AMO - Advanced Modeling and Optimization, Volume 11, Number 4, 2009

On Solving Fully Fuzzified Linear Fractional Programs

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Abstract

In an earlier work [Pop and Stancu Minasian, 2008], we proposed a method of solving the fully fuzzified linear fractional programming (FFLFP) problem. In this paper, we propose another method of solving the FFLFP problem. First, analogically using the Charnes-Cooper method, we transform the linear fractional programming problem into a linear one. Next, problem of maximizing a function with triangular fuzzy value is transformed into a deterministic multiple objective linear programming problem with quadratic constraints. We apply the extension principle of Zadeh to add fuzzy numbers, an approximate version of the same principle to multiply and divide fuzzy numbers and the Kerre's method to evaluate a fuzzy constraint. Disjunctive constraints are also taken into consideration here. An illustrative numerical example is given to clarify the developed theory and the proposed algorithm.

Keywords: Fuzzy programming, Triangular fuzzy number, Fractional programming, Disjunctive constraints.

***AMO - Advanced Modeling and Optimization. ISSN: 1841-4311**

1 Introduction

There have been significant developments in the theory and applications of fractional programming in the last decades. For more informations about fractional programming problems, the reader may consult the bibliography with 491 entries presented by Stancu-Minasian, 2006, covering mainly the years from 1997 to 2005, and which gives a clear idea of the amount of work that is been in the field in the recent years.

In most real-world situations, the possible values of coefficients of a linear fractional programming problem are often only imprecisely or ambiguously known to the expert [Sakawa et al., 1992]. With this observation in mind, it would be certainly more appropriate to interpret the coefficients as fuzzy numerical data.

Generally, two types of problems implying fuzzy uncertainity are studied in the literature. Fuzzy approaches to solve deterministic problems could be developped and also fuzzy models, implying fuzzy goals and fuzzy coefficients, could be defined and solved.

A fuzzy satisficing method was used by Sakawa and Yano, 1988, to solve multiple objective linear fractional programming problem (MOLFPP). Sakawa et al., in 1992, introduced a general concept of Pareto optimal solution and treated two types of fuzzy goals (called fuzzy equal and fuzzy min). Luhandjula, 1984, used a linguistic variable approach to solve MOLFPP. Imprecise aspirations of the decision-maker are represented by structured linguistic variable.

The concept of linguistic variable has been introduced by Zadeh (see, for example [Dubois and Prade, 1987]). The aggregation of membership functions is done with a compensatory operator which does not guarantee the efficiency of the optimal solution. Dutta, Tiwari and Rao, 1992, modified the linguistic approach of Luhandjula [op.cit.] such to obtain efficient solution. In [Stancu-Minasian and Pop, 2003] some shortcomings in the work [Dutta et al., 1992] are pointed out and a correct proof of Dutta's main theorem is given. Moreover, Stancu-Minasian and Pop, in 2003, noticed that the method presented in [Dutta et al., 1992] only works efficiently if some quite restrictive hypotheses are satisfied. Chakraborty and Gupta, 2002, described a new fuzzy method to solving MOLFPP improuving the complexity of computations by defining fuzzy goals for a deterministic multiple objective linear programming problem.

In [Li and Chen, 1996] a fuzzy linear fractional programming model with fuzzy coefficients is established and the concept and mathematical definition of the fuzzy optimal value were presented. A new method and an approximate algorithm for solving the model were also given. In [Sakawa and Nishizaki, 2000] an interactive method is presented to solving two-level linear fractional programming problems with fuzzy parameters. In [Sakawa and Kato, 1998] multiple objective linear fractional programming problems with block angular structure involving fuzzy numbers were formulated and, through the introduction of extended Pareto optimality concepts an interactive fuzzy satisficing method for linear programming was presented.

Sakawa et al., in 2001, developed an interactive fuzzy programming method for multi-level 0-1 programming problems with fuzzy parameters through genetic algorithms. The authors derived efficiently a satisfactory solution by updating satisfactory levels of the decision makers with considerations of overall satisfactory balance among all levels (the fuzzy goals of the decision makers at all levels being determined in preamble).

Sakawa and Kato, in 1998, presented an interactive satisficing method for structured MOLFPP with fuzzy numbers in the objective functions and in the constraints. The authors changed the fuzzy problem in a deterministic one using fuzzy intervals. Sakawa et al., in 1999, developped an interactive fuzzy method for two-level linear fractional programming problems with fuzzy parameters by defining fuzzy goals for decidents on both levels. The fiting goal of the first decident and the slak between levels for each α-cut are evaluated.

Buckley and Feuring, in 2000, considered the fully fuzzified linear programming problem (FFLP) by establishing all the coefficients and variables of a linear program as being fuzzy quantities. They transform the fully fuzzified programming problem in a multiple objective deterministic problem (MODP) which, in the general case treated, is non-linear. Then, the problem is transformed in a multiple objective fuzzy problem with the help of which the authors explore the entire set of the Pareto-optimal solutions of the MODP. The solving of the multiple objective fuzzy problem is being made by using a genetic algorithm leading to feasible solutions for the initial problem.

In order to find the properties describing the set of efficient solutions of the multiple objective problem, two different methods are used in [Buckley and Feuring, 2000]: one is a classic parametric method and the other, called method of flexible programming, is similar to the method of goal programming.

We are going to allow all the parameters to be fuzzy and we obtain what we have called the fully fuzzified linear fractional programming (FFLP)problem. Hashemi et al., in 2006, proposed a two-phase approach to find the optimal solutions of the FFLP problem based on the comparison of mean and standard deviation of fuzzy numbers. In the first phase maximizes the possibilistic mean value of fuzzy objective function and obtains a set of feasible solutions.The second phase minimizes the standard deviation of the original fuzzy objective function, by considering all basic feasible solutions obtained at the end of the first phase.

