Combinatorial analysis by the Ihara zeta function of graphs

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Abstract

We analyze here some new results about the asymptotic behaviour of the ratio which compares the cardinal of labeled Directed Acyclic Graphs and the corresponding cardinal of its equivalence classes in the sense of Markov. And also the parallel and comparative study for the unlabeled case. All them by Zeta Functions; more concretely, the Ihara-Selberg Zeta Function of a Graph.

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1. Riemann zeta function

A *Zeta Function* (denoted by ζ) will be given by a sum of infinite powers. See, for instance, about its foundations [3] [4] [5] [12] [13] [24].
More concretely, it can be expressable by a Dirichlet series of this type

$$\zeta(s) \equiv \sum_{n\in\mathbf{N}} [f\ (n)]^{s}$$

There exists different functions, in Mathematics, known as Zeta functions, all them included into the slot of *"special functions"*.

The more known is due to Riemann, but the more useful, in our case, is the so called *Ihara-Selberg function of a graph.* So, among the Dirichlet Series, we found a very useful tool, in fields as Number Theory, Probability or Cryptography. See, for emerging applications [33]. Because in Number Theory we are interested in the properties of the primes. With this purpose, Euler was perhaps the first to consider the so called *Riemann Zeta Function,*

$$\varsigma(s) = \sum_{n\in\mathbf{N}} \frac{1}{n^{s}} = \prod_{p\ prime} \left(1 - p^{-s}\right)^{-1}$$

$$with\ s \in \mathbf{C}$$

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Where it appears as *product of Euler.*

In fact, Euler consider the case when $s \in \mathbf{R}$; later generalization to $s \in \mathbf{C}$ is due to Riemann.

And because it has a bounded sequence of coefficients, these series converge absolutely to an analytical function, on the complex open half-plane of s such that

$$Re(s) > 1$$

It diverges on the symmetrical open half-plane of s, in the complex plane

$$Re(s) < 1$$

About the defined function on the first region, it admits analytic continuation to all $\mathbf{C}$, except when $s = 1$.

For $s = 1$, this series is formally identical to the Harmonic series, which diverges.

As a consequence, it is a meromorphic function of s, being in particular, holomorphic in a region of the complex plane, showing one pole in $s = 1$, with residue equal to 1.

Recall that a function is *holomorphic* when it is complex differentiable, and will be *meromorphic* when it is holomorphic on almost all $\mathbf{C}$, except in a set of isolated points, which are called the *poles* of the function.

Euler found a closed formula for $\zeta(2k)$, when $k \in \mathbf{N}$.

It will be expressed by

$$\zeta(2k) = \frac{(-1)^{k-1} (2\pi)^{2k} B_{2k}}{2\ (2k)!}$$

denoting by B_{2k} the *Bernoulli numbers.*

Such numbers can be defined of different modes.

So, for instance,

- as independent terms of *Bernoulli polynomials,* $B_n(x)$,

- *by a generating function,*

$$G(x) = \frac{x}{e^x - 1} = \sum_{i \in \mathbf{N}^*} B_n \frac{x^i}{i!}$$

with

$$|x| < 2\pi$$

and where each coefficient of the Taylor Series is the *n-th Bernoulli number.*

- *by the recursive formula*

$$B_0 = 1$$

$$B_m = -\sum_{j=0}^{m-1} C_{m,j} \frac{B_j}{m - j + 1}$$

As the Bernoulli numbers can be expressed in terms of the Riemann Zeta function, they are indeed values of such function to negative arguments.

The connection between the Zeta function and the set of prime numbers is given by the Euler product,

$$\varsigma(s) = \prod_{p \in P} \frac{1}{1-p^{-s}} \equiv \prod_{p \in P} \frac{1}{1-\frac{1}{p^s}}$$

which also converges for all $s > 1$.

If we write the precedent expression for the generalized product as

$$\prod_{p \in P} \left(1 + p^{-s} + \left(p^2\right)^{-s} + \left(p^3\right)^{-s} + ...\right) \equiv \prod_{p \in P} \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + ...\right)$$

it is easy to observe that this product is precisely the analytical expression of the *FTA* (acronym of the Fundamental Theorem of Arithmetic).

And taking into account the behaviour of the limit approaching to one from the right,

$$\lim_{s \to 1^+} \zeta(s) = +\infty$$

we see that there must be infinitely many factors in such product. Therefore, there are infinitely many prime numbers.

But the zeta function is not only useful to proof this important fact, because it permits many other applications, as for instance, to study and attempt to describe their distribution.

Some of the *Zeta Function special values* are

$$\varsigma(0) = -1/2$$

$$\varsigma(2) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \ ...$$

$$\Rightarrow \varsigma(2) = \frac{\pi^2}{6}$$

$$\varsigma(4) = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \ ...$$

$$\Rightarrow \varsigma(4) = \frac{\pi^4}{90}$$

$$\varsigma(6) = 1 + \frac{1}{2^6} + \frac{1}{3^6} + \ ...$$

$$\Rightarrow \varsigma(6) = \frac{\pi^6}{945}$$

$$\varsigma(8) = 1 + \frac{1}{2^8} + \frac{1}{3^8} + ...$$
$$\varsigma(8) = \frac{\pi^8}{9450}$$
$$...$$

