Invariant Graph Characteristics from the Zeta Function

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Abstract

We describe here the problem of enumerating graphs, in particular, Directed Acyclic Graphs (DAGs, in acronym), or Bayesian Networks (BNs). All them will be analyzed by the elegant and useful Ihara Zeta Function.

Keywords: Graph Theory, Combinatorics, Enumeration of graphs, Graph labeling, Asymptotic Analysis.

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1. Introduction

The *Ihara zeta function* was firstly defined by Ihara studying discrete subgroups of the two-by-two special linear groups. Zeta functions of graphs were studied not only by Ihara [34] [35], but many other works on it, as may be Sunada, Hashimoto, Bartholdi, and Bass. So, Jean Pierre Serre [47] suggested can be reinterpreted graph-theoretically, in his book *Trees.* And it was Toshikazu Sunada, in 1985, who put this suggestion into practice.

Storm [54] defined the Ihara-Selberg zeta function for *Hypergraphs.*

The *Ihara zeta function* is denoted by ς_G, and it will be defined by

$$\varsigma_G(s) \equiv \left[\prod_p \left(1 - s^{L(p)} \right) \right]^{-1}$$

or equivalently,

$$\prod_p \left(1 - s^{L(p)} \right) \equiv \frac{1}{\varsigma_G(s)}$$

Such formula is analogous to the Euler product for the Riemann zeta function.

In fact, we have an infinite product to work with.

The product is taken over all prime walks, p, on the graph G, being $L(p)$ the *length of the prime p.*

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Bass [3] [4] [5] proved, among other results, that this product is in fact a rational function.

Let G be a graph, and

$$A \equiv \left(a_{ij}\right)$$

its adjacency matrix, which as we known, will be a

$$\left(c\left\{V\left(G\right)\right\}\ x\ c\left\{V\left(G\right)\right\}\right) - matrix$$

with entries

$$a_{ij} \equiv \begin{cases} cardinal\ of\ undirected\ edges\ connecting\ n_i\ to\ n_j,\ being\ i \neq j \\ double\ of\ the\ cardinal\ of\ loops\ at\ the\ node\ n_i,\ if\ i = j \end{cases}$$

As our graphs have no loops neither multiple edges, such entries will be either zero or one, according to the adjacency or not adjacency of its respective pairs of nodes.

Suppose that we take now D, as the diagonal matrix such that its entry d_i is the degree of the i-th node minus one, and let

$$r - 1 = c\left\{E\left(G\right)\right\} - c\left\{V\left(G\right)\right\}$$

Then,The Ihara zeta function will be expressed as

$$\zeta_G\left(s\right)^{-1} \equiv \left(1 - s^2\right)^{r-1} \det\left(I - As + Du^2\right)$$

It is very interesting to look at the logarithmic derivative of the Ihara zeta function,

$$s\tfrac{d}{ds}\ln\left[\zeta_G\left(s\right)\right]$$

We have

$$\ln\left[\zeta_G\left(s\right)\right] = \ln\left[\prod_p\left(1 - s^{L(p)}\right)\right]^{-1} =$$

$$= \ln\left[\prod_p\left(1 - s^{L(p)}\right)^{-1}\right] = -\sum_p \ln\left(1 - s^{L(p)}\right)$$

and taking the derivative,

$$\frac{d}{ds}\left[\zeta_G(s)\right] = \frac{d}{ds}\left[-\sum_p \ln\left(1-s^{L(p)}\right)\right] = -\sum_p \frac{1}{1-s^{L(p)}}\frac{d}{ds}\left(1-s^{L(p)}\right) =$$

$$= -\sum_p \frac{1}{1-s^{L(p)}}\left[-L(p)\right]\ L(p) = \sum_p \frac{L(p)\ s^{L(p)-1}}{1-s^{L(p)}}$$

and now multiplying by s,

$$s\frac{d}{ds}\left[\zeta_G(s)\right] = s\sum_p \frac{L(p)\ s^{L(p)-1}}{1-s^{L(p)}} = \sum_p \frac{L(p)\ s^{L(p)}}{1-s^{L(p)}}$$

But such expression may be improved by the *geometric series* identity,

$$\sum_{n\in\mathbf{N}^*} s^n = \frac{1}{1-s}$$

giving

$$s\frac{d}{ds}\left[\zeta_G(s)\right] = \sum_p L(p)\ s^{L(p)}\left[1+s^{L(p)}+s^{2L(p)}+s^{3L(p)}+\ ...\right] =$$

$$= \sum_p L(p)\left[s^{L(p)}+s^{2L(p)}+s^{3L(p)}+s^{4L(p)}+\ ...\right]$$

If we denote

$$N_k = \sum_{p:\ L(p)|k} L(p)$$

it holds

$$s\frac{d}{ds}\left[\zeta_G(s)\right] = \sum_{k\in\mathbf{N}} N_k\ s^k$$

Where the coefficient N_k, being associated with the term s^k, will report us the number of prime paths with a number of nodes which divides k.