In [Ghatee and Hashemi, 2007] authors considered the fully fuzzified minimal cost flow problem(MCFP), that is, the problem of finding the least transportation cost of a single commodity through a capaciated network in which the supply and demand of nodes and the capacity and cost of edges are fuzzy numbers. They sort fuzzy numbers by an order using a ranking function and transform the fully fuzzified MCFP into three crisp problems solvable in polynomial time.

In [Mikaeilvand et al., 2008] is proposed a new method to solve FFLP by using a linear ranking function for defuzzifying the FFLP problem.

[Mehra et al., 2007] proposed a new method to compute an (α, β)-acceptable optimal solution and an (α, β)-acceptable optimal value of the FFLP problem where $\alpha \in [0, 1]$ and $\beta \in [0, 1]$ is the grade of satisfacion associated with the fuzzy objective function and with the fuzzy constraints, respectively.

In [Li and Fang, 2009] is consider the problem of minimizing a linear fractional function subject to a system of sup-T equations with a continuous Archimedean triangular norm T and show that this can be reduced to a 0-1 linear fractional optimization problem in polynomial time.

In 2008 Toksari suggests the use of a Taylor series for fuzzy multiobjective linear fractional programming problems.

In a previous paper [Pop and Stancu Minasian, 2008], we proposed a method to solve the fully fuzzified linear fractional programming problem, where all the variables and parameters are represented by triangular fuzzy numbers. The proposed approach is first to transform the original fuzzy problem into a deterministic multiple objective linear fractional programming problem with quadratic constraints. This transformation is obtained by using the extension principle of Zadeh and Keree's method for the evaluation of the fuzzy constraints.

The idea is also of transforming the fully fuzzified linear fractional problem in a multiple objective deterministic problem, by establishing the coefficients and the variables of the problem as triangular fuzzy numbers aggregated with fuzzy operators obtained through applying the Zadeh's extension principle (for additions and subtractions) and an approximate version of it (for multiplications and divisions).

The Charnes-Cooper transformation (see [Charnes and Cooper, 1962]) to obtain a fully fuzzified linear programming problem is applied before aggregating fuzzy quantities.

For the fuzzy inequalities evaluated by using the Kerre's method [Kerre, 1982], we find the general form of the deterministic equivalent problem. Generally, a mathematical programming problem with many objective functions and with disjunctive non-linear constraints is obtained. Using the method proposed by Patkar and Stancu-Minasian in 1982, the system of disjunctive constraints is replaced with an "and" classic system. The non-linearity of some constraints, kept the same after the transformation, will be treated with the classical methods. The case of a fully fuzzified linear fractional programming (FFLFP) is considered. The model of such a problem is presented in Section 2.

2 FFLFP model

Let us consider the fully fuzzified linear fractional programming problem

$$\max \left(\overline{Z} = \frac{\sum\limits_{j=1}^{n} \overline{C_j X_j} + \overline{C_0}}{\sum\limits_{j=1}^{n} \overline{D_j X_j} + \overline{D_0}} \right) \tag{1}$$

subject to

$$\begin{cases} \overline{M_i} = \sum\limits_{j=1}^{n} \overline{A_{ij} X_j} - \overline{B_i} \leq \overline{0}, & i = 1, ..., m \\ \overline{X_j} \geq \overline{0}, \qquad j = 1, ..., n \end{cases} \tag{2}$$

where

- $\left(\overline{C_j}\right)_{j=1,\ldots,n}$, $\overline{C_0}$ and $\left(\overline{D_j}\right)_{j=1,\ldots,n}$, $\overline{D_0}$ represent the coefficients of the linear fractional objective function,
- $\left(\overline{A_{ij}}\right)_{i=1,\ldots,m}^{j=1,\ldots,n}$ and $\left(\overline{B_i}\right)_{i=1,\ldots,m}$ represent the coefficients and the right hand side of the linear constraints respectively,

- $\left(\overline{X_j}\right)_{j=1,\ldots,n}$ represents the decision variables.

Here it is customary to assume that the denominator in (1) is strictly positive for any $\overline{X_j}$ in the feasible region. Moreover, in this paper we will assume that the nominator in (1) is strictly positive. The meaning of "strictly positive" will be explained later. The notation $\overline{Y}$ means that Y represents a fuzzy quantity.

The paper is divided into 6 sections. The aggregation and comparision of triangular fuzzy numbers is presented in Section 3. As we will see in the next section, if $\overline{C_j}$, $\overline{C_0}$, $\overline{D_j}$, $\overline{D_0}$, $\overline{B_i}$, $\overline{X_j}$, $\overline{A_{ij}}$ are triangular fuzzy numbers for each $i = 1, ..., m$ and $j = 1, ..., n$, then $\overline{Z}$ and $\overline{M_i}$ are also triangular fuzzy numbers for each $i = 1, ..., m$. The inequalities in constraints (2) are considered as evaluated according to Kerre's method.

In Section 4 we propose a method of solving problem (1)-(2) when all initial fuzzy quantities are described with triangular fuzzy numbers. According to [Buckley and Feuring, 2000], the solving of problem (1)-(2) assumes to define what the "maximum" of a fuzzy number, i.e. $max\overline{Z}$, means and the evaluation of the fuzzy inequality $\overline{M_i} \le \overline{0}$. In Section 5, to illustrate our method, we consider a numerical example which we are then going to solve by the new method. Short concluding remarks are made in Section 6.

3 Triangular fuzzy numbers - aggregation and comparision

The purpose of this section is to recall some concepts which will be needed in the sequel.

Definition 1 ([4]) A triangular fuzzy number $\overline{Y}$ is a triplet $\left(y^1, y^2, y^3\right) \in R^3$. The membership function of $\overline{Y}$ is defined in connection with the real numbers y^1, y^2, y^3 as follows:

$$\overline{Y}(x) = \begin{cases} 0, & x \in (-\infty, y^1) \\ \dfrac{x - y^1}{y^2 - y^1}, & x \in [y^1, y^2] \\ \dfrac{x - y^3}{y^2 - y^3}, & x \in (y^2, y^3] \\ 0, & x \in (y^3, \infty) \end{cases}$$

$\overline{Y}(x)$ represents a number in $[0, 1]$, which is the membership function of $\overline{Y}$ evaluated in x.

