STABLE OPTIMAL MODEL REDUCTION OF LINEAR DISCRETE
TIME SYSTEMS VIA INTEGRAL SQUARED ERROR
MINIMIZATION: COMPUTER-AIDED APPROACH

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Abstract: A computer-aided technique to obtain a reduced-order approximant of a given
(stable) single-input single-output discrete time system based on the minimization of a
integral squared error (ISE) pertaining to a unit step input is presented. Both the
numerator and denominator coefficients of the model are treated as free parameters in the
process of optimization. The method has a built-in
stability-preserving feature.

Keywords: Model reduction, Padé approximation, Routh criterion, Pareto optimal
solution, VEGA.

1 Introduction

The techniques for model reduction of discrete-time systems are limited and may be
classified into two groups. The first group contains the methods [1-12] which derive
approximant $G_r(z)$ from a given high-order transfer function $G(z)$ exploiting the already
existing continuous-time algorithms [1-5]. Some of the methods of this group are very
attractive, because a stable reduced model is obtained if the original system is stable [5-
12]. The second group contains the so called direct methods that derive $G_r(z)$ directly
from $G(z)$ without using the transformation but they do not usually ensure stability of
the reduced model even though the original system is stable [13-15]. Farasi et al. [16]
have proposed a method in which the Routh stability criterion is employed to reduce the
order of discrete system transfer functions. It is shown that the Routh approximation is
well suited to reduce both denominator and the numerator polynomials, although
alternative methods such as Padé approximation can also be used to fit the model
numerator coefficients. In [17], a Routh type approximation for discrete system is
presented. The denominator of the reduced model is directly obtained from a Routh type
table and the numerator of the reduced model is obtained either by matching the discrete

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time moments or by minimizing the step response error in $z$ domain. A common feature in above methods [1-17] is that the values of the denominator coefficients of the reduced-order transfer function are chosen arbitrarily. This feature appears to be largely motivated by the consideration of achieving computational simplicity. If the denominator coefficients are numerically specified, the resulting equations for optimization become a set of linear equations in terms of the numerator coefficients and then these coefficients are easily determined. However, selecting the denominator coefficients arbitrarily may generally mean a loss of a considerable degree of freedom for optimization. It is sometimes suggested [18] that the denominator coefficients may be chosen so as to retain the dominant poles of the system. However, it may not usually be straightforward to say that the poles of the optimal reduced-order model may have some definite relationship with those of the original system.

Thus, the problem is to derive, subject to preserving stability, a model via the minimization of an objective function (such as ISE) while allowing both the numerator and the denominator coefficients as free parameters in the optimization. An algorithm is presented by Puri and Lin [19]. The proposed algorithm [19] minimizes a weighted mean squared impulse (or step) response error between the original system and reduced-order model. The procedure guarantees a stable model and can be extended to MIMO systems.

An alternative method is presented in this paper. In the proposed work, a stable reduced-order approximant is derived for a given (stable) single-input single-output (SISO) system via minimization of ISE pertaining to a unit-step input. In this approach, which allows both the numerator and denominator coefficients of the model as free parameters in the process of optimization, the problem of construction of objective function is circumvented by tacitly introducing, using an early idea due to Astrom [20], a set of equality constraints. By way of utilizing the ideas due to Astrom [20], the approach has the built-in stability-preserving feature for any value of $r$. The minimization of ISE is carried out by applying the algorithm due to Luss and Jaakola [21]. Two examples are included that bring out the systematic nature of the algorithm.

In this context, it is worth mentioning that a similar attempt is made in [22] for continuous-time SISO systems.

## 2 Background

Consider a higher order stable system

$$
G(z) = \frac{Y(z)}{R(z)} = \frac{a_1 z^{-1} + a_2 z^{-2} + \ldots + a_n z + a_n}{z^n + b_1 z^{n-1} + \ldots + b_n z + b_n} \tag{1}
$$

The problem is to determine its stable reduced-order ($r$-th-order) approximant of the form

$$
G_r(z) = \frac{Y_r(z)}{R_r(z)} = \frac{\hat{a}_1 z^{-r} + \hat{a}_2 z^{-r-1} + \ldots + \hat{a}_r z + \hat{a}_r}{z^r + \hat{b}_1 z^{r-1} + \ldots + \hat{b}_r z + \hat{b}_r} \tag{2}
$$

such that the ISE of unit-step response given by
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\[ J = \sum_{k=0}^{\infty} e^2(k) \]  

is minimum where

\[ e(k) = y_r(k) - y(k) \]  

\( y(k) \) and \( y_r(k) \) denote the unit-step responses of the system and model, respectively.

