STABLE OPTIMAL MODEL REDUCTION OF LINEAR DISCRETE TIME SYSTEMS VIA INTEGRAL SQUARED ERROR MINIMIZATION : COMPUTER-AIDED APPROACH

S. K. Mittal[1)], Dinesh Chandra[2)]
[1)] S. K. Mittal, Research Scholar,
U. P. Technical University, Lucknow, India
(e-mail: mittalsai@rediffmail.com).
Postal Address: 83, Sharda Nagar,
Near Vidyanagari Post Office, Deopur,
Dhule- 424004, Maharashtra, India
Phone No. : +919422788382, +9102562271943
[2)] Dinesh Chandra, Department of Electrical Engineering,
Motilal Nehru National Institute of Technology,
Allahabad 211004, India (e-mail: dinuchandra@rediffmail.com).

Abstract: A computer-aided technique to obtain a reduced-order approximant of a given (stable) single-input single-output discrete time system based on the minimization of a integral squared error (ISE) pertaining to a unit step input is presented. Both the numerator and denominator coefficients of the model are treated as free parameters in the process of optimization. The method has a built-in
stability-preserving feature.

Keywords: Model reduction,Padé approximation, Routh criterion, Pareto optimal soulution,VEGA.

1 Introduction

The techniques for model reduction of discrete-time systems are limited and may be classified into two groups. The first group contains the methods [1-12] which derive approximant $G_r(z)$ from a given high-order transfer function $G(z)$ exploiting the already existing continuous-time algorithms [1-5]. Some of the methods of this group are very attractive, because a stable reduced model is obtained if the original system is stable [5-12]. The second group contains the so called direct methods that derive $G_r(z)$ directly from $G(z)$ without using the transformation but they do not usually ensure stability of the reduced model even though the original system is stable [13-15]. Farasi et al. [16] have proposed a method in which the Routh stability criterion is employed to reduce the order of discrete system transfer functions. It is shown that the Routh approximation is well suited to reduce both denominator and the numerator polynomials, although alternative methods such as Padé approximation can also be used to fit the model numerator coefficients. In [17], a Routh type approximation for discrete system is presented. The denominator of the reduced model is directly obtained from a Routh type table and the numerator of the reduced model is obtained either by matching the discrete

[1] **AMO – Advanced Modeling and Optimization. ISSN : 1841-4311**

time moments or by minimizing the step response error in z domain. A common feature in above methods [1-17] is that the values of the denominator coefficients of the reduced-order transfer function are chosen arbitrarily. This feature appears to be largely motivated by the consideration of achieving computational simplicity. If the denominator coefficients are numerically specified, the resulting equations for optimization become a set of linear equations in terms of the numerator coefficients and then these coefficients are easily determined. However, selecting the denominator coefficients arbitrarily may generally mean a loss of a considerable degree of freedom for optimization. It is sometimes suggested [18] that the denominator coefficients may be chosen so as to retain the dominant poles of the system. However, it may not usually be straight forward to say that the poles of the optimal reduced-order model may have some definite relationship with those of the original system.

Thus, the problem is to derive, subject to preserving stability, a model via the minimization of a objective function (such as ISE) while allowing both the numerator and the denominator coefficients as free parameters in the optimization. An algorithm is presented by Puri and Lin [19]. The proposed algorithm [19] minimizes a weighted mean squared impulse (or step) response error between the original system and reduced-order model. The procedure guarantees a stable model and can be extended to MIMO systems.

An alternative method is presented in this paper. In the proposed work, a stable reduced-order approximant is derived for a given (stable) single-input single-output (SISO) system via minimization of ISE pertaining to a unit-step input. In this approach, which allows both the numerator and denominator coefficients of the model as free parameters in the process of optimization, the problem of construction of objective function is circumvented by tacitly introducing, using an early idea due to Astrom [20], a set of equality constraints. By way of utilizing the ideas due to Astrom [20], the approach has the built-in stability-preserving feature for any value of r. The minimization of ISE is carried out by applying the algorithm due to Luss and Jaakola [21]. Two examples are included that bring out the systematic nature of the algorithm.

In this context, it is worth mentioning that a similar attempt is made in [22] for continuous-time SISO systems.

2 Background

Consider a higher order stable system

$$G(z) = \frac{Y(z)}{R(z)} = \frac{a_1 z^{n-1} + a_2 z^{n-2} + \dots + a_{n-1} z + a_n}{z^n + b_1 z^{n-1} + \dots + b_{n-1} z + b_n} \tag{1}$$

The problem is to determine its stable reduced-order (r th-order) approximant of the form

$$G_r(z) = \frac{Y_r(z)}{R_r(z)} = \frac{\hat{a}_1 z^{r-1} + \hat{a}_2 z^{r-2} + \dots + \hat{a}_{r-1} z + \hat{a}_r}{z^r + \hat{b}_1 z^{r-1} + \dots + \hat{b}_{r-1} z + \hat{b}_r} \tag{2}$$

such that the ISE of unit-step response given by

$$J = \sum_{k=0}^{\infty} e^2(k) \qquad (3)$$

is minimum where

$$e(k) = y_r(k) - y(k) \qquad (4)$$

$y(k)$ and $y_r(k)$ denote the unit-step responses of the system and model, respectively. Using Parseval's theorem the ISE can alternatively be expressed in the z domain [23-24] as

$$J = \frac{1}{2\pi j} \oint_{\substack{unit \\ circle}} E(z)E(z^{-1})z^{-1}dz, \qquad j = \sqrt{-1} \qquad (5)$$

where $E(z)$ is the $z-$transform of $e(k)$. It is assumed that the steady-state error is zero, i.e. $y(\infty) = y_r(\infty)$. This assumption requires that

$$\alpha_0 = \beta_0$$

where $\alpha_0 = (z-1)\left.\frac{Y(z)}{z}\right|_{z=1}$ and $\beta_0 = (z-1)\left.\frac{Y_r(z)}{z}\right|_{z=1}$

$Y(z)$ and $Y_r(z)$ denote the unit-step responses of the system and model, respectively in z domain.