The extension principle was formulated by Zadeh (see, for example [Zimmermann, 1985]) in order to extend the known models implying fuzzy elements to the case of fuzzy entities. Applying this principle the following definitions of the addition and subtraction of triangular fuzzy numbers result:

Definition 2 Being given two triangular fuzzy numbers $\overline{A} = \left(a^1, a^2, a^3\right), \overline{B} = \left(b^1, b^2, b^3\right)$, $a^1, a^2, a^3, b^1, b^2, b^3 \in R$, we define the sum and the subtraction as:

(i) $\overline{A} + \overline{B} = \left(a^1 + b^1, a^2 + b^2, a^3 + b^3\right),$

(ii) $\overline{A} - \overline{B} = \left(a^1 - b^3, a^2 - b^2, a^3 - b^1\right).$

Applying the principle of extension to multiply triangular fuzzy numbers it is not obtained a triangular fuzzy number. According to [Uhrig and Tsoukalas, 1997] we could use $\alpha-$cuts method to describe the membership function of the result. We will use the approximate version of Zadeh principle in order to multiply and to divide two fuzzy numbers. We recall here the coresponding definition introduced in [Pop and Stancu, 2008].

Definition 3 Being given two triangular fuzzy numbers $\overline{A} = \left(a^1, a^2, a^3\right), \overline{B} = \left(b^1, b^2, b^3\right)$, $a^1, a^2, a^3, b^1, b^2, b^3 \in R$, we define the multiplication and division as:

(i) $\overline{A} \cdot \overline{B} = \left(a^1 b^1, a^2 b^2, a^3 b^3\right),$

(ii) $\dfrac{\overline{A}}{\overline{B}} = \left(\dfrac{a^1}{b^3}, \dfrac{a^2}{b^2}, \dfrac{a^3}{b^1}\right).$

One definition for the inequality between two fuzzy numbers was introduced by Kerre in 1982. The main concept of comparison of fuzzy numbers is based on the comparison of areas determined by membership functions.

The inequality

$$(m_1, m_2, m_3) \leq (0, 0, 0) \tag{3}$$

of triangular fuzzy numbers was introduced in [Pop and Stancu-Minasian, 2008]. The following proposition describes it through a system of deterministic disjunctive constraints.

Proposition 1 *([Pop and Stancu-Minasian, 2008]) Given a triangular fuzzy number* $\overline{M} = (m_1, m_2, m_3)$*, relation (3) holds if and only if the following system of disjunctive constraints is satisfied:*

$$m_3 \leq 0, \tag{4}$$

or

$$\begin{cases} m_1 \leq 0 \leq m_2 \\ m_1 (m_1 + m_2 + m_3) \geq m_2 m_3 \end{cases}, \tag{5}$$

or

$$\begin{cases} m_2 \leq 0 \leq m_3 \\ m_3 (m_1 + m_2 + m_3) \leq m_1 m_2 \end{cases}. \tag{6}$$

By describing equality through a double inequality, we can describe now, in the same context, the "Kerre equality" between two fuzzy numbers which do not have the same parameters, which is an equality of the form $\overline{M} = \overline{N}$. The equality with "zero" of a triangular fuzzy number is general enough to describe the equality of two triangular fuzzy numbers because the deduction of the two triangular fuzzy numbers can be done at any time in order for the equality to be equal to "zero".

The following proposition gives necessary and sufficient conditions for equality

$$(m_1, m_2, m_3) = (0, 0, 0) \tag{7}$$

to hold.

Proposition 2 *Given the triangular fuzzy number* $\overline{M} = (m_1\,,\ m_2,\ m_3)$*, the Kerre equality (7) holds if and only if the parameters of* $\overline{M}$ *satisfy the following disjunctive system of constraints:*

$$m_1 = m_2 = m_3 = 0,$$

or

$$\begin{cases} m_1 \leq 0 \leq m_2 \\ m_1 (m_1 + m_2 + m_3) = m_2 m_3 \end{cases},$$

or

$$\begin{cases} m_2 \leq 0 \leq m_3 \\ m_3 (m_1 + m_2 + m_3) = m_1 m_2 \end{cases}.$$

4 Solving method for FFLFP

Based on the concepts discussed in the previous section, in this paragraph we describe an alternative method of building a deterministic problem equivalent with Problem (1)-(2).

First, we transform problem (1)-(2) into a fully fuzzified linear programming problem using the Charnes-Cooper transformation, and obtain the following problem

$$\max\left(\sum_{j=1}^{n}\overline{C_jY_j}+\overline{C_0T}\right) \tag{8}$$

subject to

$$\begin{cases}\sum\limits_{j=1}^{n}\overline{A_{ij}Y_j}-\overline{B_iT}\leq 0, i=1,...,m\\ \sum\limits_{j=1}^{n}\overline{D_jY_j}+\overline{D_0T}=\overline{1},\\ \overline{T}\geq 0\\ \overline{Y_j}\geq 0, j=1,...,n.\end{cases} \tag{9}$$

After aggregating the fuzzy quantities according to Definition 2 and Definition 3 we change the maximization of the objective function described by a triangular fuzzy number with the maximization of the three components of the fuzzy number, as described in [Buckley and Feuring, 2000], and we obtain the following deterministic multiple objective linear program with fuzzy constraints

$$\max\left(\sum_{j=1}^{n}c_j^1y_j^1+c_0^1t^1,\sum_{j=1}^{n}c_j^2y_j^2+c_0^2t^2,\sum_{j=1}^{n}c_j^3y_j^3+c_0^3t^3\right) \tag{10}$$

subject to

$$\begin{cases}\left(\sum\limits_{j=1}^{n}a_{ij}^1y_j^1-b_i^3t^3,\sum\limits_{j=1}^{n}a_{ij}^2y_j^2-b_i^2t^2,\sum\limits_{j=1}^{n}a_{ij}^3y_j^3-b_i^1t^1\right)\leq\overline{0}, i=1,...,m\\ \left(\sum\limits_{j=1}^{n}d_j^1y_j^1+d_0^1t^1,\sum\limits_{j=1}^{n}d_j^2y_j^2+d_0^2t^2,\sum\limits_{j=1}^{n}d_j^3y_j^3+d_0^3t^3\right)=\overline{1}\\ 0\leq y_j^1\leq y_j^2\leq y_j^3,\ j=1,...,n\\ 0\leq t^1\leq t^2\leq t^3\end{cases} \tag{11}$$

System (11) consists of m inequalities between fuzzy numbers, $n+1$ non-negativity constraints of variables and $2(n+1)$ relations between components of the fuzzy numbers which represent the decision variables.