Using Parseval’s theorem the ISE can alternatively be expressed in the \( z \) domain \([23-24]\) as

\[ J = \frac{1}{2\pi j} \int_{\text{unit circle}} E(z)E(z^{-1})z^{-1}dz, \quad j = \sqrt{-1} \]  

where \( E(z) \) is the \( z \)-transform of \( e(k) \). It is assumed that the steady-state error is zero, i.e. \( y(\infty) = y_r(\infty) \). This assumption requires that

\[ \alpha_0 = \beta_0 \]

where \( \alpha_0 = \left. \frac{Y(z)}{z} \right|_{z=1} \) and \( \beta_0 = \left. \frac{Y_r(z)}{z} \right|_{z=1} \)

\( Y(z) \) and \( Y_r(z) \) denote the unit-step responses of the system and model, respectively in \( z \) domain.

Alternatively, \( \alpha_0 \) and \( \beta_0 \) can be calculated as follows

Putting \( z = p + 1 \) in polynomial (1) and expanding about \( p = 0 \), (1) becomes

\[ G(p) = \frac{a_0(p+1)^n + a_1(p+1)^{n-1} + \ldots + a_n(p+1)}{(p+1)^n + b_1(p+1)^{n-1} + \ldots + b_n} \]

\[ = \frac{A_0 p + A_1 p + \ldots + A_n}{p^n + B_1 p^{n-1} + \ldots + B_n} \]

\[ = \alpha_0 + \alpha_1 p + \alpha_2 p^2 + \ldots \]

\[ = \alpha_0 + \alpha_1(z-1) + \alpha_2(z-1)^2 + \ldots \]  

where \( \alpha_0 \) is given by

\[ \alpha_0 = \frac{A_n}{B_n} \]
Again on putting $z = p + 1$ in (2) and expanding about $p = 0$, (2) becomes

$$G_r(p) = \frac{\hat{A}_1 (p + 1)^{r-1} + \hat{A}_2 (p + 1)^{r-2} + \ldots + \hat{A}_{r-1} (p + 1) + \hat{A}_r}{(p + 1)^r + \hat{b}_1 (p + 1)^{r-1} + \ldots + \hat{b}_{r-1} (p + 1) + \hat{b}_r}$$

$$= \frac{A_1 p^{r-1} + A_2 p^{r-2} + \ldots + A_{r-1} p + A_r}{p^r + \hat{b}_1 p^{r-1} + \ldots + \hat{b}_{r-1} p + \hat{b}_r}$$

$$= \beta_0 + \beta_1 p + \beta_2 p^2 + \ldots$$

$$= \beta_0 + \beta_1 (z - 1) + \beta_2 (z - 1)^2 + \ldots$$

where $\beta_0$ is given by

$$\beta_0 = \frac{\hat{A}_r}{\hat{B}_r}$$

For steady state matching

$$\alpha_0 = \beta_0$$

i.e.

$$\frac{A_n}{B_n} = \frac{\hat{A}_r}{\hat{B}_r}$$

Let $N(z)$ and $D(z)$ denote, respectively, the numerator and denominator polynomials of (1) and $N_r(z)$ and $D_r(z)$ the respective polynomials of (2). The error function $E(z)$ takes the form [19]

$$E(z) = \left( \frac{z}{z-1} \right) \left[ \frac{D(z)N_r(z) - D_r(z)N(z)}{D_r(z)D(z)} \right]$$

$$E(z) = \frac{g_0 z^p + g_1 z^{p-1} + \ldots + g_{p-1} z + g_p}{h_0 z^p + h_1 z^{p-1} + \ldots + h_{p-1} z + h_p}$$

where $p = n + r$. The coefficients $g_i$ are given by
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\[ \begin{align*}
g_0 &= 0 \\
g_1 &= c_1 \\
g_2 &= c_2 \\
&\vdots \\
g_p &= 0 \\
g_{p-1} &= c_{p-1} \\
g_{p-2} &= c_{p-2} \\
&\vdots \\
g_1 &= c_1 - a_1 \\
g_2 &= \hat{a}_1 b_1 + \hat{a}_2 - a_1 \hat{b}_1 - a_2 + c_1 \\
&\vdots \\
\end{align*} \]