We may translate the results from closed geodesics to cycles, considering the elements of a group, or instead, working on graphs.

We say that a *cycle* is *primitive,* if it is not the r-multiple of some other cycles, being $r \geq 2$.

A closed geodesic which is not the power of another is called a *prime geodesic.*

It is possible to establish an equivalence relation on the set of prime paths in the graph. According such relation, two closed paths are *equivalent,* if they are the same path with a different starting point.

Mathematically expressed, let E and F two cycles,

$$E = (e_i)_{i=1}^{n}$$
$$and$$
$$F = (f_i)_{i=1}^{n}$$

Both will be equivalent, if there is a fixed $k \in \mathbf{Z}/n\mathbf{Z}$, such that

$$e_i = f_{i+k}$$
$$\forall i \in \mathbf{Z}/n\mathbf{Z}$$

This says that all indices are considered module n *(mod n).* So, we are reducing to the first cycle. And we are imposing the aforementioned equivalence relation via cyclic permutation.

Therefore, two prime geodesics are said to be *equivalent,* if one is obtained from another by a cyclic permutation of edges.

An *equivalence class of prime geodesics* is called a *prime geodesic class,* or simply a *prime,* $\wp$. Given a path, γ, we denote by $L(\gamma)$ its *length.* The length of a prime, $\wp$, is the length of any of its representatives. A *prime cycle* is the equivalence class of primitive cycles which have no backtracking or tails.

Given a graph, G, with a symmetric digraph, D_G, we may associate to each of its edges (e) an invariant, s_e, and so, define the function

$$g(C) \equiv \prod_{e\ edge\ in\ C} s_e$$

for a prime cycle. Such function g inform about the edges involved in a particular prime cycle, and also about how many times they are used.

The *edge zeta function of* G is a function of $s_e \in \mathbf{C}$, given by

$$\varepsilon_G(s) \equiv \prod_{prime\ cycles} [1 - g(C)]^{-1}$$

And specializing each s_e to s, we obtain the *Ihara function of G,* by

$$\zeta_G(s) \equiv \prod_{prime\ cycles} \left[1 - s^{L(C)}\right]^{-1}$$

being denoted as $L(C)$ the *length of a representative of the prime cycle,* $[C]$.

For a finite graph where every node has at least degree two, *the zeta function give us the cardinal of edges in the graph.*

And what happens in the case of degree one? When the edges are incident to a degree one node, they are ignored by the zeta function. It is because in this situation, only admits two possibilities, either have a backtracking or a tail.

If we consider as I the identity matrix, it holds

$$\zeta_G(s) = (1-s)^{\chi(G)} \det\left[I - sA + s^2(D-I)\right]$$

where $\chi(G)$ will be the *Euler characteristics,* or Euler number (a notable topological invariant), of the graph G.

Recall that this number is reachable by

$$\chi(G) = c[V(G)] - c[E(G)]$$

The *Ihara zeta function,* ς_G, will be always representable as the reciprocal of a polynomial

$$\varsigma_G(s) \equiv \frac{1}{\det\ (I\ -\ T\ s)}$$

that is,

$$\varsigma_G(s)^{-1} \equiv \det\ \ (I\ -\ T\ s) = \det\ \ (I\ -\ s\ T)$$

where T is the *edge adjacency operator* (Hashimoto, 1990).

Therefore, the edge zeta function is the reciprocal of a multivariate polynomial in, at most, $2\ c(E(G))$ variables.

An important fact indeed, because it implies the possibility of to be computed in polynomial time.

Observe that the maximum degree of the reciprocal of the Zeta function is the double of the number of edges in the graph, its size, i. e. $2\ c(E(G))$.

Recall that the *adjacency operator, A,* is acting on the space of functions defined on the set of nodes of $G = (V,\ E)$.

Being *o(e)* and *t(e)* the *origin* and *terminus* of *e,* respectively, it is defined by

$$(Af)\ \ (x) = \sum_{e \in E_x} f\ \ [t(e)]\,,\ where\ E_x \equiv \{e \in E : o(e) = x\}$$

We may define the *directed edge matrix, T*, of a graph, G. For this, firstly we need a labeling of the directed edges (in the associated digraph).

