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A monotonicity result related to clamped triangular elastic membranes

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Abstract

In this note we prove a monotonicity result related to clamped triangular membranes. Using the continuous Steiner symmetrization we show that the average displacement increases as the design of the membrane becomes symmetric about a line.

Key words: Continuous Steiner symmetrization, Monotonicity, Elastic membrane, Boundary value problem.

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1. Introduction

Consider a thin flat elastic membrane, fixed around the edge, subject to a vertical load distribution. Naturally, the load forces the membrane to displace from the rest position. We are interested in the average displacement across the membrane. This problem can be mathematically described as follows. Let $\Omega \subset \mathbb{R}^2$ stand for the region occupied by the elastic membrane and $f : \Omega \to \mathbb{R}^+ := [0, \infty)$ denote the load distribution. Suppose $u : \Omega \to \mathbb{R}$ denotes the displacement function; it is well known that:

$$\begin{cases} -\Delta u = f(x), & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{0.1}$$

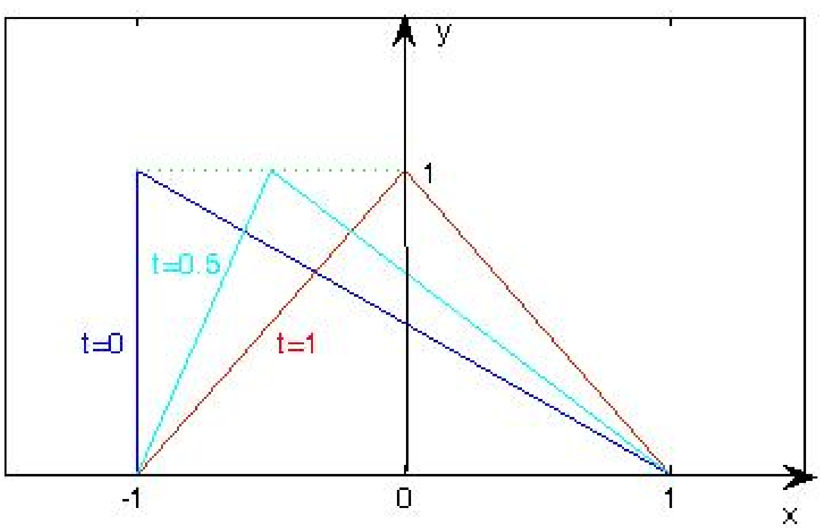
where the boundary condition in (0.1) corresponds to our assumption that the membrane is fixed around the edge. We assume henceforth that Ω is a bounded domain in $\mathbb{R}^2$ with Lipschitz boundary. The average displacement, denoted $A(\Omega)$, is defined by the integral of u over Ω; that is, $A(\Omega) = \int_\Omega u(x)\, dx$.

Our interest in this note is a design problem which is described as follows. For $0 \le t \le 1$, consider the triangle Δ_t (design) with vertices $(-1, 0), (1, 0)$, and $(t - 1, 1)$, see figure 1. Let $u_t \in H_0^1(\Delta_t)$ denote the unique solution of the following boundary value problem:

$$(BVP) \quad \begin{cases} -\Delta u = 1, & \text{in } \Delta_t \\ u = 0, & \text{on } \partial\Delta_t. \end{cases}$$

Remark 1. Note that in (BVP), the load distribution $f(x)$ is constant across the membrane.

Figure 1



Our main result is the following monotonicity result.

Theorem. *The function $\xi(t) := A(\Delta_t)$ for $t \in [0, 1]$ is an increasing function.*

Remark 2. Note that as t changes from 0 to 1, the designs Δ_t become more and more symmetric about the y-axis. In fact, the final design, Δ_1, is symmetric about the y-axis. Therefore, the physical implication of the theorem is that the average displacement of an elastic membrane with initial design Δ_0, subject to a constant load distribution, increases as the design gradually becomes symmetric.

2. Proof of the Theorem

In this section we prove the Theorem. The main tool in the proof is the continuous Steiner symmetrization (CSS). We only need to use the CSS as introduced in [Polya & Szego, 1951], since Δ_0 is convex. For general measurable sets there is a generalization of the CSS, see for example [Broc, 1995; Broc, 2000].