and \( c_i \) and \( h_i \) are given by

\[ \begin{align*}
c_i &= \hat{a}_1 - a_1 \\
c_2 &= \hat{a}_1 b_1 + \hat{a}_2 - a_1 \hat{b}_1 - a_2 + c_1 \\
&\vdots \\
c_{p-2} &= \hat{a}_1 b_{n-2} + \hat{a}_2 b_{n-1} + \hat{a}_3 b_n - a_1 \hat{b}_{r-2} - a_2 \hat{b}_{r-1} - a_3 \hat{b}_r + c_{p-3} \\
c_{p-1} &= \hat{a}_1 b_{n-1} + \hat{a}_2 b_n - a_1 \hat{b}_{r-1} - a_2 \hat{b}_r + c_{p-2} \\
&\vdots \\
\end{align*} \]

\[ \begin{align*}
h_0 &= 1 \\
h_1 &= \hat{b}_1 + b_1 \\
h_2 &= \hat{b}_2 + \hat{b}_1 b_1 + b_2 \\
&\vdots \\
h_{p-1} &= \hat{b}_r b_{n-1} + \hat{b}_r b_n \\
h_p &= \hat{b}_r b_n \\
&\vdots \\
\end{align*} \]

Now integral (5) will always exist if all the poles of polynomial \( E(z) \) are inside the unit circle. This assumption requires that
where \( h_0, h_0^{p-1}, h_0^{p-2}, \ldots, h_0^1, h_0^0 > 0, \) (17)

The first row of H-table and G-table is obtained by coefficients of denominator and numerator of (13) respectively. Each even row in the H-table is obtained by writing the coefficients of the proceeding row in reverse order. The even rows of the H- and B-tables are the same. The coefficients of the odd rows of both tables are obtained from the two as follows

\[
g_{i-1}^{k} = g_{i}^{k} - \beta_{k} h_{k-i}^{k}, \quad \alpha_{k} = h_{k}^{k} / h_{0}^{k}
\]

(18)

\[
h_{i-1}^{k} = h_{i}^{k} - \alpha_{k} h_{k-i}^{k}, \quad \beta_{k} = g_{k}^{k} / h_{0}^{k}
\]

(19)

where \( k = p, p-1, p-2, \ldots, 2, 1 \)

with the initial conditions

\[
h_{i}^{p} = h_{i}
\]

\[
g_{i}^{p} = g_{i}
\]

(20)

It is found that the integral in (5), using above tables can be evaluated [20] as

\[
J_{p} = \frac{1}{h_{0}^{p}} \sum_{i=0}^{p} \frac{(g_{i})^{2}}{h_{0}^{i}}
\]

(21)

The necessary and sufficient conditions for all the roots of denominator polynomial of the model (2) to be strictly inside the unite circle are preserved in (17).
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Therefore the problem is to minimize $J_p$ given by (21) subject to (12), (14), (15), (16) and (17). The algorithm due to Luss and Jaakola [21] suits this situation. Two examples are chosen to illustrate the steps involved in arriving at the solution.

3 Examples

Example 1

Consider a fourth-order system [25] given by (22).

\[ G(z) = \frac{0.3124z^3 - 0.5743z^2 + 0.3879z - 0.0889}{z^4 - 3.233z^3 + 3.9869z^2 - 2.2209z - 0.4723} \] (22)

Suppose a second-order approximant given of the form

\[ G_c(z) = \frac{\hat{a}_1 z + \hat{a}_2}{z^2 + \hat{b}_1 z + \hat{b}_2} \] (23)

is desired.

For this example, (12), (14), (15), (16) and (17) are identified to be, respectively, (24), (25), (26), (27) and (30)

\[ \hat{a}_2 = 7(1 + \hat{b}_1 + \hat{b}_2) - \hat{a}_1 \] (24)

\[
\begin{align*}
g_0 &= 0 \\
g_1 &= c_1 \\
g_2 &= c_2 \\
g_3 &= c_3 \\
g_4 &= c_4 \\
g_5 &= c_5 \\
g_6 &= 0
\end{align*}
\] (25)