Then, such matrix, T, has as its *(i, j)-entry,*

$$t_{ij} = \begin{cases} 1,\ if\ t(e_i) = o(e_j)\,,\ and\ e_i \neq e_j \\ 0,\ otherwise \end{cases}$$

(Bass, 1992) also gave a determinant formula involving the adjacency operator.

Recall that the set of eigenvalues of a matrix, *A,* is called its *Spectrum,* usualy denoted either by $Spec(A)$, or $\lambda(A)$,

$$\lambda(A) \equiv \{\lambda_i\}_{i=1}^{n}$$

being λ_i its eigenvalues, for every index $i \in I$.

Then, we have as *determinant of A,*

$$\det(A) = \prod_{i=1}^{n} \lambda_i = \begin{pmatrix} \lambda_1 & 0 & ... & 0 \\ 0 & \lambda_2 & ... & 0 \\ 0 & 0 & ... & 0 \\ 0 & 0 & ... & \lambda_n \end{pmatrix}$$

And the set of graph eigenvalues of the adjacency matrix is known as the *Spectrum of the Graph.*

If we have a graph, G, with $n_i - fold$ degenerate eigenvalues, λ_i, the usual expression for its spectrum will be

$$Spec(G) = \prod_{i=1}^{m} (\lambda_i)^{n_i}$$

Sometimes it will be denoted by

$$\begin{pmatrix} \lambda_1 & \lambda_2 & ... & \lambda_m \\ n_1 & n_2 & ... & n_m \end{pmatrix}$$

Let G and H be two graphs. They are called *cospectral*, if its adjacency matrices have the same spectra.

An important result of the theory of Ihara zeta function, characterizing this question on *k-regular graphs*, is the following

Theorem ($Mellein$).
Suppose G and H are both *k-regular graphs.*
Then, G and H are *cospectral* if and only if

$$\varsigma_G(s) = \varsigma_H(s), \ \forall s$$

So, *whenever a k-regular graph is uniquely determined by its spectrum,* it is possible to conclude that *its Ihara zeta function is also uniquely determined.*

For any *Graph, G,* the function ς_G can be expressed in terms of the Riemann zeta function, ς, for different dimension values, n.

So,

If $n = 1$, *then* $\varsigma_G(s) = 2\varsigma(s)$.

If $n = 2$, *then* $\varsigma_G(s) = 4\varsigma(s-1)$.

If $n = 3$, *then* $\varsigma_G(s) = 4\varsigma(s-2) + 2\varsigma(s)$.

If $n = \infty$, *then* $\varsigma_G(s) = \frac{8}{3}\varsigma(s-3) + \frac{16}{3}\varsigma(s-1)$.

Recall that $\varsigma_G(s)$ is a *decreasing function* of s.

That is,

$$\varsigma_G(s_1) > \varsigma_G(s_2)\,,\ if\ s_1 < s_2$$

And in the limit, if $n \to \infty$, when s is next to the *transition point,* it holds

$$\varsigma_G(s) = \frac{2^n\ \varsigma(s - n + 1)}{\Gamma(n)}$$

If the average degree of nodes, also called *mean coordination number of the graph,* is finite, then there exists exactly a value of s, denoted $s_{transition}$, where the zeta function changes from infinite to finite, or vice versa.

It is also called *dimension of the Graph.*

2. Enumerating Graphs

About the foundations of Graph Theory, there exists many adequate surveys, as [9] [10] [27].

For *Graphical Enumeration,* it may be convenient to see for instance [6] [11] [14] [21] [28], among others.

A very elegant construct, if certainly difficult, may be through the *Ihara Zeta function*.

It is also possible to use *Generating Functions* to count labeled DAGs. For this mathematical construct, it is necessary to make intervene the *Inclusion-Exclusion Principle (IEP).*

So, if we take the set of *n-essential graphs,* and denote its cardinal by a_n, applying the aforementioned IEP, we may obtain

$$a_n = \sum_{s=1,\ldots,p} (-1)^{s+1} \sum_{\substack{i_j \\ j\in\{1,\ldots,s\}}} c\left(A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_s}\right)$$

where

$$A_k = \{G \in E : k \text{ is a } \textit{terminal node} \text{ of } G\}\,,\ with\ k = 1,\ 2,\ \ldots,\ n \quad [*]$$

Let a_n and $a_{n'}$ be the $number\ of\ essential\ labeled\ n - DAGs$, and the $number\ of\ labeled\ n-DAGs$, respectively. Then, a_n is given by the recurrence equation

$$a_n = \sum_{s=1}^{n} (-1)^{s+1} C_{n,s} \left(2^{n-s} - n + s\right)^s a_{n-s}\,,\ with\ a_0 = 1$$

Whereas

$$a_n' = \sum_{s=1}^{n} (-1)^{s+1} C_{n,s} \left(2^{n-s}\right)^s a_{n-s}',\ with\ a_0' = 1$$

The new formula would be recursive, and it is a direct application of the *IEP.* From which, we can reach directly the equation.