Note that we take here *s even.*

Because for *odd* values of *s,* it appears troubles and also irrational numbers; for instance,

$$\varsigma(1) = 1 + \frac{1}{2} + \frac{1}{3} + ... \to \infty \; (\textit{harmonic} \text{ series})$$
$$\varsigma(3) = 1 + \frac{1}{2^3} + \frac{1}{3^3} + ... \simeq 1.2 \; (\textit{Apéry constant})$$
$$\textit{and also}$$
$$\varsigma(1/2) \simeq -1.46$$
$$\varsigma(3/2) \simeq 2.6$$
$$\varsigma(5/2) \simeq 0.134$$
$$\varsigma(7/2) \simeq 1.127, ...$$

The logarithm of the Zeta Function will be

$$\log \varsigma(s) = \sum_{n \geq 2} \left(\frac{\Lambda(n)}{\log \; n} \right) \left(\frac{1}{n^s} \right)$$
$$\textit{being}$$
$$\mathrm{Re}(s) > 1$$

Here, $\Lambda(n)$ denote the *Lambda Function,* also sometimes called *Von Mangoldt function,* defined by

$$\Lambda(n) = \log \; p, \; \textit{if} \; n = p^k, \textit{for} \; n \in \mathbf{N}$$
$$\textit{and some prime number, p;}$$
$$\textit{and 0, otherwise}$$

It is an arithmetic function that is neither additive nor multiplicative,

$$\Lambda(n + n') \neq \Lambda(n) + \Lambda(n')$$
$$\Lambda(n \cdot n') \neq \Lambda(n) \cdot \Lambda(n')$$

Such Lambda function satisfies

$$\log \; n = \sum_{d|n} \Lambda(d)$$

where the summation will be extended to all integers, *d,* dividing to *n.*

Related with the above series, we have the popular *Riemann Hypothesis,* still an important open problem in current Mathematics. It is about the distribution of zeroes of such Zeta Function.

This Zeta Function admits many variations, with different names, Selberg, Ihara, etc. So, for instance, we may consider its multiplicative inverse, expressible as a series by the *Möbius Function.* It can be reached, from the known series, by tools as the *Möbius Inversion* and the *Dirichlet Convolution.*

The values produced by such function from integer arguments are called *"zeta constants".*

We can observe their convergence to one from the right, $\varsigma(s) \to 1^{+}$.

Also, this *functional equation* is satisfied

$$\varsigma(s) = 2^{s} \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \varsigma(1-s)$$

which is true in all the complex field, relating its values in s and $1-s$.

This equation has a pole simple at $s = 1$, with residuum equal to one.

It was proved by Bernhard Riemannn (1859).

Euler conjectured an equivalent relation to the function

$$\sum_{n \in \mathbf{N}^*} \frac{(-1)^{n+1}}{n^{s}}$$

Also there exists a symmetric version of the precedent functional equation, reachable by the change

$$s \longmapsto 1-s$$

This gives

$$\varsigma(s) \Gamma\left(\frac{s}{2}\right) \pi^{-\frac{s}{2}} = 2^{s} \pi^{-\frac{1-s}{2}} \Gamma\left(\frac{1-s}{2}\right) \varsigma(1-s)$$

Sometimes, we define the very related *Eta Function,* denoted by ξ, as

$$\xi(s) \equiv \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s)$$

It holds

$$\xi(s) = \xi(1-s)$$

The value of the Zeta function for negative even real values is zero,

$$\varsigma(-2) = \varsigma(-4) = \varsigma(-6) = \ ... \ = \varsigma(-2k) = 0$$
$$with \ k \in \mathbf{N}$$

They are called *trivial zeroes of* ζ.

Furthermore, it will be cancelled on values of s that belongs to the *critic rang*

$$\{s \in \mathbf{C} : 0 < \operatorname{Re}(s) < 1\}$$

In this case, we call of *non-trivial zeroes.* It is because the difficulties to find its position into the critical rang.

To obtain zeta function values for negative and non integer arguments, we proceed by

$$\varsigma(-1/2) = \frac{2^{-3/2} \pi^{-1/2} \Gamma(-1/2)}{\sin\left(\frac{\pi s}{2}\right) \varsigma(1-s)}$$
$$\Rightarrow \varsigma(-1/2) \simeq \frac{-4 \ \pi}{2.6} \simeq - \ 0.2069$$

Γ represents the *Gamma Function of Euler,* defined by an integral expression,

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt$$

If $n \in N$, then

$$\Gamma(n+1) = n!$$

For this reason, it is considered as an extension of the factorial.

It holds

$$\Gamma(s+1) = s \ \Gamma(s)$$

and as

$$\Gamma(1) = 1$$

we have

$$\Gamma(n+1) = n \ \Gamma(n) = ... = n! \ \Gamma(1) = n!$$

Such functional equation also gives an asymptotic limit, proposed by (Nemes, 2007),

$$\varsigma\left(1-s\right)=\left(\frac{s}{2\pi l}\right)^{s}\sqrt{\frac{8\pi}{s}}\cos\left(\frac{\pi s}{2}\right)\left(1+O\left(\frac{1}{s}\right)\right)$$

Among its applications, they are useful to search compact formulas for a sequence given by a recurrence equation; to find relations among sequences, because the form of a generating function may suggest us a recurrence formula; to explore the asymptotic behaviour of sequences, as in our case; to prove identities involving sequences; or to solve enumeration problems in Combinatorics.