\[
\begin{align*}
c_1 &= \hat{a}_1 - 0.3124 \\
c_2 &= -3.233\hat{a}_1 + \hat{a}_2 - 0.3124\hat{b}_1 + 0.5743 + c_1 \\
c_3 &= 3.9869\hat{a}_1 - 3.233\hat{a}_2 + 0.5743\hat{b}_1 - 0.3124\hat{b}_2 - 0.3879 + c_2 \\
c_4 &= -2.2209\hat{a}_1 + 3.9869\hat{a}_2 - 0.3879\hat{b}_1 + 0.5743\hat{b}_2 + 0.0889 + c_3 \\
c_5 &= 0.4723\hat{a}_1 - 2.2209\hat{a}_2 - 0.0889\hat{b}_1 - 0.3879\hat{b}_2 + c_4
\end{align*}
\] (26)
\[ \begin{align*}
\hat{h}_0 &= 1 \\
\hat{h}_1 &= \hat{b}_1 - 3.233 \\
\hat{h}_2 &= -3.233\hat{b}_1 + \hat{b}_2 + 3.9869 \\
\hat{h}_3 &= 3.9869\hat{b}_1 - 3.233\hat{b}_2 - 2.2209 \\
\hat{h}_4 &= -2.2209\hat{b}_1 + 3.9869\hat{b}_2 + 0.4723 \\
\hat{h}_5 &= 0.4723\hat{b}_1 - 2.2209\hat{b}_2 \\
\hat{h}_6 &= 0.4723\hat{b}_2 
\end{align*} \] (27)

H-table and G-table are formed as:

<table>
<thead>
<tr>
<th>H-table</th>
<th>G-table</th>
</tr>
</thead>
<tbody>
<tr>
<td>( h_0 )</td>
<td>( g_0 )</td>
</tr>
<tr>
<td>( h_1 )</td>
<td>( g_1 )</td>
</tr>
<tr>
<td>( h_2 )</td>
<td>( g_2 )</td>
</tr>
<tr>
<td>( h_3 )</td>
<td>( g_3 )</td>
</tr>
<tr>
<td>( h_4 )</td>
<td>( g_4 )</td>
</tr>
<tr>
<td>( h_5 )</td>
<td>( g_5 )</td>
</tr>
<tr>
<td>( h_6 )</td>
<td>( g_6 )</td>
</tr>
</tbody>
</table>

where
\[ h_k^{i-1} = h_k^i - \alpha_i h_{k-i}^i, \quad \text{and} \quad \alpha_k = h_k^k / h_0^k, \quad k = 6, 5, 4, 3, 2, 1 \] (28)

\[ g_i^{k-1} = g_i^k - \beta_i h_{k-i}^k, \quad \text{and} \quad \beta_k = g_k^k / h_0^k, \quad k = 6, 5, 4, 3, 2, 1 \] (29)

and the initial conditions
\[ h_i^6 = h_i, \quad i = 0, 1, 2, 3, 4, 5, 6 \]
\[ g_i^6 = g_i, \quad i = 0, 1, 2, 3, 4, 5, 6 \]

while (17) and (21) take the following forms respectively
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\[ h_6^0 > 0, h_5^0 > 0, h_4^0 > 0, h_3^0 > 0, h_2^0 > 0, h_1^0 > 0 \text{ and } h_0^0 > 0 \]  \hspace{1cm} (30)

\[
J_6 = \frac{1}{h_0^6} \sum_{i=0}^{6} \left( \frac{g_i^6}{h_0^6} \right)^2
\]

\[
= \frac{1}{h_0^6} \left[ \left( \frac{g_0^6}{h_0^6} \right)^2 + \left( \frac{g_1^6}{h_0^6} \right)^2 + \left( \frac{g_2^6}{h_0^6} \right)^2 + \left( \frac{g_3^6}{h_0^6} \right)^2 + \left( \frac{g_4^6}{h_0^6} \right)^2 + \left( \frac{g_5^6}{h_0^6} \right)^2 \right]
\]  \hspace{1cm} (31)

The minimization of (31) subject to (24) - (30) using Luss-Jaakola’s algorithm [21], yields the optimal solution

\[
\hat{a}_1^* = 0.129732, \quad \hat{a}_2^* = 0.182190, \quad \hat{b}_1^* = -1.743148, \quad \hat{b}_2^* = 0.787708
\]

