}$$

$$r_i(\sum_{i=1}^{n} a_{ij} Z_j - 1) = 0, \quad i = 1, 2, \ldots, m \tag{29}$$

$$r_i \geq 0, \quad i = 1, 2, \ldots, m \tag{30}$$

$$Z_j \geq 0, \quad j = 1, 2, \ldots, n \tag{31}$$

After calculating $Z_j$ from **Model 5** say $Z_j^*$ we determine the value of game $v^*$ and optimal strategy $z^*$ for player II by the following relation.

$$\frac{1}{v^*} = \sum_{j=1}^{n} Z_j^* \tag{32}$$

$$z_j^* = v^* Z_j^*, \quad j = 1, 2, \ldots, n \tag{33}$$

Similarly to solve **Model 4** it is clear that, the **Model 4** is non-linear programming model with convex objective function and convex feasible region so the the K.K.T. conditions must gives optimum solution, which is minimum. For this purpose we construct the Lagrangian of this model is as follows.

$$L1(Y, T, U) = \sum_{i=1}^{m} Y_i \; ln(\frac{Y_i}{p_i^0}) - \sum_{j=1}^{n} t_j(\sum_{i=1}^{m} a_{ij} Y_i - 1 + u_j^2) \tag{34}$$

where $T = (t_1, t_2, \ldots, t_n)$ is Lagrange multipliers and $U = (u_1, u_2, \ldots, u_n)$associated with constraints (18). The optimality condition gives

$$Y_i = p_i^0 \exp(\sum_{j=1}^{n} a_{ij} t_j - 1), \quad i = 1, 2, \ldots, m \tag{35}$$

$$\sum_{i=1}^{m} a_{ij} Y_i \geq 1, \quad j = 1, 2, \ldots, n \tag{36}$$

$$t_j(\sum_{i=1}^{m} a_{ij} Y_i - 1) = 0, \quad j = 1, 2, \ldots, n \tag{37}$$

$$t_j \geq 0, \quad j = 1, 2, \ldots, n \tag{38}$$

$$Y_i \geq 0, \quad i = 1, 2, \ldots, m \tag{39}$$

Hence the objective function (17) is minimum for those values of $Y$ and $T$ which satisfy the relation (35) - (39). After that we determine the value of the game namely $w$ and the optimal strategy $y$ for player I. But this relations (35)-(39) give implicit relation so it is not easy to determine $Y$. There are various methods available to determine the solution but to solve equation (35)-(39)we use Lingo package to obtain the solution by considering the following model.

**Model 6**

$$\min: \quad L1(Y,T,U) = \sum_{i=1}^{m} Y_i \; ln(\frac{Y_i}{p_i^0}) - \sum_{j=1}^{n} t_j (\sum_{i=1}^{m} a_{ij} Y_i - 1 + u_j^2) \tag{40}$$

subject to

$$Y_i = p_i^0 \exp(\sum_{j=1}^{n} a_{ij} t_j - 1), \quad i = 1, 2, \ldots, m \tag{41}$$

$$\sum_{i=1}^{m} a_{ij} Y_i \geq 1, \quad j = 1, 2, \ldots, n \tag{42}$$

$$t_j (\sum_{i=1}^{m} a_{ij} Y_i - 1) = 0, \quad j = 1, 2, \ldots, n \tag{43}$$

$$t_j \geq 0, \quad j = 1, 2, \ldots, n \tag{44}$$

$$Y_i \geq 0, \quad i = 1, 2, \ldots, m \tag{45}$$

After calculating $Y_i$ from **Model 6** say $Y_i^*$ we determine the value of game $w^*$ and optimal strategy $y^*$ for player I by the following relation.

$$\frac{1}{w^*} = \sum_{i=1}^{m} Y_i^* \tag{46}$$

$$y_i^* = w^* Y_i^*, \quad i = 1, 2, \ldots, m \tag{47}$$

# 4 Numerical Example

Let us consider a matrix game as follows:

$$A = \begin{bmatrix} 9 & 1 & 4 \\ 0 & 6 & 3 \\ 5 & 2 & 8 \end{bmatrix} \tag{48}$$

Then from **Model 5** we get the following non-linear model.