We may rewrite the equation as

$$\sum_{s\,=\,0}^{n} (-1)^{n-s} C_{n,s} \left(2^{s} - s\right)^{n-s} a_{s} = 0,\ \textit{with}\ n \geq 1$$

Another case of application of *IEP* is to find the cardinal of the set of essential DAGs, *E,* with a set of labeled nodes, with labelings that belongs to $\{1, 2, \ldots, n\}$. For this, we start with a family of sets, as the aforementioned $\{A_{k}\}_{k=1}^{n}$. See the precedent formula [*], where A_{k} represents the subclass of graphs concluding at the node labeled by *k.*

Therefore, to know the cardinal of *E,* first we compute the intersection that appears in the last summatory, for *j = 1, 2, . . . , n,* being these

$$\sum_{\substack{i_j \\ j\in\{1,\ldots,s\}}} c\left(A_{i_1} \cap A_{i_2} \cap ... \cap\ A_{i_s}\right)$$

related with the aforementioned principle *(IEP).*
With the total allowed connection numbers, from a given node being

$$2^{n-s} - n + s$$

So, the number of possible ways of adding directed edges from the essential graph until all the s terminal nodes will be

$$[2^{n-s} - n + s]^{s}$$

3. Asymptotical behaviour

Analyzing the asymptotic behaviour of its ratios, i.e. studying the convergence of the quotient of cardinals, among the number of essential graphs, and the number of DAGs (acronym of Directed Acyclic Graphs), we may develop this so

$$A\left(n\right) \equiv \tfrac{a_n}{a_n}, \Rightarrow$$

$$\Rightarrow \lim_{n\to\infty} A\left(n\right) = \lim_{n\to\infty} \frac{\sum_{s=1}^{n} (-1)^{s+1} C_{n,s} \left(2^{n-s} - (n\ -\ s)\right)^{s} a_{n-s}}{\sum_{s=1}^{n} (-1)^{s+1} C_{n,s} \left(2^{n-s}\right)^{s} a_{n-s}},$$

being

$$A\left(n-s\right)\equiv\frac{a_{n-s}}{a_{n-s}}{}^{\prime}$$

and

$$\varsigma_G\left(n-s\right)\equiv\lim_{n\to\infty}\sum_{s=1}^{n}\left(\frac{(n\ -\ s)}{2^{n-s}}\right)^{s}$$

and turning to our initial step,

$$\lim_{n\to\infty}A\left(n\right)=\left[1-\lim_{n\to\infty}\varsigma_{G}\ \left(n-s\right)\right]\left[\lim_{n\to\infty}A\left(n-s\right)\right]$$

we can consider the series

$$\sum_{s=1}^{n}\ \frac{n-s}{2^{n-s}}\Rightarrow\sum_{s=1}^{n}\ \left[1-\frac{n\ -\ s}{2^{n-s}}\right]=n-\sum_{s=1}^{n}\ \frac{n\ -\ s}{2^{n-s}}$$

and its asymptotical behaviour, when $n\to\infty$. These may establish an analytical correspondence with a version of the Riemann zeta function, the so called *Ihara zeta function of the n-graph,* G_n.

But operating here on the increasing value of n – s, i.e. with $\zeta_G\ (n-s)$. Nevertheless, this proof would be very complex. Instead, we may apply here an interesting result, which permits to finalize our demonstration.

So, we obtain the ratio among terms of the series (by applying the precedent Lemma),

$$\lim_{n\to\infty}\frac{\left\{\sum_{s=1}^{n}\ (-1)^{s+1}\ C_{n,s}\left(2^{n-s}\ -(n\ -\ s)\right)^{s}\ a_{n-s}\right\}}{\left\{\sum_{s=1}^{n}\ (-1)^{s+1}C_{n,s}\left(2^{n-s}\right)^{s}\ a_{n-s}{}^{\prime}\right\}}=$$

$$=\lim_{n\to\infty}\frac{\left\{(-1)^{s+1}\ C_{n,s}\left(2^{n-s}\ -(n\ -\ s)\right)^{s}\ a_{n-s}\right\}}{\left\{(-1)^{s+1}\ C_{n,s}\left(2^{n-s}\right)^{s}\ a_{n-s}{}^{\prime}\right\}}=$$