Following are some typical results

- Initial conditions: \( \hat{a}_1 = 1.0, \hat{a}_2 = 1.0, \hat{b}_1 = 1.0, \hat{b}_2 = 1.0 \)
  Range : 1.0, 1.0, 1.0, 1.0
  Optimal solution: \( \hat{a}_1^* = 0.129732, \hat{a}_2^* = 0.182190, \hat{b}_1^* = -1.743148, \hat{b}_2^* = 0.787708 \)
  Objective function: \( J^* = 0.303030 \)

- Initial conditions: \( \hat{a}_1 = 0.5, \hat{a}_2 = 0.5, \hat{b}_1 = 0.5, \hat{b}_2 = 0.5 \)
  Range : 10.0, 10.0, 10.0, 10.0
  Optimal solution: \( \hat{a}_1^* = 0.129499, \hat{a}_2^* = 0.182591, \hat{b}_1^* = -1.743076, \hat{b}_2^* = 0.787661 \)
  Objective function: \( J^* = 0.303048 \)

- Initial conditions: \( \hat{a}_1 = 1.0, \hat{a}_2 = -1.0, \hat{b}_1 = -0.5, \hat{b}_2 = -0.5 \)
  Range : 0.5, 0.5, 0.5, 0.5
  Optimal solution: \( \hat{a}_1^* = 0.129728, \hat{a}_2^* = 0.182071, \hat{b}_1^* = -1.743230, \hat{b}_2^* = 0.787773 \)
  Objective function: \( J^* = 0.303037 \)

- Initial conditions: \( \hat{a}_1 = -1.0, \hat{a}_2 = 0.5, \hat{b}_1 = 1.0, \hat{b}_2 = -0.5 \)
  Range : 3.0, 3.0, 3.0, 3.0
  Optimal solution: \( \hat{a}_1^* = 0.129150, \hat{a}_2^* = 0.182989, \hat{b}_1^* = -1.743002, \hat{b}_2^* = 0.787593 \)
  Objective function: \( J^* = 0.303050 \)

- Initial conditions: \( \hat{a}_1 = 0.2, \hat{a}_2 = 0.4, \hat{b}_1 = 0.6, \hat{b}_2 = 0.8 \)
Range : 5.0, 5.0, 5.0, 5.0
Optimal solution: $\hat{a}_1^* = 0.130692, \hat{a}_2^* = 0.181049, \hat{b}_1^* = -1.743225, \hat{b}_2^* = 0.787759$
Objective function: $J^* = 0.303067$

Therefore the model takes the form

$$G_2(z) = \frac{0.129732z + 0.18219}{z^2 - 1.743148z + 0.787708}$$ (32)

On the other hand, the model obtained by the technique of [25]

$$G_2(z) = \frac{0.3124z - 0.0298}{z^2 - 1.7369z + 0.7773}$$ (33)

![Fig. 1 Step responses of the systems given by (22) and its reduced order models given by (32) and (33)](image)

<table>
<thead>
<tr>
<th>Model</th>
<th>ISE of unit-step response</th>
</tr>
</thead>
<tbody>
<tr>
<td>(32)</td>
<td>0.303030</td>
</tr>
<tr>
<td>(33)</td>
<td>1.481662</td>
</tr>
</tbody>
</table>

Table 1. Comparison of integral-squared error (ISE)
The step responses of the system (22) and the models (32) when plotted (Fig.1) were seen to be close to each other but model (33) when plotted was seen to be very poor against model (32). This is also confirmed by examining the ISE corresponding to (32) and (33) given in Table1.

Example 2

Consider a fifth-order system [26] given by (34).

\[
G(z) = \frac{z^4 - 1.0616z^3 + 0.7545z^2 + 0.0015z - 0.0349}{z^5 - 0.3z^4 - 0.87z^3 + 0.307z^2 + 0.082z - 0.022}
\]  
(34)

Suppose a second order approximant of the form

\[
G_r(z) = \frac{\hat{a}_2 + \hat{a}_1 z}{z^2 + \hat{b}_1 z + \hat{b}_2}
\]

is desired.