By applying Proposition 1 to the inequality of index i from (11) we obtain the following disjunctive system of constraints:

$$S_i = S_i^1 \cup S_i^2 \cup S_i^3$$

where

$$S_i^1 = \begin{cases} \sum\limits_{j=1}^{n} a_{ij}^3 y_j^3 - b_i^1 t^1 \le 0,\ i = 1, ..., m \\ 0 \le t^1 \le t^2 \le t^3,\ 0 \le y_j^1 \le y_j^2 \le y_j^3, j = 1, ..., n \end{cases}$$

$$S_i^2 = \begin{cases} \sum\limits_{j=1}^{n} a_{ij}^1 y_j^1 - b_i^3 t^3 \le 0 \le \sum\limits_{j=1}^{n} a_{ij}^2 y_j^2 - b_i^2 t^2,\ i = 1, ..., m \\ \left(\sum\limits_{j=1}^{n} a_{ij}^1 y_j^1 - b_i^3 t^3\right)\left(\sum\limits_{j=1}^{n} a_{ij}^1 y_j^1 - b_i^3 t^3 + \sum\limits_{j=1}^{n} a_{ij}^2 y_j^2 - b_i^2 t^2 + \sum\limits_{j=1}^{n} a_{ij}^3 y_j^3 - b_i^1 t^1\right) \ge \\ \qquad \ge \left(\sum\limits_{j=1}^{n} a_{ij}^2 y_j^2 - b_i^2 t^2\right)\left(\sum\limits_{j=1}^{n} a_{ij}^3 y_j^3 - b_i^1 t^1\right),\ i = 1, ..., m \\ 0 \le t^1 \le t^2 \le t^3,\ 0 \le y_j^1 \le y_j^2 \le y_j^3, j = 1, ..., n \end{cases}$$

$$S_i^3 = \begin{cases} \sum\limits_{j=1}^{n} a_{ij}^2 y_j^2 - b_i^2 t^2 \le 0 \le \sum\limits_{j=1}^{n} a_{ij}^3 y_j^3 - b_i^1 t^1,\ i = 1, ..., m \\ \left(\sum\limits_{j=1}^{n} a_{ij}^3 y_j^3 - b_i^1 t^1\right)\left(\sum\limits_{j=1}^{n} \left(a_{ij}^1 y_j^1 + a_{ij}^2 y_j^2 + a_{ij}^3 y_j^3\right) - b_i^3 t^3 - b_i^2 t^2 - b_i^1 t^1\right) \le \\ \qquad \le \left(\sum\limits_{j=1}^{n} a_{ij}^1 y_j^1 - b_i^3 t^3\right)\left(\sum\limits_{j=1}^{n} a_{ij}^2 y_j^2 - b_i^2 t^2\right),\ i = 1, ..., m \\ 0 \le t^1 \le t^2 \le t^3,\ 0 \le y_j^1 \le y_j^2 \le y_j^3, j = 1, ..., n \end{cases}$$

By applying Proposition 2 to the equality from (11) we obtain the disjunctive system of constraints $S_0 = S_0^1 \cup S_0^2 \cup S_0^3$ where

$$S_0^1 = \begin{cases} \sum\limits_{j=1}^{n} d_j^1 y_j^1 + d_0^1 t^1 = 1 \\ \sum\limits_{j=1}^{n} d_j^2 y_j^2 + d_0^2 t^2 = 1 \\ \sum\limits_{j=1}^{n} d_j^3 y_j^3 + d_0^3 t^3 = 1 \\ 0 \le y_j^1 \le y_j^2 \le y_j^3, j = 1, ..., n \\ 0 \le t^1 \le t^2 \le t^3 \end{cases}$$

$$S_0^2 = \begin{cases} \sum\limits_{j=1}^{n} d_j^1 y_j^1 + d_0^1 t^1 \leq 1 \\ \sum\limits_{j=1}^{n} d_j^2 y_j^2 + d_0^2 t^2 \geq 1 \\ \left(\sum\limits_{j=1}^{n} d_j^1 y_j^1 + d_0^1 t^1 \right) \left(\sum\limits_{j=1}^{n} \left(d_j^1 y_j^1 + d_j^2 y_j^2 + d_j^3 y_j^3 \right) + d_0^1 t^1 + d_0^2 t^2 + d_0^3 t^3 - 3 \right) \\ \qquad = \left(\sum\limits_{j=1}^{n} d_j^3 y_j^3 + d_0^3 t^3 - 1 \right) \left(\sum\limits_{j=1}^{n} d_j^2 y_j^2 + d_0^2 t^2 \right) \\ 0 \leq t^1 \leq t^2 \leq t^3,\ 0 \leq y_j^1 \leq y_j^2 \leq y_j^3, j = 1, ..., n \end{cases}$$

$$S_0^3 = \begin{cases} \sum\limits_{j=1}^{n} d_j^2 y_j^2 + d_0^2 t^2 \leq 1 \\ \sum\limits_{j=1}^{n} d_j^3 y_j^3 + d_0^3 t^3 \geq 1 \\ \left(\sum\limits_{j=1}^{n} d_j^3 y_j^3 + d_0^3 t^3 - 1 \right) \left(\sum\limits_{j=1}^{n} \left(d_j^1 y_j^1 + d_j^2 y_j^2 + d_j^3 y_j^3 \right) + d_0^1 t^1 + d_0^2 t^2 + d_0^3 t^3 - 3 \right) = \\ \qquad = \left(\sum\limits_{j=1}^{n} d_j^1 y_j^1 + d_0^1 t^1 \right) \left(\sum\limits_{j=1}^{n} d_j^2 y_j^2 + d_0^2 t^2 \right) \\ 0 \leq t^1 \leq t^2 \leq t^3,\ 0 \leq y_j^1 \leq y_j^2 \leq y_j^3, j = 1, ..., n \end{cases}$$