It will be very useful in different problems, as to accelerate the convergence of sums of this form

$$\sum_{n\in A\subset \mathbf{Z}} f\ \left(\frac{1}{n}\right)$$

where f would be an analytic function. Because many mathematical constants are slowly convergent series of these form.

To accelerate such convergence, it mabe very easy once known the values at the integers of the Zeta Function, $\zeta_{_A}\left(s\right).$

For this purpose, it is suffice to take

$$\sum_{m} f_m\ \zeta_{_A}\left(m\right)$$

being

$$\zeta_{_A}\left(m\right)=\sum_{n\in A\subset \mathbf{Z}}\left(\frac{1}{m}\right)^{s}$$

It is possible once values at integers of such zeta function are known.

In such expression,

$$f\ \left(s\right)=\sum_{m} f_m\ z^{m}$$

represent the Taylor Series, or its expansion, of f at 0.

This schema may very particularly useful, and effective, when we works in the high precision evaluation of mathematical constants.

Note.

The famous *Riemann Hypothesis* says that for every non-trivial zero, *s,* of ζ, it holds

$$\operatorname{Re}\ \left(s\right)=\frac{1}{2}$$

That is, *all non-trivial zeroes are situated on the critical line,* i.e.

$$x=\ \frac{1}{2}$$

2. Ihara zeta function of a graph

The *Zeta Function* is *generalizable to graphs,* according to the theory elaborated by (Ihara, 1966), and some others [3] [4] [5] [12] [13] [24] [30] [31] [32] [37] [40] [46] [54].

Indeed, it is a Zeta function associated with a finite graph. But also generalizable to infinite graphs. Such function was first defined in terms of discrete subgroups. It closely resembles the so called Selberg zeta function, being used to relate closed paths to the spectrum of the adjacency matrix.

The Ihara zeta function was firstly defined by the aforementioned Ihara studying discrete subgroups of the two-by-two special linear groups.

Zeta functions of graphs were studied not only by Ihara [34] [35], but many other works on it, as may be Sunada, Hashimoto, Bartholdi, and Bass.

So, Jean Pierre Serre [47] suggested can be reinterpreted graph-theoretically, in his book *Trees.*

And it was Toshikazu Sunada, in 1985, who put this suggestion into practice.

Storm defined the Ihara-Selberg zeta function of a *Hypergraph.*

Recall that a *hypergraph*

$$H \equiv (V(H),\ E(H))$$

is a pair of a set of hyper-nodes, $V(H)$, and a set of hyper-edges, $E(H)$, which the union of all hyper-edges is $V(H)$.

Note that a regular graph is a *Ramanujan Graph* if and only if the Ihara zeta function of such graph satisfies an analogue of Riemann Hypothesis, translated to Graph Theory.

The *Ihara zeta function* is denoted by ς_G, and it will be defined by

$$\varsigma_G(s) \equiv \left[\prod_p \left(1 - s^{L(p)}\right)\right]^{-1}$$

or equivalently,

$$\prod_p \left(1 - s^{L(p)}\right) \equiv \frac{1}{\varsigma_G(s)}$$

Such formula is analogous to the Euler product for the Riemann zeta function.

This product is taken over all prime walks, *p,* on the graph *G,* being *L(p)* the length of the *prime p.*

Recall that a *closed geodesic* is a closed path such there is no backtracking, if we around twice, i. e. it is a closed proper walk with the initial and final edges different.

If γ is a closed geodesic, we denote by γ^{r} the obtained by repeating γ r times.

A closed geodesic which is not the power of another is called a *prime geodesic.*

An *equivalence class of prime geodesics* is called a *prime geodesic class,* or simply a *prime,* $\wp$.

Given a path, γ, we denote by $L(\gamma)$ its *length.*

Therefore, two prime geodesics are said to be *equivalent,* if one is obtained from another by a cyclic permutation of edges.

The length of a prime, $\wp$, is the length of any of its representatives.

Also ς_G is always representable as the reciprocal of a polynomial

$$\varsigma_G(s) \equiv \frac{1}{\det\ (I\ -\ T\ s)}$$

where T is the *edge adjacency operator* (Hashimoto, 1990).

Recall that the *adjacency operator, A,* is acting on the space of functions defined on the set of nodes of $G = (V,\ E)$.

Being *o(e)* and *t(e)* the *origin* and *terminus* of *e,* respectively, it is defined by

$$(Af)(x) = \sum_{e \in E_x} f\ [t(e)],$$

$$where\ E_x \equiv \{e \in E : o(e) = x\}$$

(Bass, 1992) also gave a determinant formula involving the adjacency operator.

For any *Complex Network (CN),* or any *Graph, G,* the function ς_G can be expressed in terms of ς, for different dimension values, n.

So,

$If\ \ n = 1,\ then\ \varsigma_G(s) = 2\varsigma(s).$

$If\ \ n = 2,\ then\ \varsigma_G(s) = 4\varsigma(s-1).$

$If\ \ n = 3,\ then\ \varsigma_G(s) = 4\varsigma(s-2) + 2\varsigma(s).$

$If\ \ n = \infty,\ then\ \varsigma_G(s) = \frac{8}{3}\varsigma(s-3) + \frac{16}{3}\varsigma(s-1).$

Recall that $\varsigma_G(s)$ is a *decreasing function* of s.