For this example (12), (14), (15), (16) and (17) are identified to be, respectively, (36), (37), (38), (39) and (42)

\[
\begin{align*}
\hat{a}_2 &= 3.347(1 + \hat{b}_1 + \hat{b}_2) - \hat{a}_1 \\
g_0 &= 0 \\
g_1 &= c_1 \\
g_2 &= c_2 \\
g_3 &= c_3 \\
g_4 &= c_4 \\
g_5 &= c_5 \\
g_6 &= c_6 \\
g_7 &= 0 \\
c_1 &= \hat{a}_1 - 1 \\
c_2 &= 0.3\hat{a}_1 - \hat{a}_2 - \hat{b}_1 + 1.0616 + c_1 \\
c_3 &= 0.3\hat{a}_1 - 0.3\hat{a}_2 + 1.0616\hat{b}_1 - \hat{b}_2 - 0.7545 + c_2 \\
c_4 &= 0.307\hat{a}_1 - 0.87\hat{a}_2 - 0.7545\hat{b}_1 + 1.0616\hat{b}_2 - 0.0015 + c_3 \\
c_5 &= 0.082\hat{a}_1 + 0.307\hat{a}_2 - 0.0015\hat{b}_1 - 0.7545\hat{b}_2 + 0.0349 + c_4 \\
c_6 &= -0.022\hat{a}_1 + 0.082\hat{a}_2 + 0.0349\hat{b}_1 - 0.0015\hat{b}_2 + c_4 \\
\end{align*}
\]  
(38)
\[ h_0 = 1 \]
\[ h_1 = \hat{b}_1 - 0.3 \]
\[ h_2 = -0.3\hat{b}_1 + \hat{b}_2 - 0.87 \]
\[ h_3 = -0.87\hat{b}_1 - 0.3\hat{b}_2 + 0.307 \]
\[ h_4 = 0.307\hat{b}_1 - 0.87\hat{b}_2 + 0.082 \]
\[ h_5 = 0.082\hat{b}_1 + 0.307\hat{b}_2 - 0.022 \]
\[ h_6 = -0.022\hat{b}_1 + 0.082\hat{b}_2 \]
\[ h_7 = -0.022\hat{b}_2 \]

For this example H-table and G-table is formed as:

<table>
<thead>
<tr>
<th>H-table</th>
<th>G-table</th>
</tr>
</thead>
<tbody>
<tr>
<td>( h_0 )</td>
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<td>( h_1 )</td>
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<td>( h_6 )</td>
<td>( g_6 )</td>
</tr>
<tr>
<td>( h_7 )</td>
<td>( g_7 )</td>
</tr>
</tbody>
</table>

with the initial conditions
\[ h_i^0 = h_i, \quad i = 0, 1, 2, 3, 4, 5, 6, 7 \]

where
\[ h_i^{k+1} = h_i^k - \alpha_k h_{i-k}^k, \quad \text{and} \quad \alpha_k = h_k^k / h_0^k, \quad k = 7, 6, 5, 4, 3, 2, 1 \]  
(40)

\[ g_i^{k+1} = g_i^k - \beta_k h_{i-k}^k, \quad \text{and} \quad \beta_k = g_k^k / h_0^k, \quad k = 7, 6, 5, 4, 3, 2, 1 \]  
(41)

with the initial conditions
\[ h_i^7 = h_i, \quad i = 0, 1, 2, 3, 4, 5, 6, 7 \]
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\[ g_i^7 = g_i, \quad i = 0,1,2,3,4,5,6,7 \]

The constraints (17) for this example are identified as

\[ h_i^7 > 0, h_i^6 > 0, h_i^5 > 0, h_i^4 > 0, h_i^3 > 0, h_i^2 > 0, h_i^1 > 0 \quad \text{and} \quad h_i^0 > 0 \]

and (21) takes the form

\[
J_i = \frac{1}{h_i^0} \sum_{i=0}^{7} \left( g_i^j \right)^2
\]

\[
= \frac{1}{h_i^0} \left[ \left( g_i^0 \right)^2 + \left( g_i^1 \right)^2 + \left( g_i^2 \right)^2 + \left( g_i^3 \right)^2 + \left( g_i^4 \right)^2 + \left( g_i^5 \right)^2 + \left( g_i^6 \right)^2 + \left( g_i^7 \right)^2 \right] \]

The minimization of (43) subject to constraints (36) - (43) using Luss-Jaakola’s algorithm [21], yield the optimal solution

\[ \hat{a}_1^* = 1.138388, \quad \hat{a}_2^* = -0.194374, \quad \hat{b}_1^* = 0.085556, \quad \hat{b}_2^* = -0.803568 \]

**Following are some typical results**

- **Initial conditions:** \( \hat{a}_1 = 1.0, \hat{a}_2 = 1.0, \hat{b}_1 = 1.0, \hat{b}_2 = 1.0 \)
  