Consequently, Problem (1)-(2) is reduced to the following deterministic multiple objective linear programming problem (MOLP) subject to a conjunctive system of disjunctive non-linear constraints:

$$\max \left(\sum_{j=1}^{n} c_j^1 y_j^1 + c_0^1 t^1, \sum_{j=1}^{n} c_j^2 y_j^2 + c_0^2 t^2, \sum_{j=1}^{n} c_j^3 y_j^3 + c_0^3 t^3 \right) \tag{12}$$

subject to

$$\left(\bigcap_{i=1}^{m} \left(S_i^1 \cup S_i^2 \cup S_i^3 \right) \right) \cap \left(S_0^1 \cup S_0^2 \cup S_0^3 \right). \tag{13}$$

According to the method described in [Patkar and Stancu-Minasian, 1985], we shall consider the indicator variables $\delta_i^1,\ \delta_i^2,\ \delta_i^3$ $(i = 0, 1, ..., m)$ in order to eliminate the disjuncture and to obtain (14) as system of conjunctive constraints, where M represents the upper bounds for all expressions which appear in constraints and L represents the lower bounds for expressions which appear in equalities.

$$\sum_{j=1}^{n} a_{ij}^3 y_j^3 - b_i^1 t^1 \le \left(1-\delta_i^1\right) M,\ i=1,...,m$$
$$\sum_{j=1}^{n} a_{ij}^1 y_j^1 - b_i^3 t^3 \le \left(1-\delta_i^2\right) M,\ i=1,...,m$$
$$-\sum_{j=1}^{n} a_{ij}^2 y_j^2 + b_i^2 t^2 \le \left(1-\delta_i^2\right) M,\ i=1,...,m$$
$$-\left(\sum_{j=1}^{n} a_{ij}^1 y_j^1 - b_i^3 t^3\right)\left(\sum_{j=1}^{n} a_{ij}^1 y_j^1 - b_i^3 t^3 + \sum_{j=1}^{n} a_{ij}^2 y_j^2 - b_i^2 t^2 + \sum_{j=1}^{n} a_{ij}^3 y_j^3 - b_i^1 t^1\right) +$$
$$+\left(\sum_{j=1}^{n} a_{ij}^2 y_j^2 - b_i^2 t^2\right)\left(\sum_{j=1}^{n} a_{ij}^3 y_j^3 - b_i^1 t^1\right) \le \left(1-\delta_i^2\right) M,\ i=1,...,m$$
$$\sum_{j=1}^{n} a_{ij}^2 y_j^2 - b_i^2 t^2 \le \left(1-\delta_i^3\right) M,\ i=1,...,m$$
$$-\sum_{j=1}^{n} a_{ij}^3 y_j^3 + b_i^1 t^1 \le \left(1-\delta_i^3\right) M,\ i=1,...,m$$
$$\left(\sum_{j=1}^{n} a_{ij}^3 y_j^3 - b_i^1 t^1\right)\left(\sum_{j=1}^{n} a_{ij}^1 y_j^1 - b_i^3 t^1 + \sum_{j=1}^{n} a_{ij}^2 y_j^2 - b_i^2 t^2 + \sum_{j=1}^{n} a_{ij}^3 y_j^3 - b_i^1 t^1\right) -$$
$$-\left(\sum_{j=1}^{n} a_{ij}^1 y_j^1 - b_i^3 t^3\right)\left(\sum_{j=1}^{n} a_{ij}^2 y_j^2 - b_i^2 t^2\right) \le \left(1-\delta_i^3\right) M,\ i=1,...,m$$
$$\delta_i^1+\delta_i^2+\delta_i^3 \ge 1,\ \delta_i^1,\delta_i^2,\delta_i^3 \in \{0,1\},\ i=1,...,m$$
$$\sum_{j=1}^{n} d_j^1 y_j^1 + d_0^1 t^1 - 1 \le \left(1-\delta_0^1\right) M$$
$$\sum_{i=1}^{n} d_j^1 y_j^1 + d_0^1 t^1 - 1 \le \left(1-\delta_0^1\right) L$$
$$\sum_{j=1}^{n} d_j^2 y_j^2 + d_0^2 t^2 - 1 \le \left(1-\delta_0^1\right) M$$
$$\sum_{j=1}^{n} d^2 y_j^2 + d_0^2 t^2 - 1 \ge \left(1-\delta_0^1\right) L$$

$$\sum_{j=1}^{n} d_j^3 y_j^3 + d_0^3 t^3 - 1 \le \left(1-\delta_0^1\right) M$$
$$\sum_{j=1}^{n} d_j^3 y_j^3 + d_0^3 t^3 - 1 \ge \left(1-\delta_0^1\right) L$$
$$\sum_{j=1}^{n} d_j^1 y_j^1 + d_0^1 t^1 - 1 \le \left(1-\delta_0^2\right) M$$
$$-\sum_{j=1}^{n} d_j^2 y_j^2 + d_0^2 t^2 + 1 \le \left(1-\delta_0^2\right) L$$
$$\left(\sum_{j=1}^{n} d_j^1 y_j^1 + d_0^1 t^1\right)\left(\sum_{j=1}^{n} \left(d_j^1 y_j^1 + d_j^2 y_j^2 + d_j^3 y_j^3\right) + d_0^1 t^1 + d_0^2 t^2 + d_0^3 t^3 - 3\right) -$$
$$-\left(\sum_{j=1}^{n} d_j^3 y_j^3 + d_0^3 t^3 - 1\right)\left(\sum_{j=1}^{n} d_j^2 y_j^2 + d_0^2 t^2\right) \le \left(1-\delta_0^2\right) M$$