Before proving the theorem we give some preliminaries. For any $(x,y) \in \mathbb{R}^2$, let $V(x,y) = y\vec{i}$, where $\vec{i}$ stands for the standard unit vector along the x-axis. We define $\Phi_t(x,y) = (I + tV)(x,y)$, $t \in [0,1]$, where I is the identity map on $\mathbb{R}^2$. Where no confusion arises we write $\Phi_t(\Omega)$ to denote $\Phi_t(x,y)$, where $(x,y) \in \Omega \subseteq \mathbb{R}^2$. Since $\Phi_0(\Delta_0) = \Delta_0$, $\Phi_1(\Delta_0) = \Delta_1$, and Δ_1 is Steiner symmetric about the y-axis, the map $[0,1] \ni t \to \Phi_t(\Delta_0) \in \mathcal{D}$, defines a continuous Steiner symmetrization of Δ_0 with respect to the y-axis. Recall that u_0 satisfies

$$\begin{cases} -\Delta u_0 = 1, & \text{in } \Delta_0 \\ u_0 = 0, & \text{on } \partial\Delta_0. \end{cases}$$

The continuous Steiner symmetrization of u_0 with respect to the y-axis is the (essentially) unique function $u_0^t \in H_0^1(\Delta_t)$ such that

$$\{u_0^t > \alpha\} = \Phi_t(\{u_0 > \alpha\}), \quad t \in [0,1],$$

where $\{u_0 > \alpha\}$ is used in place of $\{x \in \Delta_0 : \ u_0(x) > \alpha\}$, for simplicity. The following properties of u_0^t are well known for every $t \in [0,1]$:

a) $\int_{\Delta_0} u_0 \, dx = \int_{\Delta_t} u_0^t \, dx$.
b) $\int_{\Delta_0} |\nabla u_0|^2 \, dx \geq \int_{\Delta_t} |\nabla u_0^t|^2 \, dx$.

Proof of the Theorem. Let us begin with the following observation:

$$-\frac{1}{2}A(\Delta_0) = \frac{1}{2}\int_{\Delta_0} |\nabla u_0|^2 \, dx - \int_{\Delta_0} u_0 \, dx.$$

Hence, by applying properties (a) and (b), we deduce

$$-\frac{1}{2}A(\Delta_0) \geq \frac{1}{2}\int_{\Delta_t} |\nabla u_0^t|^2 \, dx - \int_{\Delta_t} u_0^t \, dx \geq \frac{1}{2}\int_{\Delta_t} |\nabla u_t|^2 \, dx - \int_{\Delta_t} u_t \, dx,$$

where the last inequality is a consequence of the fact that $u_t \in H_0^1(\Delta_t)$ is the unique minimizer of the functional

$$F(u) = \frac{1}{2}\int_{\Delta_t} |\nabla u|^2 \, dx - \int_{\Delta_t} u \, dx,$$

relative to $u \in H_0^1(\Delta_t)$. Therefore,

$$-\frac{1}{2}A(\Delta_0) \geq \frac{1}{2}\int_{\Delta_t} |\nabla u_t|^2 \, dx - \int_{\Delta_t} u_t \, dx = -\frac{1}{2}\int_{\Delta_t} u_t \, dx = -\frac{1}{2}A(\Delta_t).$$

Thus $A(\Delta_0) \leq A(\Delta_t)$, for every $t \in [0, 1]$. Since the above argument can be repeated with the initial design, Δ_0, replaced by any intermediate design, $\Delta_{t'}$, we infer that $A(\Delta_t)$ is increasing as a function of t, as desired. □