Alternatively, α_0 and β_0 can be calculated as follows
Putting $z = p+1$ in polynomial (1) and expanding about $p = 0$, (1) becomes

$$G(p) = \frac{a_1(p+1)^{n-1} + a_2(p+1)^{n-2} + \ldots + a_{n-1}(p+1) + a_n}{(p+1)^n + b_1(p+1)^{n-1} + \ldots + b_{n-1}(p+1) + b_n} \qquad (6)$$

$$= \frac{A_1 p^{n-1} + A_2 p^{n-2} + \ldots + A_{n-1}p + A_n}{p^n + B_1 p^{n-1} + \ldots + B_{n-1}p + B_n}$$

$$= \alpha_0 + \alpha_1 p + \alpha_2 p^2 + \ldots$$

$$= \alpha_0 + \alpha_1(z-1) + \alpha_2(z-1)^2 + \ldots \qquad (7)$$

where α_0 is given by

$$\alpha_0 = \frac{A_n}{B_n} \qquad (8)$$

Again on putting $z = p+1$ in (2) and expanding about $p = 0$, (2) becomes

$$G_r(p) = \frac{\hat{a}_1(p+1)^{r-1} + \hat{a}_2(p+1)^{r-2} + \ldots + \hat{a}_{r-1}(p+1) + \hat{a}_r}{(p+1)^r + \hat{b}_1(p+1)^{r-1} + \ldots + \hat{b}_{r-1}(p+1) + \hat{b}_r} \tag{9}$$

$$= \frac{\hat{A}_1 p^{r-1} + \hat{A}_2 p^{r-2} + \ldots + \hat{A}_{r-1} p + \hat{A}_r}{p^r + \hat{B}_1 p^{r-1} + \ldots + \hat{B}_{r-1} p + \hat{B}_r}$$

$$= \beta_0 + \beta_1 p + \beta_2 p^2 + \ldots$$

$$= \beta_0 + \beta_1 (z-1) + \beta_2 (z-1)^2 + \ldots \tag{10}$$

where β_0 is given by

$$\beta_0 = \frac{\hat{A}_r}{\hat{B}_r} \tag{11}$$

For steady state matching

$$\alpha_0 = \beta_0$$

i.e.

$$\frac{A_n}{B_n} = \frac{\hat{A}_r}{\hat{B}_r} \tag{12}$$

Let $N(z)$ and $D(z)$ denote, respectively, the numerator and denominator polynomials of (1) and $N_r(z)$ and $D_r(z)$ the respective polynomials of (2). The error function $E(z)$ takes the form [19]

$$E(z) = \left(\frac{z}{z-1}\right)\left[\frac{D(z)N_r(z) - D_r(z)N(z)}{D_r(z)D(z)}\right]$$

$$E(z) = \frac{g_0 z^p + g_1 z^{p-1} + \ldots + g_{p-1} z + g_p}{h_0 z^p + h_1 z^{p-1} + \ldots + h_{p-1} z + h_p} \tag{13}$$

where $p = n + r$. The coefficients g_i are given by

$$\left.\begin{array}{l} g_0 = 0 \\ g_1 = c_1 \\ g_2 = c_2 \\ . \\ . \\ . \\ g_{p-1} = c_{p-1} \\ g_p = 0 \end{array}\right\} \quad (14)$$

and c_i and h_i are given by

$$\left.\begin{array}{l} c_1 = \hat{a}_1 - a_1 \\ c_2 = \hat{a}_1 b_1 + \hat{a}_2 - a_1 \hat{b}_1 - a_2 + c_1 \\ . \\ . \\ . \\ c_{p-2} = \hat{a}_r b_{n-2} + \hat{a}_{r-1} b_{n-1} + \hat{a}_{r-2} b_n - a_n \hat{b}_{r-2} - a_{n-1} \hat{b}_{r-1} - a_{n-2} \hat{b}_r + c_{p-3} \\ c_{p-1} = \hat{a}_r b_{n-1} + \hat{a}_{r-1} b_n - a_n \hat{b}_{r-1} - a_{n-1} \hat{b}_r + c_{p-2} \end{array}\right\} \quad (15)$$

$$\left.\begin{array}{l} h_0 = 1 \\ h_1 = \hat{b}_1 + b_1 \\ h_3 = \hat{b}_2 + \hat{b}_1 b_1 + b_2 \\ . \\ . \\ . \\ h_{p-1} = \hat{b}_r b_{n-1} + \hat{b}_{r-1} b_n \\ h_p = \hat{b}_r b_n \end{array}\right\}$$

(16)

Now integral (5) will always exist if all the poles of polynomial $E(z)$ are inside the unit circle. This assumption requires that

$$h_0, h_0^{p-1}, h_0^{p-2}, \ldots, h_0^1, h_0^0 \;\; >0, \tag{17}$$

where $h_0, h_0^{p-1}, h_0^{p-2}, \ldots, h_0^1, h_0^0$ are determined by following tables [20]

