New analyses of fuzzy entropies

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Abstract

Our paper analyzes here, and now in a more detailed way, some new lines to introduce the evolving concept of a very important Uncertainty Measure, the so-called Entropy. We need to obtain these new ways to model adequate conditions, departing from vague pieces of information. For this, it will be very necessary to classify these different type of measure, with very profound theoretical implications, and many interesting applications, as for instance, in Economics.

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1. Some initial quotations about Entropy

We must initiate this analysis by a perhaps well-known historical quotations about the idea of Entropy. At least, from my precedent paper in this same journal. But they are very appropiate, in my personal opinion.

According to the story Tribus tells us, Shannon didn't know what to call his measure, so he asked Von Neumann,

"- My greatest concern was what to call it. I thought of calling it 'information', but the word was overly used, so I decided to call it 'uncertainty'. When I discussed it with John, he had a better idea.

John told me:

'- You should call it *entropy,* for two reasons. In the first place, your uncertainty function has been used in Statistical Mechanics under that name, so it already has a name. In the second place, and more important, nobody knows what entropy really is, so in a debate you will always have the advantage".

And a final quotation. "The fundamental problem of communication is that of reproducing at one point, either exactly or approximately, a message selected at another point." It is said by *Claude E. Shannon* (1916-2001), in his "A Mathematical Theory of Communication", the now very famous paper of 1948.

Because the basic problem is to reconstruct, as closely as possible, the input signal after observing the received signal at the output.

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The development of the idea of entropy of random variables and processes by Shannon, provided the foundations of *IT (Information Theory),* and also of *Ergodic Theory.* Recall that it is a branch of Mathematics that studies Dynamical Systems with an invariant measure.

Entropy and related information measures provide descriptions of the long term behavior of random processes, and that this behavior is a key factor in developing aspects as the *Coding Theorems of Information Theory.*

About that the Shannon information measure is the only possible one, it must be clear that it will be only valid within the more restricted scope of coding problems which the own C. E. Shannon had see in his time.

As pointed out by *Alfred Rényi* (1961), in his essential paper on generalized information measures, in other sort of problems other quantities may serve just as well, or even better, as measures of information.

This should be supported either by their operational significance or by a set of natural postulates characterizing them, or, preferably, by both.

Thus, the idea of generalized entropies arises in the scientific literature.

2. The maximum entropy principle

The maximum entropy principle (expressed as MEP, in acronym) is a postulate about a universal feature of any probability assignment on a given set of propositions: events, hypotheses, etc.

Let some testable information about a probability distribution function be given. Consider the set of all trial probability distributions that encode this information.

Then, *the probability distribution that maximizes the information entropy is the true probability distribution* with respect to the testable information prescribed.

This principle was first expounded by E. T. Jaynes in his seminal papers of 1957, where he emphasized some natural correspondence between Statistical Mechanics and Information Theory.

In particular, E. T. Jaynes offered a new and more general rationale why the known as "Gibbsian" method of Statistical Mechanics works.

He suggested that the Entropy (denoted by S) in Statistical Mechanics, and H (Information Entropy, in Information Theory), are basically the same thing.

Consequently, Statistical mechanics should be seen just as a particular application of a general tool of Logical Inference and Information Theory.

Given testable information, the maximum entropy procedure consists of seeking the probability distribution which maximizes the information entropy, subject to the constraints of the information.

This constrained optimization problem will be typically solved using the analytical method known as Lagrange Multipliers.

Entropy maximization (max H) with no testable information takes place under a single constraint: the sum of the probabilities must be one. Under this constraint, the maximum entropy probability distribution is the well-known U (uniform distribution).

The *MEP* can, thus, be considered as a generalization of the classical *Principle of Indifference* (due to Laplace), also known as the *Principle of Insufficient Reason.*

And it is very interesting the way open by the Romanian Mathematician *Nicolae Georgescu-Roegen,* suggesting the application of the 2nd Law of Thermodynamics to Economics. This and subsequent developments give rise to essential fields as Bioeconomics, Ecological Entropy. Also to the essential study, in Economics, of Equilibrium Theory, and so on.