$$=\lim_{n\to\infty}\frac{\left(2^{n-s}\ -(n\ -\ s)\right)^{s}\ a_{n-s}}{\left(2^{n-s}\right)^{s}\ a_{n-s}{}^{\prime}}=$$

$$=\left[\lim_{n\to\infty}\left(\frac{2^{n-s}\ -(n-s)}{2^{n-s}}\right)^{s}\right]\left[\lim_{n\to\infty}\frac{a_{n-s}}{a_{n-s}}{}^{\prime}\right]=$$

$$=\left[\lim_{n\to\infty}\left\{1-\varsigma_{G}\ \left(n-s\right)\right\}\right]\left[\lim_{n\to\infty}A\left(n-s\right)\right]$$

4. Applying the Ihara zeta function

We establish from now these auxiliary and useful notation

$$f\left(n-s\right)=1-\frac{n-s}{2^{n-s}}\Rightarrow f\left(n\right)=1-\frac{n}{2^{n}}$$

that is,

$$1-f\left(n-s\right)=\frac{n-s}{2^{n-s}}\Rightarrow 1-f\left(n\right)=\frac{n}{2^{n}}$$

But as we known

$$\lim_{n\to\infty}\frac{n-s}{2^{n-s}}=\lim_{n\to\infty}\frac{n}{2^{n}}=0^{+}$$

and by this procedure,

$$\left[1-\left\{f\left(n-s\right)\right\}\right]^{s}=1-\left(\frac{(n-s)}{2^{n-s}}\right)^{s}$$

Hence

$$\lim_{n\to\infty}\left[1-\left\{f\left(n-s\right)\right\}\right]^{s}=\lim_{n\to\infty}\left\{1-\left(\frac{(n-s)}{2^{n-s}}\right)^{s}\right\}=1^{-}$$

and

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{2^{n-s}-(n-s)}{2^{n-s}}\right)^{s}=n-\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{(n-s)}{2^{n-s}}\right)^{s}=$$

$$=n-\lim_{n\to\infty}\sum_{i=1}^{n}\left[1-f\left(n-s\right)\right]^{s}=n-\left[n-\zeta_{G}\left(n-s\right)\right]=\zeta_{G}\left(n-s\right)$$

These last terms must regulate the asymptotical behaviour, by its limit values.

And respect to its reciprocal function

$$\zeta_{G}^{-1}\left(n-s\right)=\lim_{n\to\infty}\left[\sum_{i=1}^{n}\left(1-\frac{(n-s)}{2^{n-s}}\right)^{-s}\right]$$

whith this may appears Φ_{1}^{-1} from Φ_{1}, as they are described in the subsequent step.

Note that we can take

$$f\left(n-s\right)=\frac{2^{n-s}-(n-s)}{2^{n-s}},\ \textit{for each } n\in\mathbf{N},\ \textit{once fixed } s$$

So, by

$$\zeta_{G}(n-s)=\lim_{n\to\infty}\sum_{i=1}^{n}\left[f(n-s)\right]^{s}$$

it holds

$$\begin{aligned}\exists\Phi_{A}&=\left[n-\lim_{n\to\infty}\sum_{i=1}^{n}\left[f(n-s)\right]^{s}\right]\left[\lim_{n\to\infty}A(n-s)\right]=\\&=\left[n-\lim_{n\to\infty}\varsigma_{G}(n-s)\right]\left[\lim_{n\to\infty}A(n-s)\right],\\&\textit{fixed } s\textit{, when } n \textit{ increases to } \infty\Rightarrow\\&\Rightarrow\Phi_{A}=\frac{1}{10\ \varsigma(5/2)}=\frac{1}{5\varsigma_{G}(5/2)}\simeq 0.07\end{aligned}$$

DAGs for each equivalence class, or equivalently,

$$\Phi_{A}^{-1}=10\ \varsigma(5/2)$$

hence, passing from Riemann to Ihara zeta function,

$$\Phi_{A}^{-1}=5/2\ \varsigma_{G}(7/2)\simeq 13.6$$

This will be the number of equivalence classes for each DAG.

So far, we have supposed *dimension one.*

Because in case of *dimension two,* where the new functions are denoted Φ_{B} and Φ_{B}^{-1}, respectively, it holds

$$\Phi_{B}=\frac{1}{10\ \varsigma(5/2)}=\frac{1}{5/2\ \varsigma_{G}(7/2)},\ \textit{and}\ \Phi_{B}^{-1}=4\ \varsigma(7/2-1)=4\ \varsigma(5/2)$$

And translating this from Riemann to Ihara Zeta Function, we obtain

$$\Phi_{B}^{-1}=5/2\ \varsigma_{G}(7/2)$$

Note that the “labeled” or “unlabeled” character of the considered graphs is relevant, because they are very distinct situations, giving so different ratios.