H-table

$$\begin{array}{ccccccc}
h_0 & h_1 & . & . & . & h_{p-1} & h_p \\
h_p & h_{p-1} & . & . & . & h_1 & h_0 \\
h_0^{p-1} & h_1^{p-1} & . & . & . & h_{p-1}^{p-1} & \\
h_{p-1}^{p-1} & h_{p-2}^{p-1} & . & . & . & h_0^{p-1} & \\
. & & & & & & \\
. & & & & & & \\
. & & & & & & \\
h_0^1 & h_1^1 & & & & & \\
h_1^1 & h_0^1 & & & & & \\
h_0^0 & & & & & &
\end{array}$$

G-table

$$\begin{array}{ccccccc}
g_0 & g_1 & . & . & . & g_{p-1} & g_p \\
h_p & h_{p-1} & . & . & . & h_1 & h_0 \\
g_0^{p-1} & g_1^{p-1} & . & . & . & g_{p-1}^{p-1} & \\
h_{p-1}^{p-1} & h_{p-2}^{p-1} & . & . & . & h_0^{p-1} & \\
. & & & & & & \\
. & & & & & & \\
. & & & & & & \\
g_0^1 & g_1^1 & & & & & \\
h_1^1 & h_0^1 & & & & & \\
g_0^0 & & & & & &
\end{array}$$

The first row of H- table and G- table is obtained by coefficients of denominator and numerator of (13) respectively. Each even row in the H- table is obtained by writing the coefficients of the proceeding row in reverse order. The even rows of the H- and B-tables are the same. The coefficients of the odd rows of both tables are obtained from the two as follows

$$g_i^{k-1} = g_i^k - \beta_k h_{k-i}^k, \qquad \alpha_k = h_k^k / h_0^k \tag{18}$$

$$h_i^{k-1} = h_i^k - \alpha_k h_{k-i}^k, \qquad \beta_k = g_k^k / h_0^k \tag{19}$$

where $k = p, p-1, p-2, \ldots, 2, 1$

with the initial conditions

$$\begin{aligned} h_i^p &= h_i \\ g_i^p &= g_i \end{aligned} \tag{20}$$

It is found that the integral in (5), using above tables can be evaluated [20] as

$$J_p = \frac{1}{h_0^p} \sum_{i=0}^{p} \frac{(g_i^i)^2}{h_0^i} \tag{21}$$

The necessary and sufficient conditions for all the roots of denominator polynomial of the model (2) to be strictly inside the unite circle are preserved in (17).

Therefore the problem is to minimize J_p given by (21) subject to (12), (14), (15), (16) and (17). The algorithm due to Luss and Jaakola [21] suits this situation. Two examples are chosen to illustrate the steps involved in arriving at the solution.

3 Examples

Example 1

Consider a fourth-order system [25] given by (22).

$$G(z) = \frac{0.3124z^3 - 0.5743z^2 + 0.3879z - 0.0889}{z^4 - 3.233z^3 + 3.9869z^2 - 2.2209z - 0.4723} \tag{22}$$

Suppose a second-order approximant given of the form

$$G_r(z) = \frac{\hat{a}_1 z + \hat{a}_2}{z^2 + \hat{b}_1 z + \hat{b}_2} \tag{23}$$

is desired.

For this example, (12), (14), (15), (16) and (17) are identified to be, respectively, (24), (25), (26), (27) and (30)

$$\hat{a}_2 = 7(1 + \hat{b}_1 + \hat{b}_2) - \hat{a}_1 \tag{24}$$

$$\left.\begin{aligned} g_0 &= 0 \\ g_1 &= c_1 \\ g_2 &= c_2 \\ g_3 &= c_3 \\ g_4 &= c_4 \\ g_5 &= c_5 \\ g_6 &= 0 \end{aligned}\right\} \tag{25}$$

$$\left.\begin{aligned} c_1 &= \hat{a}_1 - 0.3124 \\ c_2 &= -3.233\hat{a}_1 + \hat{a}_2 - 0.3124\hat{b}_1 + 0.5743 + c_1 \\ c_3 &= 3.9869\hat{a}_1 - 3.233\hat{a}_2 + 0.5743\hat{b}_1 - 0.3124\hat{b}_2 - 0.3879 + c_2 \\ c_4 &= -2.2209\hat{a}_1 + 3.9869\hat{a}_2 - 0.3879\hat{b}_1 + 0.5743\hat{b}_2 + 0.0889 + c_3 \\ c_5 &= 0.4723\hat{a}_1 - 2.2209\hat{a}_2 - 0.0889\hat{b}_1 - 0.3879\hat{b}_2 + c_4 \end{aligned}\right\} \tag{26}$$

$$\left.\begin{aligned}
h_0 &= 1 \\
h_1 &= \hat{b}_1 - 3.233 \\
h_2 &= -3.233\hat{b}_1 + \hat{b}_2 + 3.9869 \\
h_3 &= 3.9869\hat{b}_1 - 3.233\hat{b}_2 - 2.2209 \\
h_4 &= -2.2209\hat{b}_1 + 3.9869\hat{b}_2 + 0.4723 \\
h_5 &= 0.4723\hat{b}_1 - 2.2209\hat{b}_2 \\
h_6 &= 0.4723\hat{b}_2
\end{aligned}\right\} \tag{27}$$

H-table and G-table are formed as:

H-table

h_0	h_1	h_2	h_3	h_4	h_5	h_6
h_6	h_5	h_4	h_3	h_2	h_1	h_0
h_0^5	h_1^5	h_2^5	h_3^5	h_4^5	h_5^5	
h_5^5	h_4^5	h_3^5	h_2^5	h_1^5	h_0^5	
h_0^4	h_1^4	h_2^4	h_3^4	h_4^4		
h_4^4	h_3^4	h_2^4	h_1^4	h_0^4		
h_0^3	h_1^3	h_2^3	h_3^3			
h_3^3	h_2^3	h_1^3	h_0^3			
h_0^2	h_1^2	h_2^2				
h_2^2	h_1^2	h_0^2				
h_0^1	h_1^1					
h_1^1	h_0^1					
h_0^0						

G-table

g_0	g_1	g_2	g_3	g_4	g_5	g_6
h_6	h_5	h_4	h_3	h_2	h_1	h_0
g_0^5	g_1^5	g_2^5	g_3^5	g_4^5	g_5^5	
h_5^5	h_4^5	h_3^5	h_2^5	h_1^5	h_0^5	
g_0^4	g_1^4	g_2^4	g_3^4	g_4^4		
h_4^4	h_3^4	h_2^4	h_1^4	h_0^4		
g_0^3	g_1^3	g_2^3	g_3^3			
h_3^3	h_2^3	h_1^3	h_0^3			
g_0^2	g_1^2	g_2^2				
h_2^2	h_1^2	h_0^2				
g_0^1	g_1^1					
h_1^1	h_0^1					
g_0^0						

where

$$h_i^{k-1} = h_i^k - \alpha_k h_{k-i}^k, \quad \text{and} \quad \alpha_k = h_k^k / h_0^k, \ \ k = 6,5,4,3,2,1 \tag{28}$$

$$g_i^{k-1} = g_i^k - \beta_k h_{k-i}^k, \quad \text{and} \quad \beta_k = g_k^k / h_0^k, \ \ k = 6,5,4,3,2,1 \tag{29}$$

and the initial conditions

$$h_i^6 = h_i, \quad i = 0,1,2,3,4,5,6$$

$$g_i^6 = g_i, \quad i = 0,1,2,3,4,5,6$$

while (17) and (21) take the following forms respectively

$h_0^6 > 0, h_0^5 > 0, h_0^4 > 0, h_0^3 > 0, h_0^2 > 0, h_0^1 > 0$ and $h_0^0 > 0$ (30)

$$J_6 = \frac{1}{h_0^6}\sum_{i=0}^{6}\frac{(g_i^i)^2}{h_0^i}$$

$$= \frac{1}{h_0^6}\left[\frac{(g_0^0)^2}{h_0^0} + \frac{(g_1^1)^2}{h_0^1} + \frac{(g_2^2)^2}{h_0^2} + \frac{(g_3^3)^2}{h_0^3} + \frac{(g_4^4)^2}{h_0^4} + \frac{(g_5^5)^2}{h_0^5} + \frac{(g_6^6)^2}{h_0^6}\right] \tag{31}$$

The minimization of (31) subject to (24) - (30) using Luss-Jaakola's algorithm [21], yields the optimal solution

$\hat{a}_1^* = 0.129732,\ \hat{a}_2^* = 0.182190,\ \hat{b}_1^* = -1.743148,\ \hat{b}_2^* = 0.787708$

Following are some typical results

- Initial conditions: $\hat{a}_1 = 1.0,\ \hat{a}_2 = 1.0,\ \hat{b}_1 = 1.0,\ \hat{b}_2 = 1.0$
  Range : 1.0, 1.0, 1.0, 1.0
  Optimal solution: $\hat{a}_1^* = 0.129732,\ \hat{a}_2^* = 0.182190,\ \hat{b}_1^* = -1.743148,\ \hat{b}_2^* = 0.787708$
  Objective function: $J^* = 0.303030$

- Initial conditions: $\hat{a}_1 = 0.5,\ \hat{a}_2 = 0.5,\ \hat{b}_1 = 0.5,\ \hat{b}_2 = 0.5$
  Range : 10.0, 10.0, 10.0, 10.0
  Optimal solution: $\hat{a}_1^* = 0.129499,\ \hat{a}_2^* = 0.182591,\ \hat{b}_1^* = -1.743076,\ \hat{b}_2^* = 0.787661$
  Objective function: $J^* = 0.303048$

- Initial conditions: $\hat{a}_1 = 1.0,\ \hat{a}_2 = -1.0,\ \hat{b}_1 = -0.5,\ \hat{b}_2 = -0.5$
  Range : 0.5, 0.5, 0.5, 0.5
  Optimal solution: $\hat{a}_1^* = 0.129728,\ \hat{a}_2^* = 0.182071,\ \hat{b}_1^* = -1.743230,\ \hat{b}_2^* = 0.787773$
  Objective function: $J^* = 0.303037$

- Initial conditions: $\hat{a}_1 = -1.0,\ \hat{a}_2 = 0.5,\ \hat{b}_1 = 1.0,\ \hat{b}_2 = -0.5$
  Range : 3.0, 3.0, 3.0, 3.0
  Optimal solution: $\hat{a}_1^* = 0.129150,\ \hat{a}_2^* = 0.182989,\ \hat{b}_1^* = -1.743002,\ \hat{b}_2^* = 0.787593$
  Objective function: $J^* = 0.303050$

- Initial conditions: $\hat{a}_1 = 0.2,\ \hat{a}_2 = 0.4,\ \hat{b}_1 = 0.6,\ \hat{b}_2 = 0.8$