We may also consider the *Metric Entropy,* also called *Kolmogorov Entropy,* or *Kolmogorov-Sinai Entropy,* in acronym *K-S Entropy.*

In a dynamical system, the metric entropy is equal to zero, for nonchaotic motion, and is strictly greater than zero, for chaotic motion.

In Thermodynamics, Prigogine entropy is a very often used term referring to the splitting of Entropy into two variables, one being that which is "exchanged" with the surroundings, and the other being a result of "internal“ processes.

It holds

$$dS = d_e S + d_i S$$

This expression is sometimes referred to as the *Prigogine entropy equation.* Such equation was formulated by Ilya Prigogine in his *Study of the Thermodynamics of Irreversible Phenomenon* (1945).

This new function results, according to Prigogine, because

a) The Entropy of a System is an Extensive Property, i. e., if the system consists of several parts, the total entropy is equal to the sum of the entropies of each part,

and

(b) The change in entropy can be split into two parts, being these $d_e S$ and $d_i S$.

Denoting as $d_e S$ the flow of entropy, due to interactions with the exterior, and denoting as $d_i S$ the contributions due to changes inside the system.

3. Topological Entropy

Let (X, d) be a compact metric space, and let *f:* $X \to X$ be a continuous map.

For each $n > 0$, we define a new metric, d_n, by

$$d_n(x, y) = max\{d(f^i(x), f^i(y)) : 0 \leq i < n\}$$

Two points, x and y, are *close* with respect to (w. r. t.) this metric, if their first n iterates (given by f^i, $i=1,2,\ldots$) are close.

For $\epsilon > 0$, and n$\in$ N*, we say that S $\subset$ X is an *(n, ϵ) - separated set,* if for each pair, *x, y,* of points of *S,* we have

$$d_n(x, y) > \epsilon$$

Denote by $N(n, \epsilon)$ the *maximum cardinality of a (n, ϵ)-separated set.*

It must be finite, because X is compact. In general, this limit may exists, but it could be infinite.

A possible interpretation of this number is as a measure the average exponential growth of the number of distinguishable orbit segments

So, we could say that

the higher the topological entropy is, the more essentially different orbits we have

Topological entropy was introduced, in 1965, by Adler, Konheim and McAndrew.

Hence, $N(n, \epsilon)$ shows the number of "distinguishable" orbit segments of length *n,* assuming we cannot distinguish points that are less than ϵ apart.

The *topological entropy of f* is then defined by

$$H_{top} = \lim_{\epsilon \to 0} \lim\ \sup_{n\to\infty} \left[\frac{1}{n} \log N\left(n, \epsilon\right)\right]$$

4. Entropy on Intuitionistic Fuzzy Sets

The notion of *Intuitionistic Fuzzy Set* (in acronym, *IFS*) was introduced by Atanassov (1983), and then developed by authors as Hung & Yang, among others.

Recall that an Intuitionistic Set is an incompletely known set.

An IFS must represent the degrees of their membership and non-membership, with a certain degree of *hesitancy,* or *doubt.*

For this reason, they have widely used in many applied areas.

Therefore, the apparition here of IFS, instead of FS, permits the introduction of another degree of freedom into a set description.

Such a generalization of FS gives us a new possibility to represent imperfect knowledge. Thus, we can access to describing many more real problems in a more adequate way.

IFS-based models may be very useful in situations with distinct opinions, or human testimonies, involving answers as

- Yes,
- No,
- Doesn´t apply.

As for instance, in voting.

A very frequent measure of fuzziness is the entropy of Fuzzy Sets, which were first mentioned by Zadeh (1965). But recently two new definitions have been proposed

by Szmidt & Kacprzyk (2001),

and

by Burillo & Bustince (1996).