That is,

$$\varsigma_G(s_1) > \varsigma_G(s_2),$$

$$if\ s_1 < s_2$$

And in the limit, if $n \to \infty$, when s is next to the *transition point,* it holds

$$\varsigma_G(s) = \frac{2^n \varsigma(s - n + 1)}{\Gamma(n)}$$

If the average degree of nodes, also called *mean coordination number of the graph,* is finite, then there exists exactly a value of *s,* denoted $s_{transition}$, where the Zeta Function changes from infinite to finite, or vice versa.

It is also called *dimension of the Graph,or the Complex Network (CN).*

Also, the ς_G function possesses three *properties*,

- *monotonicity,*
- *stability,*
and
- *Lipschitz Invariance.*

According to *monotonicity,* a subset has dimension less than or equal to a superset.

According to *stability,* the dimension of the union of a family of sets is equal to the maximum cardinal among its members,

$$dim\ (\cup_{s=1}^{n} A_s) = \max_{s=1,2,\ldots,n} (\dim\ A_s)$$

And according to *Lipschitz Invariance,* the operations must intervene in the change of distances between nodes only by finite magnitudes, when the size, n, of the graph tends to infinity.

The Ihara zeta function plays an important role in many applications, such as in the study of

- *spectral graph theory*
- *dynamical systems*
- *free groups*
- *combinatorial enumeration;* for instance, of graphs,

and so on.

3. Enumerating Bayesian Networks

About the foundations of Graph Theory, there exists many adequate surveys, as [9] [10] [23] [27] [29].

Bayesian Networks are the most sucessful class of models to represent uncertain knowledge.

See, for instance, [1] [15] [16] [19] [20] [22] [49] [50] [51] [52].

But the representation of conditional independencies (*CIs,* in acronym) does not have uniqueness. The reason is that probabilistically equivalent models may have different representations.

And this problem is overcome by the introduction of the concept of *Essential Graph,* as unique representant of each equivalence class. They represent *CI models by graphs.*

For such mathematical graphical tools, see [23] [38] [39].

It may be containing both types, directed or/and undirected edges; hence, producing respectively *Directed Graphs (DGs), Undirected Graphs (UGs),* or *Chain Graphs (CGs),* in the mixed case.

So, DAG models are generally represented as *Essential Graphs (EGs).*

Knowing the ratio of EGs to DAGs is a valuable tool, because through this information we may decide in which space to search.

If the ratio is low, we may prefer to search the space of DAG models, rather than the space of DAGs directly, as it was usual until now.

The most common approach to learning DAG models is that of performing a search into the space of either DAGS or DAG models (EGs).

It is preferable, from a mathematical point of view, to obtain the more exact solution possible, studying its asymptotic behaviour.

For *Graphical Enumeration,* it may be convenient to see for instance [6] [11] [14] [21] [28] [41] [42] [48], among others.

But also it is feasible to propose a *Monte Carlo Chain Method (MCMC)* to approach the ratio, avoiding the straightforward enumeration of EGs.

And a many more elegant construct, if very difficult, through the *Ihara Zeta function* for counting graphs.

The labeled or unlabeled character of the graph means whether its nodes or edges are distinguishable or not.

The labeling will be a mathematical function, referred to a value or name assigned to its elements, either nodes, edges, or both, which makes them distinguishable.

It is possible to use *Generating Functions* to count labeled DAGs.

See, for instance, these references on the foundations of *Combinatorics,* [6] [11] [14] [21] [28] [41] [42] [48].

For this mathematical construct, it is necessary to make intervene the *Inclusion-Exclusion Principle (IEP).*

So, if we take the set of *n-essential graphs,* and denote its cardinal by a_n, applying the aforementioned IEP, we may obtain

$$a_n = \sum_{s=1,\ldots,p} (-1)^{s+1} \sum_{\substack{i_j \\ j\in\{1,\ldots,s\}}} c\left(A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_s}\right)$$

where

$$A_k = \{G \in E : k \text{ is a } \textit{terminal node} \text{ of } G\},$$
$$\textit{with } k = 1, 2, \ldots, n \quad [*]$$

4. New Research

Let a_n be the *number of essential labeled n-DAGs.*

And let $a_{n'}$ be the *number of labeled n-DAGs.*

Then, a_n is given by the recurrence equation

$$a_n = \sum_{s=1}^{n} (-1)^{s+1} C_{n,s} \left(2^{n-s} - n + s\right)^s a_{n-s}$$

$$with\ a_0 = 1$$

Whereas

$$a_n{}' = \sum_{s=1}^{n} (-1)^{s+1} C_{n,s} \left(2^{n-s}\right)^s a_{n-s}{}'$$

$$with\ a_0{}' = 1$$

The basic idea (see for this [44] [45] [49] [50] [51] [52]) is to count the number of *n-DAGs* considering each digraph as created by adding terminal nodes to a DAG with lesser number of nodes. After this addition, we obtain a new DAG.

So, the new formula would be recursive, and it is a direct application of the *IEP.* From which, we can reach directly the equation.

We may rewrite the equation as

$$\sum_{s=0}^{n} (-1)^{n-s} C_{n,s} \left(2^s - s\right)^{n-s} a_s = 0$$

$$with\ n \geq 1$$

Another case of application of *IEP* is to find the cardinal of the set of essential DAGs, *E,* with a set of labeled nodes, with labelings that belongs to $\{1, 2, \ldots, n\}$.