Range : 5.0, 5.0, 5.0, 5.0

Optimal solution: $\hat{a}_1^* = 0.130692$, $\hat{a}_2^* = 0.181049$, $\hat{b}_1^* =$-1.743225, $\hat{b}_2^* = 0.787759$

Objective function: $J^* = 0.303067$

Therefore the model takes the form

$$G_2(z) = \frac{0.129732z + 0.18219}{z^2 - 1.743148z + 0.787708} \tag{32}$$

On the other hand, the model obtained by the technique of [25]

$$G_2(z) = \frac{0.3124z - 0.0298}{z^2 - 1.7369z + 0.7773} \tag{33}$$

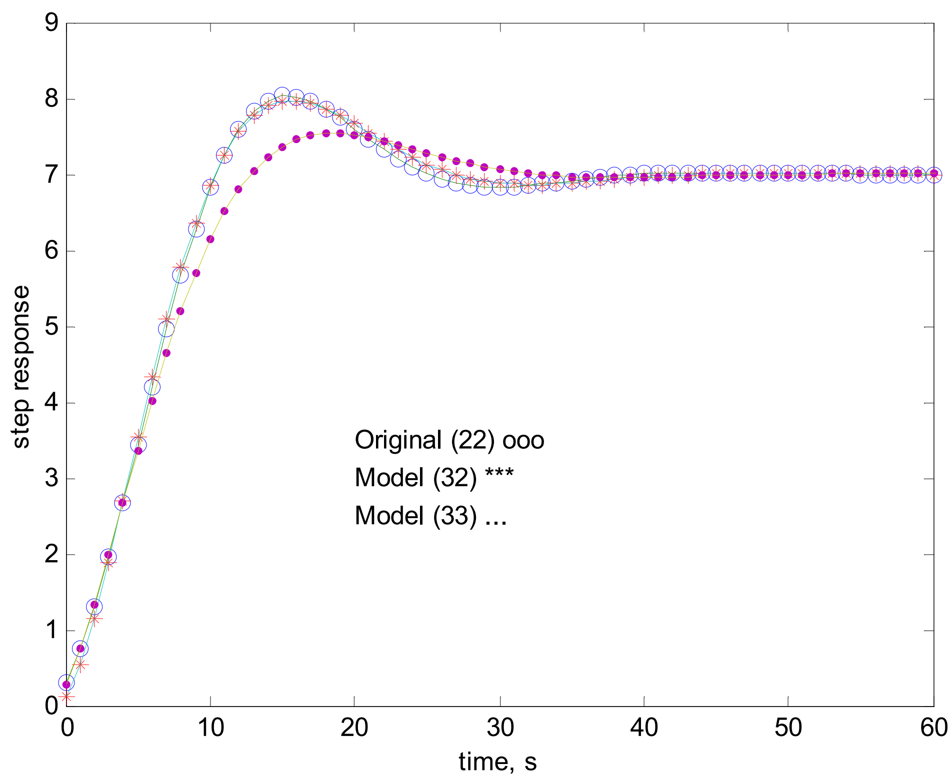

Fig. 1 Step responses of the systems given by (22) and its reduced order models given by (32) and (33)

Table1. Comparison of integral-squared error (ISE)

Model	ISE of unit-step response
(32)	0.303030
(33)	1.481662

The step responses of the system (22) and the models (32) when plotted (Fig.1) were seen to be close to each other but model (33) when plotted was seen to be very poor against model (32). This is also confirmed by examining the ISE corresponding to (32) and (33) given in Table1.

Example 2

Consider a fifth-order system [26] given by (34).

$$G(z)=\frac{z^4-1.0616z^3+0.7545z^2+0.0015z-0.0349}{z^5-0.3z^4-0.87z^3+0.307z^2+0.082z-0.022} \tag{34}$$

Suppose a second order approximant of the form

$$G_r(z)=\frac{\hat{a}_1 z+\hat{a}_2}{z^2+\hat{b}_1 z+\hat{b}_2} \tag{35}$$

is desired.

For this example (12), (14), (15), (16) and (17) are identified to be, respectively, (36), (37), (38), (39) and (42)

$$\hat{a}_2=3.347(1+\hat{b}_1+\hat{b}_2)-\hat{a}_1 \tag{36}$$

$$\left.\begin{aligned} g_0&=0\\ g_1&=c_1\\ g_2&=c_2\\ g_3&=c_3\\ g_4&=c_4\\ g_5&=c_5\\ g_6&=c_6\\ g_7&=0 \end{aligned}\right\} \tag{37}$$

$$\left.\begin{aligned} c_1&=\hat{a}_1-1\\ c_2&=-0.3\hat{a}_1-\hat{a}_2-\hat{b}_1+1.0616+c_1\\ c_3&=-0.87\hat{a}_1-0.3\hat{a}_2+1.0616\hat{b}_1-\hat{b}_2-0.7545+c_2\\ c_4&=0.307\hat{a}_1-0.87\hat{a}_2-0.7545\hat{b}_1+1.0616\hat{b}_2-0.0015+c_3\\ c_5&=0.082\hat{a}_1+0.307\hat{a}_2-0.0015\hat{b}_1-0.7545\hat{b}_2+0.0349+c_4\\ c_6&=-0.022\hat{a}_1+0.082\hat{a}_2+0.0349\hat{b}_1-0.0015\hat{b}_2+c_5 \end{aligned}\right\} \tag{38}$$

$$\left.\begin{aligned}
h_0 &= 1 \\
h_1 &= \hat{b}_1 - 0.3 \\
h_2 &= -0.3\hat{b}_1 + \hat{b}_2 - 0.87 \\
h_3 &= -0.87\hat{b}_1 - 0.3\hat{b}_2 + 0.307 \\
h_4 &= 0.307\hat{b}_1 - 0.87\hat{b}_2 + 0.082 \\
h_5 &= 0.082\hat{b}_1 + 0.307\hat{b}_2 - 0.022 \\
h_6 &= -0.022\hat{b}_1 + 0.082\hat{b}_2 \\
h_7 &= -0.022\hat{b}_2
\end{aligned}\right\} \tag{39}$$