$$\left(\sum_{j=1}^{n} d_j^1 y_j^1 + d_0^1 t^1\right)\left(\sum_{j=1}^{n}\left(d_j^1 y_j^1 + d_j^2 y_j^2 + d_j^3 y_j^3\right) + d_0^1 t^1 + d_0^2 t^2 + d_0^3 t^3 - 3\right) -$$
$$- \left(\sum_{j=1}^{n} d_j^3 y_j^3 + d_0^3 t^3 - 1\right)\left(\sum_{j=1}^{n} d_j^2 y_j^2 + d_0^2 t^2\right) \geq \left(1-\delta_0^2\right) L$$
$$\sum_{j=1}^{n} d_j^2 y_j^2 + d_0^2 t^2 - 1 \leq \left(1-\delta_0^3\right) M$$
$$-\sum_{j=1}^{n} d_j^3 y_j^3 + d_0^3 t^3 + 1 \leq \left(1-\delta_0^3\right) M$$
$$\left(\sum_{j=1}^{n} d_j^3 y_j^3 + d_0^3 t^3 - 1\right)\left(\sum_{j=1}^{n}\left(d_j^1 y_j^1 + d_j^2 y_j^2 + d_j^3 y_j^3\right) + d_0^1 t^1 + d_0^2 t^2 + d_0^3 t^3 - 3\right) -$$
$$- \left(\sum_{j=1}^{n} d_j^1 y_j^1 + d_0^1 t^1\right)\left(\sum_{j=1}^{n} d_j^2 y_j^2 + d_0^2 t^2\right) \leq \left(1-\delta_0^3\right) M$$

$$\left(\sum_{j=1}^{n} d_j^3 y_j^3 + d_0^3 t^3 - 1\right)\left(\sum_{j=1}^{n}\left(d_j^1 y_j^1 + d_j^2 y_j^2 + d_j^3 y_j^3\right) + d_0^1 t^1 + d_0^2 t^2 + d_0^3 t^3 - 3\right) -$$
$$- \left(\sum_{j=1}^{n} d_j^1 y_j^1 + d_0^1 t^1\right)\left(\sum_{j=1}^{n} d_j^2 y_j^2 + d_0^2 t^2\right) \geq \left(1-\delta_0^3\right) L$$
$$\delta_0^1 + \delta_0^2 + \delta_0^3 \geq 1,\ \delta_0^1, \delta_0^2, \delta_0^3 \in \{0,1\}$$
$$0 \leq t^1 \leq t^2 \leq t^3,\ 0 \leq y_j^1 \leq y_j^2 \leq y_j^3, j = 1, ..., n \tag{14}$$

Solving Problem (10)-(14) will allow us to obtain solution $\left(y_j^1, y_j^2, y_j^3\right)_{j=1,\ldots,n}$ $\left(t^1, t^2, t^3\right)$, $\left(\delta_i^1, \delta_i^2, \delta_i^3\right)_{i=0,\ldots,m}$ namely the triangular fuzzy numbers $\left(\overline{y_j}\right)_{j=1,\ldots,n}$ and $\bar{t}$, which represent the solution of problem (8)-(9). The optimal solution of Problem (1)-(2) is $\overline{x_j} = \dfrac{\overline{y_j}}{\bar{t}}$, $j = 1, ..., n$.

5 Computation results

In order to illustrate the method of solving fully fuzzified linear fractional programs, let us consider the deterministic linear fractional problem (15)-(16) which was also considered in [Pop and Stancu-Minasian, 2008].

$$\max\left(z = \frac{x_1 - x_2 + 1}{x_1 + x_2 + 2}\right) \tag{15}$$

subject to

$$\begin{cases} x_1 + x_2 \leq 2 \\ x_1 - x_2 \leq 1 \\ x_1, x_2 \geq 0 \end{cases} . \tag{16}$$

The optimal solution of this problem is $x^1 = 1$, $x^2 = 0$ and the optimal value of z is $\frac{2}{3} = 0.66667$.

We attach now to this problem a fully fuzzified problem, considering its real number coefficient m as being symmetric triangular fuzzy number $\overline{m}$ of spread 2, having the following form

$$\overline{m} = \left(m^1, m^2, m^3\right),\ m^1 = m - 1,\ m^2 = m,\ m^3 = m + 1.$$

Thus the fully fuzzified problem which we want to solve is

$$\max\left(z = \frac{\overline{c_1 x_1} + \overline{c_2 x_2} + \overline{c_0}}{\overline{d_1 x_1} + \overline{d_2 x_2} + \overline{d_0}}\right) \tag{17}$$

subject to

$$\begin{cases} \overline{a_{11} x_1} + \overline{a_{12} x_2} - \overline{b_1} \leq \overline{0} \\ \overline{a_{21} x_1} + \overline{a_{22} x_2} - \overline{b_2} \leq \overline{0}, \\ \overline{x_1}, \overline{x_2} \geq \overline{0} \end{cases} \tag{18}$$

where $\overline{c} = [(0,1,2), (-2,-1,0), (0,1,2)]$, $\overline{d} = [(0,1,2), (0,1,2), (1,2,3)]$,

$$\overline{a} = \begin{bmatrix} (0,1,2) & (0,1,2) \\ (0,1,2) & (-2,-1,0) \end{bmatrix}, \quad \overline{b} = \begin{bmatrix} (1,2,3) \\ (0,1,2) \end{bmatrix}.$$

are coefficients values.