For this, we start with a family of sets, as the aforementioned $\{A_k\}_{k=1}^{n}$. See the precedent formula [*], where A_k represents the subclass of graphs concluding at the node labeled by *k.*

Therefore, to know the cardinal of *E,* first we compute the intersection that appears in the last summatory, for $j = 1, 2, \ldots, n$, being these

$$\sum_{\substack{i_j \\ j \in \{1,\ldots,s\}}} c\left(A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_s}\right)$$

related with the aforementioned principle *(IEP).*

With the total allowed connection numbers, from a given node being

$$2^{n-s} - n + s$$

So, the number of possible ways of adding directed edges from the essential graph until all the s terminal nodes will be

$$[2^{n-s} - n + s]^s$$

If we denote e_n the number of *essential n-graphs,* also labeled, it holds

$$a_n \leq e_n \leq a_n{}'$$

I.e. both precedent values, a_n and a'_n, are the lower and upper bounds of e_n, for each selected order, n. So, it holds

$$\frac{1}{13.6} \leq \frac{a_n}{a'_n}$$

hence

$$\frac{a_n}{a'_n} \leq 13.6$$

or equivalently,

$$\frac{a'_n}{a_n} \geq 0.07$$

And also

$$\frac{1}{13.6} a_n{}' \leq e_n \leq a'_n$$

i.e.

$$a_n{}' \leq 13.6 \; e_n \leq 13.6 \; a'_n$$

or

$$a_n{}' \geq 0.07 \; e_n \geq 0.07 \; a'_n$$

where we obtain the lower and the upper bounds for the cardinal of essential graphs, by this expression

$$e_n \in \left(\left[\frac{1}{13.6}, \; 1 \right] \; a_n{}' \right) \equiv \left([0.07, \; 1] \; a_n{}' \right)$$

Analyzing the asymptotic behaviour of the ratio, i.e. studying the convergence of ratios among the number of equivalence classes, or essential graphs, and the number of DAGs, we may develop this so

$$A(n) \equiv \frac{a_n}{a_n{}'} \Rightarrow$$

$$\Rightarrow \lim_{n\to\infty} A(n) = \lim_{n\to\infty} \frac{\sum_{s=1}^{n} (-1)^{s+1} C_{n,s} \left(2^{n-s} - (n - s) \right)^{s} a_{n-s}}{\sum_{s=1}^{n} (-1)^{s+1} C_{n,s} \left(2^{n-s} \right)^{s} a_{n-s}{}'}$$

Observe that in each step is augmented/disminished the summatory with the apparition of a new term, ever positive and increasing , but with alternating sign. For this reason, we need promptly to introduce results on Alternating Series.

But we have the known results

$$fixed\ s,\ \frac{n-s}{2^{n-s}} \to 0^{+} \Rightarrow$$

$$\Rightarrow 1-\left(\frac{n-s}{2^{n-s}}\right)^{s} \to 1^{-}$$

Hence,

$$\lim_{n\to\infty}\left\{\sum_{s=1}^{n}\left(\frac{2^{n-s}-(n-s)}{2^{n-s}}\right)^{s} A(n-s)\right\}=$$

$$=\left\{1-\lim_{n\to\infty}\sum_{s=1}^{n}\left(\frac{(n-s)}{2^{n-s}}\right)^{s}\right\}\ \{\lim_{n\to\infty}A(n-s)\}$$

being

$$A(n-s)\equiv\frac{a_{n-s}}{a_{n-s}},$$

$$\varsigma_{G}(n-s)\equiv\lim_{n\to\infty}\sum_{s=1}^{n}\left(\frac{(n-s)}{2^{n-s}}\right)^{s}$$

So, returning to our initial step,

$$\lim_{n\to\infty}A(n)=\lim_{n\to\infty}\left\{\left[1-\varsigma_{G}(n-s)\right]\ A(n-s)\right\}=$$

$$=\left[1-\lim_{n\to\infty}\varsigma_{G}(n-s)\right][\lim_{n\to\infty}A(n-s)]$$

Considering that the series

$$\sum_{s=1}^{n}\frac{n-s}{2^{n-s}}\Rightarrow\sum_{s=1}^{n}\left[1-\frac{n-s}{2^{n-s}}\right]=n-\sum_{s=1}^{n}\frac{n-s}{2^{n-s}}\Rightarrow$$

$$\Rightarrow\sum_{s=1}^{n}\left[1-\frac{n-s}{2^{n-s}}\right]^{s}\leq n-\sum_{s=1}^{n}\left(\frac{n-s}{2^{n-s}}\right)^{s}$$

and its asymptotical behaviour, when $n \to \infty$, these may establish a correspondence with a version of the Zeta Function of Riemann, χ_G, the so called *Ihara-Selberg of the n-graph* G_n.

But operating here on the increasing value of $n - s$, i.e. with $\chi_G\ (n - s)$.

Nevertheless, this proof would be very complex.

Instead, we may apply here an interesting result, which permits to finalize our demonstration.

Lemma.