For this example H-table and G-table is formed as:

H-table

h_0	h_1	h_2	h_3	h_4	h_5	h_6	h_7
h_7	h_6	h_5	h_4	h_3	h_2	h_1	h_0
h_0^6	h_1^6	h_2^6	h_3^6	h_4^6	h_5^6	h_6^6	
h_6^6	h_5^6	h_4^6	h_3^6	h_2^6	h_1^6	h_0^6	
h_0^5	h_1^5	h_2^5	h_3^5	h_4^5	h_5^5		
h_5^5	h_4^5	h_3^5	h_2^5	h_1^5	h_0^5		
h_0^4	h_1^4	h_2^4	h_3^4	h_4^4			
h_4^4	h_3^4	h_2^4	h_1^4	h_0^4			
h_0^3	h_1^3	h_2^3	h_3^3				
h_3^3	h_2^3	h_1^3	h_0^3				
h_0^2	h_1^2	h_2^2					
h_2^2	h_1^2	h_0^2					
h_0^1	h_1^1						
h_1^1	h_0^1						
h_0^0							

G-table

g_0	g_1	g_2	g_3	g_4	g_5	g_6	g_7
h_7	h_6	h_5	h_4	h_3	h_2	h_1	h_0
g_0^6	g_1^6	g_2^6	g_3^6	g_4^6	g_5^6	g_6^6	
h_6^6	h_5^6	h_4^6	h_3^6	h_2^6	h_1^6	h_0^6	
g_0^5	g_1^5	g_2^5	g_3^5	g_4^5	g_5^5		
h_5^5	h_4^5	h_3^5	h_2^5	h_1^5	h_0^5		
g_0^4	g_1^4	g_2^4	g_3^4	g_4^4			
h_4^4	h_3^4	h_2^4	h_1^4	h_0^4			
g_0^3	g_1^3	g_2^3	g_3^3				
h_3^3	h_2^3	h_1^3	h_0^3				
g_0^2	g_1^2	g_2^2					
h_2^2	h_1^2	h_0^2					
g_0^1	g_1^1						
h_1^1	h_0^1						
g_0^0							

where

$$h_i^{k-1} = h_i^k - \alpha_k h_{k-i}^k, \quad \text{and} \quad \alpha_k = h_k^k / h_0^k, \quad k = 7,6,5,4,3,2,1 \tag{40}$$

$$g_i^{k-1} = g_i^k - \beta_k h_{k-i}^k, \quad \text{and} \quad \beta_k = g_k^k / h_0^k, \quad k = 7,6,5,4,3,2,1 \tag{41}$$

with the initial conditions

$$h_i^7 = h_i, \quad i = 0,1,2,3,4,5,6,7$$

$$g_i^7 = g_i, \quad i = 0,1,2,3,4,5,6,7$$

The constraints (17) for this example are identified as

$$h_0^7 > 0, h_0^6 > 0, h_0^5 > 0, h_0^4 > 0, h_0^3 > 0, h_0^2 > 0, h_0^1 > 0 \text{ and } h_0^0 > 0 \tag{42}$$

and (21) takes the form

$$J_7 = \frac{1}{h_0^7} \sum_{i=0}^{7} \frac{(g_i^i)^2}{h_0^i}$$

$$= \frac{1}{h_0^7} \left[\frac{(g_0^0)^2}{h_0^0} + \frac{(g_1^1)^2}{h_0^1} + \frac{(g_2^2)^2}{h_0^2} + \frac{(g_3^3)^2}{h_0^3} + \frac{(g_4^4)^2}{h_0^4} + \frac{(g_5^5)^2}{h_0^5} + \frac{(g_6^6)^2}{h_0^6} + \frac{(g_7^7)^2}{h_0^7} \right] \tag{43}$$

The minimization of (43) subject to constraints (36) - (43) using Luss-Jaakola's algorithm [21], yield the optimal solution

$$\hat{a}_1^* = 1.138388, \ \hat{a}_2^* = \text{-}0.194374, \ \hat{b}_1^* = 0.085556, \ \hat{b}_2^* = \text{-}0.803568$$

Following are some typical results

- Initial conditions: $\hat{a}_1 = 1.0, \ \hat{a}_2 = 1.0, \ \hat{b}_1 = 1.0, \ \hat{b}_2 = 1.0$
  Range : 1.0, 1.0, 1.0, 1.0
  Optimal solution: $\hat{a}_1^* = 1.138388, \ \hat{a}_2^* = \text{-}0.194374, \ \hat{b}_1^* = 0.085556, \ \hat{b}_2^* = \text{-}0.803568$
  Objective function: $J^* = 0.781373$