It is obvious that the enumeration of unlabeled essential graphs results more complex that in the labeled case. Also, its symmetries can be used.

Recall that permutating between the positions of two symmetric nodes is an operation that when acting on graphs, leaves the shape unaffected.

For the case of unlabeled graphs (Sunada, 1985), denoted here by the letter *"c"*, we have

$$c_n \equiv \textit{ cardinal of the set of unlabeled n-DAGs}$$
$$\textit{and}$$
$$c_n{}' \equiv \textit{ cardinal of the set of unlabeled essential n-DAGs}$$

And so, we can find (applying our previous *Lemma*) that

$$c_{n-s} \leq c\acute{}_{n-s}, \forall n, \ once\ fixed\ s \Rightarrow$$
$$\Rightarrow \exists \Phi_C = \lim_{n\to\infty} \left(\varsigma_G(n) \left[C\left(n-s\right)\right]\right),$$
$$fixed\ s,\ and\ when\ n \to \infty,$$
$$with\ C\left(n-s\right) \equiv \lim_{n\to\infty} \sum_{s=1}^{n} \frac{c_{n-s}}{c\acute{}_{n-s}} \Rightarrow$$
$$\Phi_C \equiv \{\lim_{n\to\infty} \varsigma_G\left(n\right)\} \{\lim_{n\to\infty} C\left(n-s\right)\} = \frac{1}{10}\ \varsigma\left(3/2\right) = \frac{1}{20}\varsigma_G\left(3/2\right) \simeq 0.26$$

essential graphs for each unlabeled DAG, in the case of dimension one.

And symmetrically,

$$\exists \left(\Phi_C\right)^{-1} \equiv \{\lim_{n\to\infty} \left[\varsigma_G\left(n\right)\right]\}^{-1} \{\lim_{n\to\infty} \left[C\left(n-s\right)\right]\}^{-1} = \frac{10}{\varsigma(3/2)} = \frac{20}{\varsigma_G(3/2)} \simeq 3.73$$

unlabeled DAGs for each essential graph, which coincides with our precedent analythical results.

In case of *dimension two,* it holds

$$\exists \Phi_C^* \equiv \lim_{n\to\infty} \left(\left\{\varsigma_G\ (n)\right\} \left[C^*\ (n-s)\right]\right) = \frac{1}{10}\ \varsigma\left(3/2\right) = \frac{1}{40}\varsigma_G\left(5/2\right) \simeq 0.26$$

and dually,

$$\exists \left(\Phi_C^*\right)^{-1} \equiv \lim_{n\to\infty} \left(\left\{\varsigma_G\ (n)\right\} \left[C^*\ (n-s)\right]\right)^{-1} = \frac{10}{\varsigma(3/2)} = \frac{4}{\varsigma_G(5/2)} \simeq 3.73$$

5. Conclusion

And it is so in the limit situation, reflecting the degree of fitness of the proposed model, based in analytical framework, to the precedent computational results, as the shown by [1], or [20].

References

[1] S. A. Anderson, D. Madigan, and M. D. Perlman (1997). A characterization of Markov equivalence classes for acyclic digraphs. *Ann. Statist.* **25**: 505-541.

[2] L. Bartholdi (1999). Counting paths in graphs. *Enseign. Math.* **45:** 83-131.

[3] H. Bass (1992). The Ihara-Selberg Zeta Function of a tree lattice. *Int. J. Math.*, **3**(6): 717-797.

[4] H. Bass (1989). Zeta functions of finite graphs and representations of p-adic groups. *Advanced Studies in Pure Math.*, **15**: 211-280.

[5] H. Bass (1992). The Ihara-Selberg zeta function of a tree lattice. *International J. Math.* **3:** 717-797.

[6] E. A. Bender, and S. Gill Williamson (2006). *Foundations of Combinatorics with Applications.* Dover Publ.

[7] E. A. Bender, L. B. Richmond, R. W. Robinson, and N. C. Wormald (1986). The asymptotic number of acyclic digraphs I. *Combinatorica* **6** (1): 15-22.

[8] E. A. Bender, and R. W. Robinson (1988). The asymptotic number of acyclic digraphs II. *J. Comb. Theory,* Serie **B44** (3): 363-369.