Let

$$\{\alpha_n\}_{n\in N}$$
$$and$$
$$\{\beta_n\}_{n\in N}$$

be two sequences, and suppose that

$$\forall n >> 0,\ \beta_n > 0,$$
$$\lim_{n\to\infty} \frac{\alpha_n}{\beta_n} = c,$$
$$and$$
$$\lim_{n\to\infty} \sum_{n=1}^{m} \beta_n = \infty.$$

Then,

$$\lim_{n\to\infty} \frac{\sum\limits_{n=1}^{m} \alpha_n}{\sum\limits_{n=1}^{m} \beta_n} = c$$

In our case, all the requirements hold.

So, we obtain the ratio among terms of the series (by applying the precedent Lemma),

$$\lim_{n\to\infty} \frac{\left\{\sum\limits_{s=1}^{n} (-1)^{s+1} C_{n,s} \left(2^{n-s} - (n-s)\right)^{s} a_{n-s}\right\}}{\left\{\sum\limits_{s=1}^{n} (-1)^{s+1} C_{n,s} \left(2^{n-s}\right)^{s} a_{n-s}{}'\right\}} =$$

$$= \lim_{n\to\infty} \frac{\left\{(-1)^{s+1} C_{n,s} \left(2^{n-s} - (n-s)\right)^{s} a_{n-s}\right\}}{\left\{(-1)^{s+1} C_{n,s} \left(2^{n-s}\right)^{s} a_{n-s}{}'\right\}} =$$

$$= \lim_{n\to\infty} \frac{\left(2^{n-s} - (n-s)\right)^{s} a_{n-s}}{\left(2^{n-s}\right)^{s} a_{n-s}{}'} =$$

$$= \left[\lim_{n\to\infty} \frac{\left(2^{n-s} - (n-s)\right)^{s}}{\left(2^{n-s}\right)^{s}}\right] \left[\lim_{n\to\infty} \frac{a_{n-s}}{a_{n-s}{}'}\right] =$$

$$= \left[\lim_{n\to\infty} \left(\frac{2^{n-s} - (n-s)}{2^{n-s}}\right)^{s}\right] \left[\lim_{n\to\infty} \frac{a_{n-s}}{a_{n-s}{}'}\right] =$$

$$= \left[\lim_{n\to\infty} \left(1 - \frac{n-s}{2^{n-s}}\right)^{s}\right] \left[\lim_{n\to\infty} A\,(n-s)\right] =$$

$$= \left[1 - \lim_{n\to\infty} \left(\frac{n-s}{2^{n-s}}\right)^{s}\right] \left[\lim_{n\to\infty} A\,(n-s)\right] =$$

$$= \left[\lim_{n\to\infty} \left\{1 - \varsigma_{G}\,(n-s)\right\}\right] \left[\lim_{n\to\infty} A\,(n-s)\right]$$

We can assert that the convergence of these series by the *Comparison Test.*
Let

$$0 \le \alpha_n \le \beta_n$$
$$with\ n \ge k$$

from some k onwards.
Then, the convergence of $\sum_{n\in\mathbf{N}} \beta_n$ implies the convergence of $\sum_{n\in\mathbf{N}} \alpha_n$.
Symmetrically, the divergence of $\sum_{n\in\mathbf{N}} \alpha_n$ implies the divergence of $\sum_{n\in\mathbf{N}} \beta_n$.
In our case, once fixed s, for each natural n,

$$a_{n-s} \le a_{n-s}{}' \Rightarrow$$

$$\Rightarrow \left(2^{n-s} - (n-s)\right)^{s} a_{n-s} \le \left(2^{n-s}\right)^{s} a_{n-s}{}' \Rightarrow$$

$$being\ C_{n,s} \equiv \binom{n}{s} \in \mathbf{N},$$

$$C_{n,s} \left(2^{n-s} - (n-s)\right)^{s} a_{n-s} \le C_{n,s} \left(2^{n-s}\right)^{s} a_{n-s}{}' \Rightarrow$$

$$\Rightarrow if\ s+1\ is\ even,\ therefore\ if\ s\ is\ odd,\ then$$

$$(-1)^{s+1} C_{n,s} \left(2^{n-s} - (n-s)\right)^{s} a_{n-s} \le (-1)^{s+1} C_{n,s} \left(2^{n-s}\right)^{s} a_{n-s}{}'$$

$$and\ if\ s+1\ is\ odd,\ therefore\ if\ s\ is\ even,\ then$$

$$(-1)^{s+1} C_{n,s} \left(2^{n-s} - (n-s)\right)^{s} a_{n-s} \geq (-1)^{s+1} C_{n,s} \left(2^{n-s}\right)^{s} a_{n-s}'$$

This permits to reach the proof of the convergence of the series, and as a consequence, by the *Comparison Test,* the convergence of the ratio among both.

As

$$(n-s) \geq 0$$

then

$$2^{n-s} - (n-s) \leq 2^{n-s}$$

So, being $n \in \mathbf{N}$,

$$\left[2^{n-s} - (n-s)\right]^{s} \leq \left(2^{n-s}\right)^{s} \Rightarrow$$

$$\Rightarrow C_{n,s} \left(2^{n-s} - (n-s)\right)^{s} \leq C_{n,s} \left(2^{n-s}\right)^{s} \Rightarrow$$

$$\Rightarrow \sum_{i=1}^{n} (-1)^{s+1} C_{n,s} \left(2^{n-s} - (n-s)\right)^{s} a_{n-s} \leq \sum_{i=1}^{n} (-1)^{s+1} C_{n,s} \left(2^{n-s}\right)^{s} a_{n-s}'$$

Hence, from the convergence of the second series we induce the convergence of the first.