- Initial conditions: $\hat{a}_1 = 0.5, \ \hat{a}_2 = 0.5, \ \hat{b}_1 = 0.5, \ \hat{b}_2 = 0.5$
  Range : 10.0, 10.0, 10.0, 10.0
  Optimal solution: $\hat{a}_1^* = 1.138074, \ \hat{a}_2^* = \text{-}0.193991, \ \hat{b}_1^* = 0.085576, \ \hat{b}_2^* = \text{-}0.803568$
  Objective function: $J^* = 0.781373$

- Initial conditions: $\hat{a}_1 = 1.0, \ \hat{a}_2 = \text{-}1.0, \ \hat{b}_1 = \text{-}0.5, \ \hat{b}_2 = \text{-}0.5$
  Range : 0.5, 0.5, 0.5, 0.5
  Optimal solution: $\hat{a}_1^* = 1.138189, \ \hat{a}_2^* = \text{-}0.194133, \ \hat{b}_1^* = 0.085570, \ \hat{b}_2^* = \text{-}0.803570$
  Objective function: $J^* = 0.781373$

- Initial conditions: $\hat{a}_1 = \text{-}1.0, \ \hat{a}_2 = 0.5, \ \hat{b}_1 = 1.0, \ \hat{b}_2 = \text{-}0.5$
  Range : 3.0, 3.0, 3.0, 3.0
  Optimal solution: $\hat{a}_1^* = 1.138188, \ \hat{a}_2^* = \text{-}0.194107, \ \hat{b}_1^* = 0.085573, \ \hat{b}_2^* = \text{-}0.803566$

Objective function: $J^* = 0.781373$

- Initial conditions: $\hat{a}_1 = 0.2,\ \hat{a}_2 = 0.4,\ \hat{b}_1 = 0.6,\ \hat{b}_2 = 0.8$
  Range : 5.0, 5.0, 5.0, 5.0
  Optimal solution: $\hat{a}_1^* = 1.138135,\ \hat{a}_2^* = \text{-}0.194122,\ \hat{b}_1^* = 0.085561,\ \hat{b}_2^* = \text{-}0.803573$
  Objective function: $J^* = 0.781373$

Therefore the model takes the form

$$G_2(z) = \frac{1.138388\, z - 0.194374}{z^2 + 0.085556z - 0.803568} \tag{44}$$

On the other hand, the models obtained by the techniques of [25], [26] and [27] are, respectively

$$G_2(z) = \frac{z - 0.1481}{z^2 + 0.0687z - 0.8142} \tag{45}$$

$$G_2(z) = \frac{0.5611z - 0.2842}{1.3132z^2 - 1.9586z + 0.7281} \tag{46}$$

and

$$G_2(z) = \frac{z - 0.052}{z^2 + 0.1027z - 0.8195} \tag{47}$$

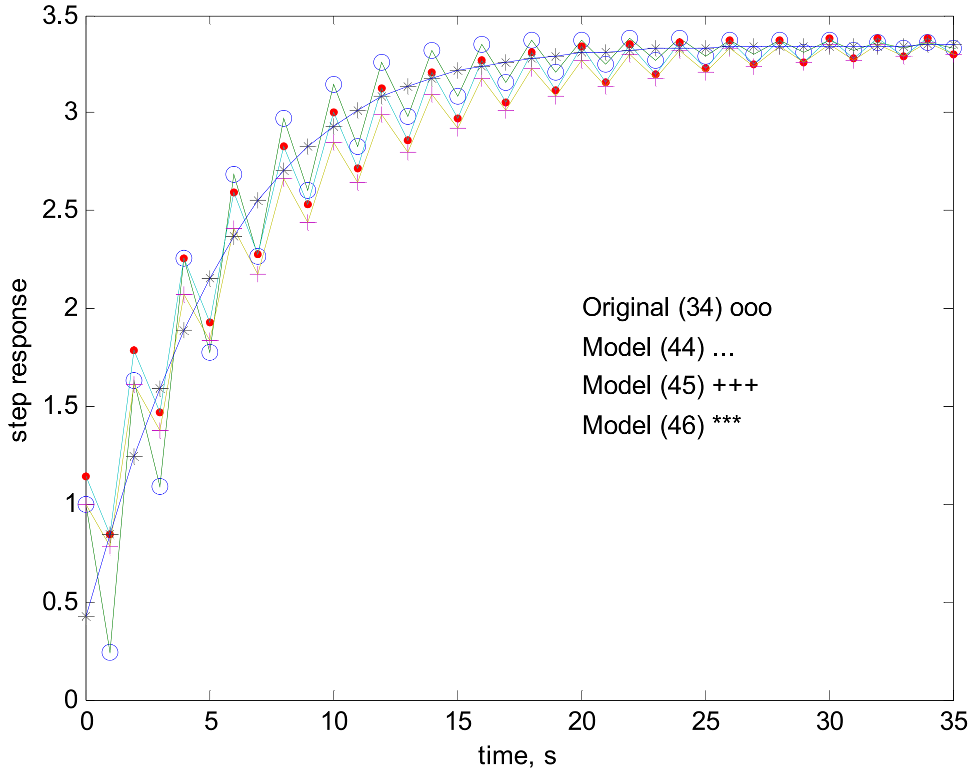

Fig. 2 Step responses of the system given by (34) and its reduced order models given by (44), (45) and (46)

Table2. Comparison of integral-squared error (ISE)

Model	ISE of unit-step response
(44)	0.7814
(45)	1.0844
(46)	1.9176
(47)	0.8554

The step responses of the system (34) and the models (44), (45) when plotted (Fig.2) were seen to be close to each other but model (46) when plotted was seen to be very poor against models (44) and (45). The ISE corresponding to (44)-(47) are shown in Table2. This shows some improvement realized from the methods [8], [24] and [25].