Now we transform problem (17)-(18) into a fully-fuzzified linear programming problem using Charnes-Cooper transformation and we obtain:

$$\max\left(\overline{c_1 y_1} + \overline{c_2 y_2} + \overline{c_0}\overline{t}\right)$$

subject to

$$\begin{cases} \overline{a_{11} y_1} + \overline{a_{12} y_2} - \overline{b_1}\overline{t} \leq \overline{0} \\ \overline{a_{21} y_1} + \overline{a_{22} y_2} - \overline{b_2}\overline{t} \leq \overline{0} \\ \overline{d_1}\overline{y_1} + \overline{d_2}\overline{y_2} + \overline{d_0}\overline{t} = \overline{1} \\ \overline{y_1}, \overline{y_2}, \overline{t} \geq \overline{0}. \end{cases} \tag{19}$$

According to the method described in Section 4, in order to obtain the solution of this problem we solve the following multiple objective linear problem still having fuzzy constraints:

$$\max\left(f_1(y,t) = -2y_2^3, f_2(y,t) = y_1^2 - y_2^2 + t^2, f_3(y,t) = 2y_1^3 + 2t^3\right) \tag{20}$$

subject to

$$\left\{\begin{array}{c}\left(-3t^3, y_1^2+y_2^2-2t^2, 2y_1^3+2y_2^3-t^1\right) \leq \overline{0} \\ \left(-2y_2^3-2t^3, y_1^2-y_2^2-t^2, 2y_1^3\right) \leq \overline{0} \\ \left(t^1-1, y_1^2+y_2^2+2t^2-1, 2y_1^3+2y_2^3+3t^3-1\right) = \overline{0} \\ \overline{y_1}, \overline{y_2}, \bar{t} \geq \overline{0}.\end{array}\right. \tag{21}$$

By evaluating the fuzzy constraints with Kerre's method, described in Section 3, we obtain the following equivalent system of disjunctive constraints

$$\begin{gathered}[R_1 \cup (R_2 \cap R_3) \cup (R_4 \cap R_5 \cap R_6)] \cap \\ \cap [R_7 \cup (R_8 \cap R_9 \cap R_{10}) \cup (R_{11} \cap R_{12})] \cap \\ \cap [(R_{13} \cap R_{14} \cap R_{15}) \cup (R_{16} \cap R_{17} \cap R_{18}) \cup (R_{19} \cap R_{20} \cap R_{21})] \cap \\ \cap R_{22} \cap R_{23} \cap R_{24} \cap R_{25} \cap R_{26} \cap R_{27} \cap R_{28} \cap R_{29} \cap R_{30},\end{gathered} \tag{22}$$

where

$$\begin{array}{lll}
R_1 & : & S_1 = 2y_1^3+2y_2^3-t^1 \leq 0 \\
R_2 & : & S_2 = -y_1^2-y_2^2+2t^2 \leq 0 \\
R_3 & : & \left\{\begin{array}{l} S_3 = \left(y_1^2+y_2^2-2t^2\right)\left(2y_1^3+2y_2^3-t^1\right)+ \\ \quad +3t^3\left(y_1^2+y_2^2-2t^2+2y_1^3+2y_2^3-t^1-3t^3\right) \leq 0 \end{array}\right. \\
R_4 & : & S_4 = y_1^2+y_2^2-2t^2 \leq 0 \\
R_5 & : & S_5 = -2y_1^3-2y_2^3+t^1 \leq 0 \\
R_6 & : & \left\{\begin{array}{l} S_6 = \left(2y_1^3+2y_2^3-t^1\right)\left(y_1^2+y_2^2-2t^2+2y_1^3+2y_2^3-t^1-3t^3\right)+ \\ \quad +3t^3\left(y_1^2+y_2^2-2t^2\right) \leq 0 \end{array}\right. \\
R_7 & : & S_7 = y_1^3 \leq 0 \\
R_8 & : & S_8 = -y_2^3-t^3 \leq 0 \\
R_9 & : & S_9 = -y_1^2+y_2^2+t^2 \leq 0 \\
R_{10} & : & S_{10} = y_1^3\left(y_1^2-y_2^2-t^2\right)+\left(y_2^3+t^3\right)\left(y_1^2-y_2^2-t^2+2y_1^3-2y_2^3-2t^3\right) \leq 0 \\
R_{11} & : & S_{11} = y_1^3\left(y_1^2-y_2^2-t^2+2y_1^3-2y_2^3-2t^3\right)+\left(y_2^3+t^3\right)\left(y_1^2-y_2^2-t^2\right) \leq 0 \\
R_{12} & : & S_{12} = y_1^2-y_2^2-t^2 \leq 0 \\
R_{13} & : & S_{13} = t^1-1 = 0 \\
R_{14} & : & S_{14} = y_1^2+y_2^2+2t^2-1 = 0
\end{array}$$

$$R_{15} \quad : \quad S_{15} = 3t^3 + 2y_1^3 + 2y_2^3 - 1 = 0$$
$$R_{16} \quad : \quad S_{16} = t^1 - 1 \leq 0$$
$$R_{17} \quad : \quad S_{17} = -y_1^2 - y_2^2 - 2t^2 + 1 \leq 0$$
$$R_{18} \quad : \quad \begin{cases} S_{18} = \left(t^1 - 1\right)\left(t^1 - 3 + y_1^2 + y_2^2 + 2t^2 + 2y_1^3 + 2y_2^3 + 3t^3\right) - \\ \qquad - \left(2y_1^3 + 2y_2^3 + 3t^3 - 1\right)\left(y_1^2 + y_2^2 + 2t^2 - 1\right) = 0 \end{cases}$$
$$R_{19} \quad : \quad S_{19} = y_1^2 + y_2^2 + 2t^2 - 1 \leq 0$$
$$R_{20} \quad : \quad S_{20} = -2y_1^3 - 2y_2^3 - 3t^3 + 1 \leq 0$$