The first one is a non-probabilistic entropy measure, which departs on a geometric interpretation of an IFS.

And by the second, it would be possible to measure the *degree of intuitionism of an IFS.*

Also we can to generalize from IFS to a Neutrosophic Set, abridgedly *N-Set,* a concept due to Smarandache (1995).

It is a set which generalizes many existing classes of sets. In particular, FS, and its first generalization, IFS.

Let U be a universe of discourse, and let M be a set included in U.

An element x from U is denoted, with respect to the set M, as

$$x\ (T,\ I,\ F)$$

And it belongs to M in the following way:

It is T % in the set (membership degree),

I % indeterminate (unknown if it is in the set),

and F % not in the set (non-membership degree).

Here *T, I, F* components are real standard/non-standard subsets, included in the non-standard unit interval, representing truth, indeterminacy and falsity percentages, repectively.

It is possible to define a measure of Neutrosophic H, or *N-Entropy,* as the summation of the respective entropies of three subsets, T, I and F.

5. Negentropy

The *Second Law of Thermodynamics* makes no distinction between living and non-living things. But living things are characterized by a very high degree of asembly and structure.

Every isolated system (as may be the Universe) moves towards a state of maximum entropy. Its entropy can never decrease.

Hence, the decrease in entropy that accompanies growth of living structures must be accompanied by an increase in entropy in the physical environment.

In his famous book *What is life?,* Erwin Schrödinger analyse the life as a state of very low probability, because the necessary energy to create and sustain it.

The “vital force” that maintains life is energy. Living things preserve their low levels of entropy throughout time, because they receive energy from the surroundings in the form of food. They gain its order at the expense of disordering the nutrients they consume.

The entropy of a system represent the amount of uncertainty one observer has about the state of the system.

The simplest example of a system will be a random variable.

Information measures the amount of correlation between two systems, and it reduces to a mere difference in entropies.

As islands of order in a sea of chaos, organisms are far superior to human-built machines. And unlike Watt´s steam engine, the body concentrates order. It continuously self-repairs, being the metabolism a sure sign of life.

Brillouin says that a living system imports negentropy and stores it.

So, the J (negentropy) function is the entropy that it exports to keep its own entropy low.

The Negentropy is used as a measure of distance to Normality.

It is always non-negative: J > 0.

And linear by any linear invertible change of coordinates.

It vanishes iff the signal is Gaussian.

Some biologists speaks in terms of the entropy of an organism, or about its antonym, negentropy, as a measure of the structural order within such organism. Being entropy defined as a measure of how close a system is to equilibrium, i. e., to perfect internal disorder.

Living things and ordered structures, as crystals for instance, appears because locally entropy can be lowered by external action. It may be reduced in different systems, as the cold chamber of a refrigator, being in such case possible by an increase of S in their surroundings.

6. Graph Entropy

It is a functional on a graph, $G = (V, E)$, with P a probability distribution on its node (or vertex) set, V. It will be denoted by *GE.*

A concept introduced by Körner, as solution of a coding problem formulated in I T.

Because its sub-additivity, has become a useful tool in proving some lower bounds results in Computational Complexity Theory.

The search for exact additivity has produced certain interesting combinatorial structures. One of such results is the characterization of perfect graphs by the additivity of GE.

It will be defined as

$$H\ (G,\ P) = min \sum p_i\ \log\ p_i$$

Observe that such function is convex. It tends to ∞ on the boundary of the non-negative orthant of R^n , and monotonically to $-\infty$ along rays from the origin.

So, such minimum is always achieved and it will be finite.

Final conclusions

Statistical entropy is a probabilistic measure of uncertainty, or ignorance about data.

Whereas *Information* is a measure of a reduction in that uncertainty.

The Entropy of a probability distribution is just the expected value of the information of such distribution.

All these ideas induces improved tools to advance in Optimization Theory, Generalized Fuzzy Measures, Economics, modeling in Biology, and so on.

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