[9] B. Bollobás (1978). *Extremal Graph Theory.* Academic Press, New York. Reedited by Dover Publ.

[10] B. Bollobás (1998). *Modern Graph Theory.* Springer Verlag, New York.

[11] Ch. A. Charalambides (2002). *Enumerative Combinatorics.* Chapman and Hall/CRC.

[12] B. Clair, S. Mokhtari-Shargi (2001). Zeta functions of discrete subgroups acting on trees. *J. Algebra* **237:** 591-620.

[13] B. Clair, S. Mokhtari-Shargi (2002). Convergence of Zeta functions of graphs. *Proc. Amer. Math. Soc.* **130:** 1881-1886.

[14] B. Clair, S. Mokhtari-Shargi (2001). Zeta functions of discrete groups acting on trees. *J. Algebra* **237:** 591-620.

[15] P. Flajolet, and R. Sedgewick (2009). *Analytic Combinatorics.* Cambridge University Press.

[16] A. Garrido (2009). Bayesian Networks and Essential Graphs. *IEEE Computer Society Press.* Enlarged and modified paper of *Proc. CANS 2008,* Ed. Barna Lászlo Iantovics. Petru Maior University Press, Tirgu Mures, 2008. Accepted, to be appear, 12 pp.

[17] A. Garrido (2009). Asymptotic behaviour of Essential Graphs. *Electronic International Journal of Advanced Modeling and Optimization (EIJ-AMO).* Vol. 18, Issue Nr. **3:** 195-210.

[18] A. Garrido (2009). Enumerating Graphs. *Electronic International Journal of Advanced Modeling and Optimization (EIJ-AMO).* Vol. 18, Issue Nr. **3:** 227-246.

[19] A. Garrido (2009). Combinatorial Analysis by the Ihara zeta function of Graphs. *Electronic International Journal of Advanced Modeling and Optimization (EIJ-AMO).* Vol. 18, Issue Nr. **3:** 253-278.

[20] S. Gill Williamson (2002). *Combinatoris for Computer Science.* Dover Publ.

[21] S. B. Gillispie, and M. D. Perlman (2002). The size distribution for Markov equivalence classes of acyclic digraph models. *AI* **141** (1/2): 137-155.

[22] S. B. Gillispie, and M. D. Perlman (2001). Enumerating Markov equivalence classes of acyclic digraph models. *UAI 2001:* 171-177.

[23] I. P. Goulden, and D. M. Jackson (1983). *Combinatorial Enumeration.* Wiley, New York.

[24] P. J. Grabner, and B. Steinsky (2005). Asymptotic behaviour of the poles of a special generating function for acyclic digraphs. *Aequationes Mathematicae* **70** (3): 268-278.

[25] J. L. Gross (1987). *Topological graph theory.* Wiley-Interscience. New York.

[26] D. Guido, T. Isola, and M. L. Lapidus (2009). A trace on fractal graphs and the Ihara zeta function. *Trans. Amer. Math. Soc.* **361:** 3041-3070.

[27] F. Harary (1955). The number of linear, directed, rooted, and connected graphs. *Proc. Amer. Math. Soc.* (AMS) **78**: 445-463.

[28] F. Harary (1957). The number of oriented graphs. *Michigan Math. J.* Volume 4, Issue **3:** 221-224.

[29] F. Harary (1969). *Graph Theory.* Addison-Wesley, Reading, Mass., USA.

[30] F. Harary, and E. M. Palmer (1973). *Graphical Enumeration.* Academic Press. New York.

[31] F. Harary (1965). *Structural Models: An Introduction to the Theory of Graphs.* Wiley, New York.

[32] K. Hashimoto (1989). Zeta functions of finite graphs and representations of p-adic groups. *Adv. Stud. Pure Math.* **15:** 211-280.

[33] K. Hashimoto (1990). On Zeta and L-functions of finite graphs. *I. J. Math.* **1**: 381-396.

[34] K. Hashimoto (1992). Artin-type L-functions and the density theorem for prime cycles of finite graphs. *Internat. J. Math.* **3:** 809-826.

[35] D. A. Hejhal; M. C. Gutzwiller; and A. M. Odlyzko (1999). *Emerging Applications of Number Theory.* Springer.

[36] Y. Ihara (1966). Discrete subgroups of PL (2, k). *Proc. Sympos. Pure MathBoulder, Colorado*: 219-235.

[37] Y. Ihara (1996). On discrete subgroups of the two by two projective linear groups over p-adic fields. *J. Math. Soc. Japan* **18**: 219-235.