**Model 7**

$$\max: \quad L(Z,r,S) = -\sum_{j=1}^{3} Z_j \ ln(Z_j) - r_1(9Z_1 + 1Z_2 + 4Z_3 + s_1^2 - 1)$$
$$-r_2(0Z_1 + 6Z_2 + 3Z_3 + s_2^2 - 1) - r_3(5Z_1 + 2Z_2 + 8Z_3 + s_3^2 - 1) \tag{49}$$

subject to

$$Z_1 = \exp(-1 - 9r_1 - 5r_3) \tag{50}$$
$$Z_2 = \exp(-1 - r_1 - 6r_2 - 2r_3) \tag{51}$$
$$Z_3 = \exp(-1 - 4r_1 - 3r_2 - 8r_3) \tag{52}$$
$$9Z_1 + 1Z_2 + 4Z_3 - 1 \leq 0 \tag{53}$$
$$0Z_1 + 6Z_2 + 3Z_3 - 1 \leq 0 \tag{54}$$
$$5Z_1 + 2Z_2 + 8Z_3 - 1 \leq 0 \tag{55}$$
$$r_1(9Z_1 + 1Z_2 + 4Z_3 - 1) = 0 \tag{56}$$
$$r_2(0Z_1 + 6Z_2 + 3Z_3 - 1) = 0 \tag{57}$$
$$r_3(5Z_1 + 2Z_2 + 8Z_3 - 1) = 0 \tag{58}$$
$$r_i \geq 0, \quad i = 1, 2, \ldots, 3 \tag{59}$$
$$Z_j \geq 0, \quad j = 1, 2, \ldots, 3 \tag{60}$$

Equation (32) and (33) becomes the following relation.

$$\frac{1}{v^*} = Z_1 + Z_2 + Z_3 \tag{61}$$
$$z_1^* = v^* Z_1^* \tag{62}$$
$$z_2^* = v^* Z_2^* \tag{63}$$
$$z_3^* = v^* Z_3^* \tag{64}$$

And from **Model 6** we get the following non-linear model.

**Model 8**

$$\min: \quad L1(Y,T,U) = \sum_{i=1}^{m} Y_i \ ln(\frac{Y_i}{p_i^0}) - t_1(9Y_1 + 0Y_2 + 5Y_3 + u_1^2 - 1)$$
$$-t_2(1Y_1 + 6Y_2 + 2Y_3 + u_2^2 - 1) - t_3(4Y_1 + 3Y_2 + 8Y_3 + u_3^2 - 1) \tag{65}$$

subject to

$$Y_1 = p_1^0 \exp(-1 + 9t_1 + t_2 + 4t_3) \tag{66}$$
$$Y_2 = p_2^0 \exp(-1 + 6t_2 + 3t_3) \tag{67}$$
$$Y_3 = p_3^0 \exp(-1 + 5t_1 + 2tY_2 + 8t_3) \tag{68}$$
$$9Y_1 + 0Y_2 + 5Y_3 - 1 \geq 0 \tag{69}$$
$$1Y_1 + 6Y_2 + 2Y_3 - 1 \geq 0 \tag{70}$$
$$4Y_1 + 3Y_2 + 8Y_3 - 1 \geq 0 \tag{71}$$
$$u_1(9Y_1 + 0Y_2 + 5Y_3 - 1) = 0 \tag{72}$$
$$u_2(1Y_1 + 6Y_2 + 2Y_3 - 1) = 0 \tag{73}$$
$$u_3(4Y_1 + 3Y_2 + 8Y_3 - 1) = 0 \tag{74}$$
$$t_j \geq 0, \quad j = 1, 2, \ldots, 3 \tag{75}$$
$$Y_i \geq 0, \quad i = 1, 2, \ldots, 3 \tag{76}$$

Equation (32) and (33) becomes the following relation.

$$\frac{1}{w^*} = Z_1 + Z_2 + Z_3 \tag{77}$$
$$y_1^* = w^* Y_1^* \tag{78}$$
$$y_2^* = w^* Y_2^* \tag{79}$$
$$y_3^* = w^* Y_3^* \tag{80}$$

## 4.1 Result

Since the the non-linear **Model 7** and **Model 8** are not easy to solve by any linear programming technique. So using Lingo 9.0 package the optimal solution for player II and player I can be represented in the following Table-1.

| *value of game* | *optimal strategy* |
| --- | --- |
| $v^* = 3.791667$ | $z^* = (0.2916667, 0.5555556, 0.1527778)$ |
| $w^* = 3.791667$ | $y^* = (0.375, 0.5416666, 0.083333333)$ |

Table - 1

### 4.2 Comparision

We solve **Model 1** and **Model 2** by linear programming method, we get the results in the following Table-2.

| *value of game* | *optimal strategy* |
| --- | --- |
| $v^* = 3.791667$ | $z^* = (0.2916667, 0.5555556, 0.1527778)$ |
| $w^* = 3.791667$ | $y^* = (0.375, 0.5416666, 0.083333333)$ |

Table - 2

## 5 Conclusions

The application of entropy optimization for two-person zero-sum matrix game problem has a vibrant research area. The entropy optimization model is non-linear model whose objective function is also an entropy function. In this model the objective function is concave function and feasible region is convex function, so the K.K.T. optimality conditions assure us that both the models must have optimal solutions. Here, it is obvious that the entropy optimization model on matrix game gives identical results with classical game model which are highly significant.

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