  Range : 1.0, 1.0, 1.0, 1.0
  
  Optimal solution: \( \hat{a}_1^* = 1.138388, \hat{a}_2^* = -0.194374, \hat{b}_1^* = 0.085556, \hat{b}_2^* = -0.803568 \)
  
  Objective function: \( J^* = 0.781373 \)

- **Initial conditions:** \( \hat{a}_1 = 0.5, \hat{a}_2 = 0.5, \hat{b}_1 = 0.5, \hat{b}_2 = 0.5 \)
  
  Range : 10.0, 10.0, 10.0, 10.0
  
  Optimal solution: \( \hat{a}_1^* = 1.138074, \hat{a}_2^* = -0.193991, \hat{b}_1^* = 0.085576, \hat{b}_2^* = -0.803568 \)
  
  Objective function: \( J^* = 0.781373 \)

- **Initial conditions:** \( \hat{a}_1 = 1.0, \hat{a}_2 = -1.0, \hat{b}_1 = -0.5, \hat{b}_2 = -0.5 \)
  
  Range : 0.5, 0.5, 0.5, 0.5
  
  Optimal solution: \( \hat{a}_1^* = 1.138189, \hat{a}_2^* = -0.194133, \hat{b}_1^* = 0.085570, \hat{b}_2^* = -0.803570 \)
  
  Objective function: \( J^* = 0.781373 \)

- **Initial conditions:** \( \hat{a}_1 = -1.0, \hat{a}_2 = 0.5, \hat{b}_1 = 1.0, \hat{b}_2 = -0.5 \)
  
  Range : 3.0, 3.0, 3.0, 3.0
  
  Optimal solution: \( \hat{a}_1^* = 1.138188, \hat{a}_2^* = -0.194107, \hat{b}_1^* = 0.085573, \hat{b}_2^* = -0.803566 \)
Objective function: \( J^* = 0.781373 \)

- Initial conditions: \( \hat{a}_1 = 0.2, \hat{a}_2 = 0.4, \hat{b}_1 = 0.6, \hat{b}_2 = 0.8 \)
  
  Range : 5.0, 5.0, 5.0, 5.0

Optimal solution: \( \* \hat{a}_1 = 1.138135, \* \hat{a}_2 = -0.194122, \* \hat{b}_1 = 0.085561, \* \hat{b}_2 = -0.803573 \)

Objective function: \( J^* = 0.781373 \)

Therefore the model takes the form

\[
G_2(z) = \frac{1.138388z - 0.194374}{z^2 + 0.085556z - 0.803568} \tag{44}
\]

On the other hand, the models obtained by the techniques of [25], [26] and [27] are, respectively

\[
G_2(z) = \frac{z - 0.1481}{z^2 + 0.0687z - 0.8142} \tag{45}
\]

\[
G_2(z) = \frac{0.5611z - 0.2842}{1.3132z^2 - 1.9586z + 0.7281} \tag{46}
\]

and

\[
G_2(z) = \frac{z - 0.052}{z^2 + 0.1027z - 0.8195} \tag{47}
\]
Fig. 2 Step responses of the system given by (34) and its reduced order models given by (44), (45) and (46)

Table 2. Comparison of integral-squared error (ISE)

<table>
<thead>
<tr>
<th>Model</th>
<th>ISE of unit-step response</th>
</tr>
</thead>
<tbody>
<tr>
<td>(44)</td>
<td>0.7814</td>
</tr>
<tr>
<td>(45)</td>
<td>1.0844</td>
</tr>
<tr>
<td>(46)</td>
<td>1.9176</td>
</tr>
<tr>
<td>(47)</td>
<td>0.8554</td>
</tr>
</tbody>
</table>

The step responses of the system (34) and the models (44), (45) when plotted (Fig.2) were seen to be close to each other but model (46) when plotted was seen to be very poor against models (44) and (45). The ISE corresponding to (44)-(47) are shown in Table 2. This shows some improvement realized from the methods [8], [24] and [25].

4 Conclusions
A novel method for obtaining a reduced-order model of a given (stable) SISO discrete time system based on minimization of ISE has been developed. The method allows both the numerator and denominator coefficients of the models as free parameters in the process of optimization, and guarantees that a stable system is reduced to a stable model.
The problem of formulating the objective function is circumvented by introducing a set of equality constraints.

References:


Stable Optimal Model Reduction Of Linear Discrete Time Systems