And also, in case of divergence of the first series, we obtain the divergence of the second.

Observe that it appears *alternating series.*

For this reason, we can see the corresponding *Alternating Series Test.*

Suppose that we have a series

$$\sum_{n} d_n$$

and either

$$d_n = (-1)^{n} c_n$$

or

$$d_n = (-1)^{n+1} c_n$$

being

$$c_n \geq 0,\ \forall n \in \mathbf{N}$$

If

$$1)\ \lim_{n\to\infty} c_n = 0$$

and

$$2)\ \{c_n\}_{n\in\mathbf{N}}\ \textit{is a decreasing sequence}$$

then the series

$$\sum_n d_n$$

is *convergent.*

In our case, as the sequences,

$$\{a_n\}_{n\in\mathbf{N}}$$

and

$$\{a_n'\}_{n\in\mathbf{N}}$$

are both increasing, it will be sufficient to take

$$c_n = \frac{1}{a_n}$$

and

$$c_n' = \frac{1}{a_n'}$$

to obtain

$$c_n \to 0^+$$

and

$$c_n' \to 0^+$$

and then, the sequences

$$\{c_n\}_{n\in\mathbf{N}}$$

and

$$\{c_n'\}_{n\in\mathbf{N}}$$

are both decreasing.

And so,

$$\sum_{n} c_n$$

and

$$\sum_{n} c_n{}'$$

both are converging series.

Recall some more details about the *convergence of alternating series.*

A series

$$S = \sum_{n \in \mathbf{N}} (-1)^{n-1} a_n = a_1 - a_2 + a_3 - a_4 + ... + (-1)^{k-1} a_k + ...$$

where $a_k \geq 0, \ \forall k \in \mathbf{N},$ is named an *alternating series.*

We have the subsequent two results,

Theorem I.

Suppose that for any $k \in N,$ the inequality

$$a_k \geq a_{k+1}$$

holds.

Then,

the alternating series converges *if and only if* $\lim_{k \to \infty} a_k = 0$

And also

Theorem II.

Suppose that an alternating series

$$\sum_{k \in \mathbf{N}} a_k$$

converges.

Then,

$$\lim_{k \to \infty} a_k = 0$$

We establish from now these auxiliary and useful notation

$$f(n-s) = 1 - \frac{n-s}{2^{n-s}}$$
$$f(n) = 1 - \frac{n}{2^n}$$

that is,

$$1 - f(n-s) = \frac{n - s}{2^{n - s}}$$
$$1 - f(n) = \frac{n}{2^{n}}$$

and so, once fixed s,

$$\lim_{n\to\infty} f(n-s) = \lim_{n\to\infty} f(n)$$

But as we known

$$\lim_{n\to\infty} \frac{n-s}{2^{n-s}} = \lim_{n\to\infty} \frac{n}{2^{n}} = 0^{+}$$

and by this procedure,

$$\left[1 - \{f(n-s)\}\right]^{s} = 1 - \left(\frac{(n - s)}{2^{n - s}}\right)^{s}$$

Hence

$$\lim_{n\to\infty} \left[1 - \{f(n-s)\}\right]^{s} = \lim_{n\to\infty} \left\{1 - \left(\frac{(n - s)}{2^{n - s}}\right)^{s}\right\} =$$
$$= 1 - \lim_{n\to\infty} \left\{1 - \left(\frac{n}{2^{n}}\right)^{s}\right\} = 1 - 0^{+} \equiv 1^{-}$$

and

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{2^{n - s} - (n - s)}{2^{n - s}}\right)^{s} = \lim_{n\to\infty} \sum_{i=1}^{n} \left(1 - \frac{(n - s)}{2^{n - s}}\right)^{s} =$$
$$= n - \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{(n - s)}{2^{n - s}}\right)^{s} = n - \lim_{n\to\infty} \sum_{i=1}^{n} \left[1 - f(n-s)\right]^{s} =$$
$$= n - \left[n - \zeta_{G}(n-s)\right] = \zeta_{G}(n-s)$$

These last terms must regulate the asymptotical bahaviour, by its limit values.

And respect to its reciprocal function

$$\zeta_G^{-1}(n-s) = \left[\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{2^{n-s}-(n-s)}{2^{n-s}}\right)^{s}\right]^{-1} =$$

$$= \lim_{n\to\infty}\left[\sum_{i=1}^{n}\left(\frac{2^{n-s}-(n-s)}{2^{n-s}}\right)^{-s}\right] =$$

$$= \lim_{n\to\infty}\left[\sum_{i=1}^{n}\left(1-\frac{(n-s)}{2^{n-s}}\right)^{-s}\right]$$

whith this may appears Φ_1^{-1} from Φ_1, as they are described in the subsequent step.