$$R_{21} \quad : \quad \begin{cases} S_{21} = \left(2y_1^3 + 2y_2^3 + 3t^3 - 1\right)\left(t^1 - 3 + y_1^2 + y_2^2 + 2t^2 + 2y_1^3 + 2y_2^3 + \right. \\ \qquad \left. + 3t^3\right)\left(t^1 - 1\right)\left(y_1^2 + y_2^2 + 2t^2 - 1\right) = 0 \end{cases}$$
$$R_{22} \quad : \quad S_{22} = y_1^2 - y_1^3 \leq 0$$
$$R_{23} \quad : \quad S_{23} = y_1^1 - y_1^2 \leq 0$$
$$R_{24} \quad : \quad S_{24} = -y_1^1 \leq 0$$
$$R_{25} \quad : \quad S_{25} = y_2^2 - y_2^3 \leq 0$$
$$R_{26} \quad : \quad S_{26} = y_2^1 - y_2^2 \leq 0$$
$$R_{27} \quad : \quad S_{27} = -y_2^1 \leq 0$$
$$R_{28} \quad : \quad S_{28} = t^2 - t^3 \leq 0$$
$$R_{29} \quad : \quad S_{29} = t^1 - t^2 \leq 0$$
$$R_{30} \quad : \quad S_{30} = -t^1 \leq 0$$

We transform the system of disjunctive constraints in a system of conjunctive constraints using the indicator variables $\delta_1, \delta_2, \ldots, \delta_9$, and we obtain

$$\begin{aligned} & S_1 \leq (1 - \delta_1) M_1 \\ & S_2 \leq (1 - \delta_2) M_2 \\ & S_3 \leq (1 - \delta_2) M_3 \\ & S_4 \leq (1 - \delta_3) M_4 \\ & S_5 \leq (1 - \delta_3) M_5 \\ & S_6 \leq (1 - \delta_3) M_6 \\ & \delta_1 + \delta_2 + \delta_3 \geq 1 \\ & S_7 \leq (1 - \delta_4) M_7 \\ & S_8 \leq (1 - \delta_5) M_8 \end{aligned}$$

$$\begin{aligned}
&S_{10} \le (1-\delta_5)\, M_{10} \\
&S_{11} \le (1-\delta_6)\, M_{11} \\
&S_{12} \le (1-\delta_6)\, M_{12} \\
&\delta_4 + \delta_5 + \delta_6 \ge 1 \\
&(1-\delta_7)\, m_{13} \le R_{13} \le (1-\delta_7)\, M_{13} \\
&(1-\delta_7)\, m_{14} \le R_{14} \le (1-\delta_7)\, M_{14} \\
&(1-\delta_7)\, m_{15} \le R_{15} \le (1-\delta_7)\, M_{15} \\
&S_{16} \le (1-\delta_8)\, M_{16} \\
&S_{17} \le (1-\delta_8)\, M_{17}\, (1-\delta_8)\, m_{18} \le R_{18} \le (1-\delta_8)\, M_{18} \\
&S_{19} \le (1-\delta_9)\, M_{19} \\
&S_{20} \le (1-\delta_9)\, M_{20} \\
&(1-\delta_9)\, m_{21} \le R_{21} \le (1-\delta_9)\, M_{21} \\
&\delta_7 + \delta_8 + \delta_9 \ge 1 \\
&\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6, \delta_7, \delta_8, \delta_9 \in \{0,1\} \\
&R_{22}, R_{23}, R_{24}, R_{25}, R_{26}, R_{27}, R_{28}, R_{29}, R_{30}
\end{aligned} \tag{23}$$

where each coefficient M_i represents a value large enough and each coefficient m_i represents a value small enough so that the left hand side of constraint R_i can take it only outside the feasible region defined by the constraints (22).

In order to obtain a synthesis function of the three objective functions from (20) and applying to it the results presented in [Stancu-Minasian, 1984], we use the importance coefficients $\pi_1 = 0.1$, $\pi_2 = 0.8$ and $\pi_3 = 0.1$ respectively. The optimum of the synthesis function $\pi_1 f_1 + \pi_2 f_2 + \pi_3 f_3$ is reached in

$$\left(\overline{y^*}, \overline{t^*}\right) = \left[\overline{y_1} = (0.05, 0.333, 0.339)\,, \overline{y_2} = (0,0,0)\,, \bar{t} = (0, 0.333, 0.4)\right],$$

$$\delta^* = (0,0,1,0,1,1,0,0,1)\,.$$

It follows that the solution of problem (17)-(18) is

$$\overline{x^*} = \left[\overline{x_1} = (0.125, 1, \infty)\,, \overline{x_2} = (0,0,0)\right],$$

which approximates very closely the pair of real numbers $(1,0)$ representing the solution (x_1, x_2) of problem (15)-(16). Also triangular fuzzy number $\overline{z^*} = (0, 0.666667, \infty)$ approximates very closely the real number $\frac{2}{3}$ which represents the optimal value of problem (15)-(16).

6 Concluding remarks

In this paper we have proposed a method of solving the fully fuzzified linear fractional programming problems where all the parameters and variables are triangular fuzzy numbers. This method differs of the method presented by us in [Pop and Stancu-Minasian, 2008].

We have transformed the fractional problem into a linear one using Charnes-Cooper method. After that, the problem of maximizing a triangular fuzzy number was transformed into a deterministic multiple objective linear programming problem with quadratic constraints. We have applied the extension principle of Zadeh and an approximate version of it to aggregate fuzzy numbers. Kerre's method was applied in order to evaluate each fuzzy constraint. The results obtained by Buckley and Feuring in 2000 have been taken into consideration here. The method has added extra zero-one variables for treating disjunctive constraints.

The example illustrated the fact that the developed method can be succesfully applied in solving fuzzy programming problems. Noticing that the division of variable triangular fuzzy numbers (which involves approximation) was avoided here, we can conclude that this method is more efficient than the previous one, presented in [Pop and Stancu-Minasian, 2008].

Few ideas for possible researches which should be explored can be mentioned. A similar method for solving fully fuzzified linear fractional programming problems, where all the parameters and variables are trapezoidal fuzzy numbers can be described. A stochastic approach could be studied for problem (1)-(2). A comparison study can be carried out between the fuzzy approach and the stochastic approach for solving problem (1)-(2). These extensions are under investigation.

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