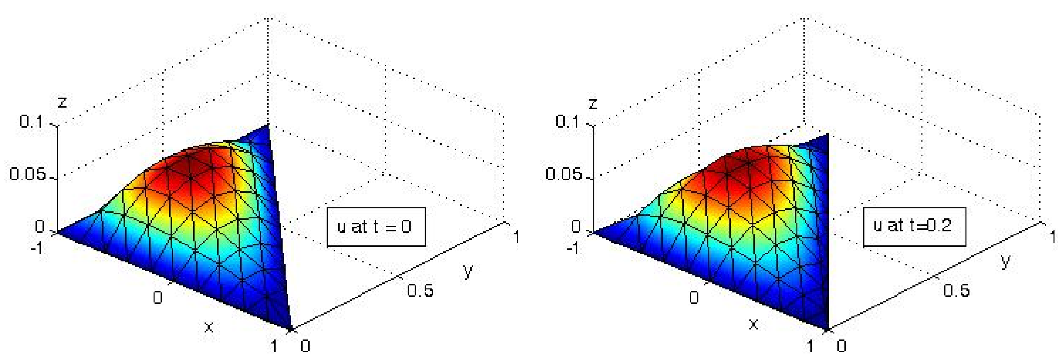
3. Numerical Results

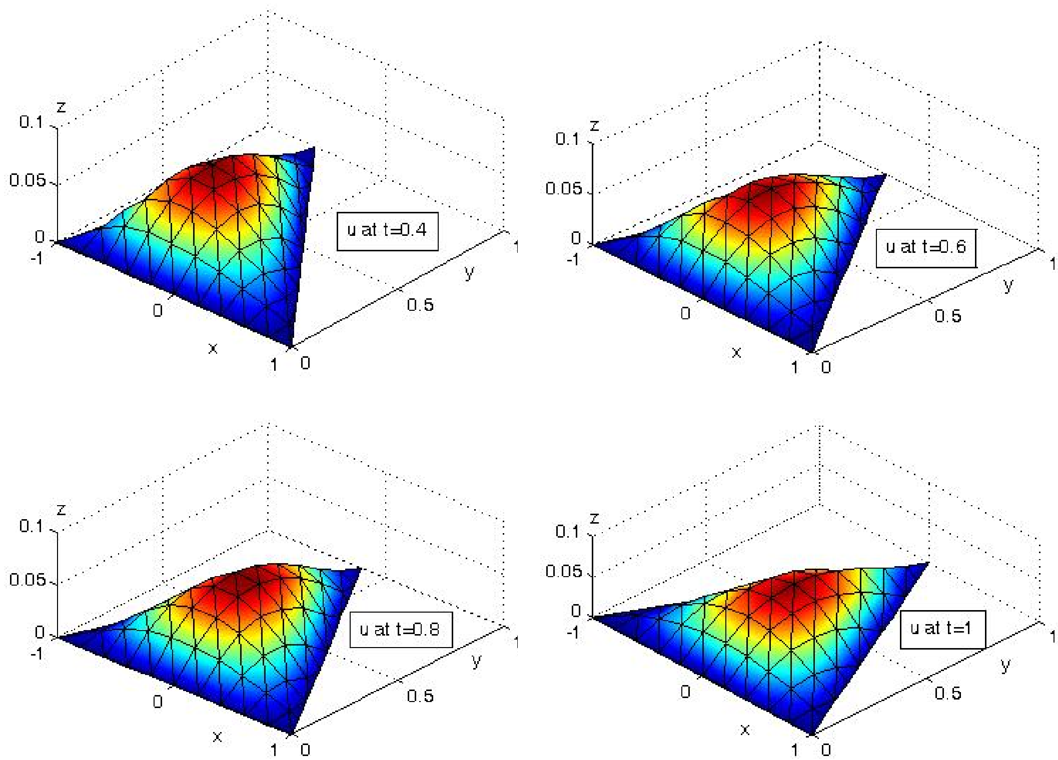
In this section we demonstrate numerically the increasing nature of $A(\Delta_t)$ on $[0, 1]$. To this end, for increasing values of $t \in [0, 1]$ we first use a standard finite element Galerkin method with triangular basis functions to obtain an approximate global solution of (BVP). Then the integration is performed exactly by integrating the basis functions to obtain an approximate value for $A(\Delta_t)$.

For our computation we choose $t = 0, 0.2, 0.4, 0.6, 0.8$ and 1. In each case, we use a mesh with grid spacing 0.2 along the $x-$axis and 0.1 along the $y-$axis. The mesh points are used to generate a sequence of sub-triangles that cover the region with only edges as overlap and with no sub-triangle having all its vertices on the boundary. We point out that a basis for the finite element space is a function of the vertices of the sub-triangles (see for example, (11) in [Alberty et al., 1999]). Thus, on integration of the approximate global solution we ensure that no sub-triangle yields a zero contribution. It should be noted that a zero contribution from any sub-triangle amounts to it not being present in the region.

Figure 2 shows the graphs of the approximate solutions u_t of (BVP) for the different values of t selected.

Figure 2





t	$A(\Delta_t)$
0	0.212–01
0.2	0.227–01
0.4	0.235–01
0.6	0.242–01
0.8	0.245–01
1.0	0.245–01

Table 1: A(Δ_t) for different values of t

On integration, we obtain the values shown in Table 1. From the table, it is clear that $A(\Delta t)$ increases as t increases from 0 to 0.8. The values when $t = 0.8$ and $t = 1$ appear to be equal. In order to see the increase in $A(\Delta t)$ in this case, a refinement of the mesh in both x- and y-directions is necessary.

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