[38] J. W. Kennedy, K. A. Mc Keon, E. M. Palmer, and R. W. Robinson (1990). Asymptotic number of symmetries in locally restricted trees. *Discrete Applied Mathematics* **26 (1):** 121-124.

[39] M. Kotani, and T. Sunada (2000). Zeta Functions of Finite Graphs. *J. Math. Si. Univ. Tokio,* **7**: 7-25.

[40] J. Matousek, and J. Nesetril (2008). *An Invitation to Discrete Mathematics.* Oxford University Press.

[41] J. Matousek, J. Nesetril, and H. Mielke (2007). *Diskrete Mathematik. Eine Entdeckungsreise.* Springer-Lehrbuch.

[42] S. Northshields (1998). A note on the zeta function of a graph. *Journal of Combinatorial Theory, Series B,* vol. 74, No. **2:** 408-410.

[43] G. Pólya (1937, 1987). Kombinatorische Anzahlbestimmungen für Grüppen, Graphen und chemische Verbindungen. *Acta Mathematica* **68:** 145-254.

Translated with commentaries by R. C. Read, in *Combinatorial enumeration of groups, graphs...* Springer.

[44] P. Ren, R. C. Wilson, and E. R. Hancock (2008). Graph Characteristics from the Ihara Zeta Function. *Lecture Notes in Computer Science (LCNS),* Vol. **5342**: 257-266. Springer-Verlag. Berlin-Heidelberg.

[45] J. Riordan (2002). *Introduction to Combinatorial Analysis.* Dover Publ.

[46] R. W. Robinson (1973). Counting labelled acyclic digraphs, in Harary, F. (ed.), *New Directions in the Theory of Graphs:* 239-279. Academic Press. New York.

[47] R. W. Robinson (1970). Enumeration of acyclic digraphs. *Proc. Second Chapel Hill Conf. on Combinatorial Mathematics and Its Applications:* 391-399. Univ. North Carolina, Chapel Hill.

[48] R. W. Robinson (1977). Counting unlabeled acyclic digraphs. *Combinatorial Mathematics V* (C. H. C. Little Ed.). Springer. *Lecture Notes in Mathematics* **622:** 28-43.

[49] I. Sato (2008). Bartholdi zeta functions for hypergraphs. *TGT 20,* Japan.

[50] J. - P. Serre (1980). *Trees.* Translated from the French by J. Stillwell. Spinger Verlag.

[51] R. P. Stanley (1986). *Enumerative Combinatorics.* Vols. I and II. Wadsworth and Brooks/Cole Advanced Books and Software.

[52] B. Steinsky (2004). Asymptotic behaviour of the number of labeled essential acyclic digraphs and labeled chain graphs. *Graphs and Combinatorics* **20** (3): 399-411.

[53] B. Steinsky (2003). Enumeration of labeled chain graphs and labeled essential directed acyclic graphs. *Discrete Mathematics* **270** (1-3): 266-277.

[54] C. Storm (2008). Some graph properties determined by edge zeta functions. *arxiv.org/PS.*

[55] M. Studeny (1987). Asymptotic behaviour of empirical multiinformation. *Kybernetika* **23** (2): 124-135.

[56] M. Studeny and M. Volf (1999). A graphical characterization of the largest chain graphs. *IJAR* **20**: 209-236.

[57] T. Sunada (1985). L-functions in geometry and some applications. *Lecture Notes in Mathematics* **1201**: 266-284.

[58] A. Terras, and H. Stark (1996). Zeta functions of Finite Graphs and Coverings. Part I. *Advances in Mathematics* **121:** 124-165.

[59] A. Terras, and H. Stark (2000). Zeta functions of Finite Graphs and Coverings. Part II. *Advances in Mathematics* **154:** 132-195.

[60] A. Terras, and H. Stark (2007). Zeta functions of Finite Graphs and Coverings. Part III. *Advances in Mathematics* **208:** 467-489.

[61] A. Terras, M. D. Horton, and H. Stark (2006). What are Zeta functions of Graphs and What are They Good For? *Contemporary Mathematics,* Vol **415:** 173-190.

[62] A. Terras, M. D. Horton, and H. Stark (2008). Zeta functions of Weighted Graphs and Covering Graphs. *Proc. Symp. Pure Math.* Vol **77**,

"Analysis on Graphs", edited by Exner, Keating, Kuchment, Sunada and Teplyaev. *AMS.*

[63] D. Zywina (2005). The Zeta Function of a Graph. Expanded lecture notes from a talk entitled *"The Prime Number Theorem for Graphs".* Mathematics Department, University of Berkeley. Belongs to the *"Many Cheerful Facts" Seminar.*