4 Conclusions

A novel method for obtaining a reduced-order model of a given (stable) SISO discrete time system based on minimization of ISE has been developed. The method allows both the numerator and denominator coefficients of the models as free parameters in the process of optimization, and guarantees that a stable system is reduced to a stable model.

The problem of formulating the objective function is circumvented by introducing a set of equality constraints.

References:

[1]. Shih, Y.P., and Wu, W.T., Simplification of z-transfer functions by continued fractions, *Int. J. Contr*, 17, (1973) 1089-1094.

[2]. Chiang, S.C., Homographic transformation for the simplification of discrete-time transfer functions by Pade approximation, *Int. J. Contr*, 23,(1975) 721-729.

[3]. Parthasarthy, R., and Jayasimha, K.N., Modeling of linear discrete-time systems using modified Cauer continued fraction, *J. Frank. Inst*., (1983) 316, 79-86.

[4]. Shamash, Y., Continued fraction methods for the reduction of discrete-time dynamic systems. *Int. Journal of Control*, 20, (1974) 267-275.

[5]. Hwang , C., Shih, Y.P., and Hwang, R.Y., A combined time and frequency domain method for model reduction of discrete systems, *J. Frank. Inst*., (1981) 311, 391.

[6]. Appiah, R.K., Pade methods of Hurwitz polynomial approximation with application to linear system reduction, *Int. J. Contr*., (1979) 29, 39-48,.

[7]. Hwang, C.and Shih, Y.P., Routh approximations for reducing order of discrete systems, *ASME Trans., Dynamic Sys. Meas. and Contr*., (1982)104, pp. 107-109.

[8]. Hwang, C. and Hsieh, C.S., Order reduction of discrete-time systems via Bilinear Routh approximation, *ASME Trans., Dynamic Sys. Meas. And Contr*.,(1990)112, 292-297.

[9]. Younseok Choo, An improvement in Routh approximation method using impulse energy criterion, *J. Elec. Eng. Sci.,* (1999) 594-599,.

[10]. Younseok Choo, Improved Bilinear Routh approximation method for discrete-time systems, *ASME Trans*., *Dynamic Sys. Meas. And Contr*., (2001) 123, 125-127,.

[11]. Younseok Choo, Equivalence ofbilinear Routh and Schwarz Approximation methods, *Electron. Lett.,* (2002) 761-762.

[12]. Younseok Choo, A new consideration for model reduction using blinear Schwarz approximation for discrete-time systems, *ASME Trans*., *Dynamic Sys. Meas. And Contr*., (2002) 124, 475-477.

[13]. Hwang, C., Shih, Y.P. and Hwang R.Y., A combined time and frequency domain method for model reduction of discrete systems, *J.Frank. Inst.,* (1981) 311, 391.

[14]. Hwang, R.Y., and Shih,Y.P., Combined methods for model reduction via discrete Laguerre polynomial, *Int. J. Contr*. (1983)37, 615-622.

[15]. Pujara, L.R., and Rattan, R.S., A frequency matching method for model reduction of digital control systems, *Int. J. Contr*. (1982) 35, 139-148.

[16]. Farsi, M., Warwich, K., and Guilandoust, M., Stable reduced-order models for discrete-time systems, IEEE Proc. D, (1986) 133, 137-141.

[17]. Bandyopadhyay, B., Kande, T.M., and Srisalam, M.C., Routh type approximation for discrete system, *IEEE, International Symposium on Circuits & SystI,* (1988) 3, 2899-2902.

[18]. Sebakhy, O.A. amd Aly, M.N., Discrete-time model reduction with optimal zero locations by norm minimization, *IEE Proc. Control Theory Appl*.,(1998) 145. No.6, November.

[19]. Puri, N.N., and. Lim, M.T, Stable optimal Model Reduction of Linear Discrete-Time Systems, *ASME Trans., Dynamic Sys., Meas. and Contr*.,(1997) 119, 300-304.

[20]. Astrom, K. J., Introduction to Stochastic Control Theory, *Mathematics in science and engineering*, 70, Academic Press, New York.

[21]. Luss, R. and Jaakola, T.H.I., Direct search and systematic reduction of size of search region, *AIChE J*., (1973) 19, 760-766.

[22] Singh, V., Chandra, D. and Kar, H., Optimal Routh approximants through integral squared error minimization: computer-aided approach, *IEE Procc. Control Theory Appl*., (Jan. 2004)151, No. 1.

[23]. Jury, E.I., A note on the evaluation of the total square integral, *IEEE Trans. Autom. Control*, (1965) AC-10, 110-111,.

[24]. Jury, E.I., Theory and application of the z-transform method, (Wiely, New York, 1964).

[25]. Warwick, K., A new approach to reduced order modeling, *IEE Proceedings - Control Theory and Applications*, (1984) 131(2), 74-78.

[26]. Younseok Choo and Jaeho Choi, A New Consideration for Disrete-Systemm Reduction via Impulse Response Gramian, *Int. Journal of Cont., Automation and Systems*, (Sept. 2004) 2, 3, 384-389.

[27]. Azou, S., Brehonnet, P., Vilbe, P. and Calvez, L.C., A new discrete impulse response Gramian and its application to model reduction, *IEEE Trans on Automat. Contr*., (2000) 45(3), 533-537.