Note that we can take

$$f(n-s) = \frac{2^{n-s}-(n-s)}{2^{n-s}}$$

$$\textit{for each } n \in \mathbf{N}, \textit{ once fixed } s,$$

So, by

$$\zeta_G(n-s) = \lim_{n\to\infty}\sum_{i=1}^{n}\left[f(n-s)\right]^{s}$$

it holds

$$a_{n-s} \le a'_{n-s}, \forall n, \textit{ once fixed } s \Rightarrow$$

$$\Rightarrow \exists\Phi_A = \left[n - \lim_{n\to\infty}\sum_{i=1}^{n}\left[f(n-s)\right]^{s}\right]\left[\lim_{n\to\infty} A(n-s)\right] =$$

$$= \left[n - \lim_{n\to\infty}\varsigma_G(n-s)\right]\left[\lim_{n\to\infty} A(n-s)\right],$$

$$\textit{fixed } s, \textit{ when } n \textit{ increases to } \infty \Rightarrow$$

$$\Rightarrow \Phi_A = \frac{1}{10\,\varsigma(5/2)} = \frac{1}{5\varsigma_G(5/2)} \simeq 0.07$$

essential graphs for each equivalence class, or equivalently,

$$\Phi_A^{-1} = 10\,\varsigma(5/2)$$

hence

$$\Phi_A^{-1} = 5/2\,\varsigma_G(7/2) \simeq 13.6$$

equivalence classes for each essential graph.

So far, we have supposed *dimension one.*

Because in case of *dimension two,* where the new functions are denoted Φ_B and Φ_B^{-1}, respectively, it holds

$$\Phi_B = \frac{1}{10\ \varsigma(5/2)} = \frac{1}{5/2\ \varsigma_G(7/2)},$$

$$and$$

$$\Phi_B^{-1} = 4\ \varsigma\left(7/2 - 1\right) = 4\ \varsigma\left(5/2\right)$$

And translating this from Riemann to Ihara Zeta Function, we obtain

$$\Phi_B^{-1} = 5/2\ \varsigma_G\left(7/2\right)$$

Note that the "labeled" or "unlabeled" character of the considered graphs is relevant, because they are very distinct situations, giving so different ratios.

It is obvious that the enumeration of unlabeled essential graphs results more complex that in the labeled case. Also, its symmetries can be used.

Recall that permutating between the positions of two symmetric nodes is a operation that when acting on graphs, leaves the shape unaffected.

For the case of unlabeled graphs (Sunada, 1985), denoted here by the letter "c", we have

$$c_n \equiv\ cardinal\ \ of\ the\ set\ of\ unlabeled\ n\text{-}graphs$$

$$and$$

$$c_n{}' \equiv\ cardinal\ \ of\ the\ set\ of\ unlabeled\ essential\ n\text{-}graphs$$

And so, we can find (applying our previous *Lemma*) that

$$c_{n-s} \leq c'_{n-s}, \forall n,\ once\ fixed\ s \Rightarrow$$

$$\Rightarrow \exists \Phi_C = \lim_{n\to\infty}\left(\varsigma_G(n)\left[C\left(n-s\right)\right]\right),$$

$$fixed\ s,\ when\ n \to \infty,$$

$$being\ C\left(n-s\right) \equiv \lim_{n\to\infty} \sum_{s=1}^{n} \frac{c_{n-s}}{c_{n-s}{}'} \Rightarrow$$

$$\Rightarrow \Phi_C \equiv \left\{\lim_{n\to\infty} \varsigma_G\left(n\right)\right\}\left\{\lim_{n\to\infty} C\left(n-s\right)\right\} \Rightarrow$$

$$\Rightarrow \Phi_C = \frac{1}{10}\ \varsigma\left(3/2\right) = \frac{1}{20}\varsigma_G\left(3/2\right) \simeq 0.26$$

essential graphs for each unlabeled DAG, in the case of dimension one.

And symmetrically,

$$\exists \left(\Phi_{C}\right)^{-1} \equiv \left\{\lim_{n\to\infty}\left[\varsigma_{G}\left(n\right)\right]\right\}^{-1}\left\{\lim_{n\to\infty}\left[C\left(n-s\right)\right]\right\}^{-1} \Rightarrow$$

$$\Rightarrow \left(\Phi_{C}\right)^{-1} = \frac{10}{\varsigma(3/2)} = \frac{20}{\varsigma_{G}(3/2)} \simeq 3.73$$

unlabeled DAGs for each essential graph, which coincides with our precedent analythical results.

In case of *dimension two,* it holds

$$\exists \Phi^{*}_{C} \equiv \lim_{n\to\infty}\left(\left\{\varsigma_{G}\left(n\right)\right\}\left[C^{*}\left(n-s\right)\right]\right) \Rightarrow$$

$$\Rightarrow \Phi^{*}_{C} = \frac{1}{10}\,\varsigma\left(3/2\right) = \frac{1}{40}\varsigma_{G}\left(5/2\right) \simeq 0.26$$

and dually,

$$\exists \left(\Phi^{*}_{C}\right)^{-1} \equiv \lim_{n\to\infty}\left(\left\{\varsigma_{G}\left(n\right)\right\}\left[C^{*}\left(n-s\right)\right]\right)^{-1} \Rightarrow$$

$$\Rightarrow \left(\Phi^{*}_{C}\right)^{-1} = \frac{10}{\varsigma(3/2)} = \frac{4}{\varsigma_{G}(5/2)} \simeq 3.73$$

5. Conclusion

And also in the limit situation, which reflects the degree of fitness of the proposed model, based in analytical framework, to the precedent computational results, as the shown by [1], or [20].

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