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# SOLVING LINEAR PROGRAMMING PROBLEM VIA A REDUCED DIMENSION METHOD

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**Abstract:** Motivated by the alternating direction method for variational inequalities, we consider a reduced dimension method in this paper for the solution of linear programming problems. Its main idea is to reformulate the complementary conditions in the primal-dual optimality conditions as a linear projection equation. By using this reformulation, we only need to make one projection and solve a linear system with reduced dimension at each iterate. Under weak conditions, the global convergence is established.

**Keywords:** Linear programming, Optimality condition, Projection equation, Global convergence.

**AMS(2000) Subject Classification 65K05, 49D37**

## 1. Introduction

In this paper, we consider an algorithm for the solution of linear programming problem in the primal form

$$\begin{array}{l} \min c^T x \\ \text{s.t. } Ax = b,\ x \geq 0. \end{array} \tag{1}$$

or in the dual form

$$\begin{array}{l} \max b^T y \\ \text{s.t. } A^T y + z = c,\ z \geq 0. \end{array} \tag{2}$$

where $A \in R^{m\times n},\ c \in R^n,\ b \in R^m$ are the given data.

Both the primal and the dual linear programming have the same optimality conditions, namely

$$\begin{cases} A^T y + z = c, \\ Ax = b, \\ x_i \geq 0, z_i \geq 0, x_i z_i = 0,\ i = 1, 2, \cdots, n. \end{cases} \tag{3}$$

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Consequently, the primal problem (1) has a optimality solution $x^\star \in R^n$ if and only if the dual problem (2) has a optimality solution. Moreover, any of these two conditions is equivalent to the solvability of the optimality conditions (3). Hence solving the optimality conditions (3) is completely equivalent to solve the original line programming problem (1).

Primal-dual interior point methods are a class of efficient methods for solving linear programming problems. The basic idea of these algorithms is based on the optimality conditions (3) and introduces a certain perturbation of (3) depending on a parameter $\tau > 0$ :

$$\begin{cases} A^T y + z = c, \\ Ax = b, \\ x_i > 0, z_i > 0, x_i z_i = \tau, \ i = 1, 2, \cdots, n. \end{cases} \tag{4}$$

The system (4) is usually called the central path conditions, under certain assumptions, there is a unique solution $\omega_\tau = (x_\tau, y_\tau, z_\tau)$ of (4) for each $\tau > 0$. The corresponding mapping

$$\tau \hookrightarrow \omega_\tau$$

is called the central path, and the main idea of interior point methods is to follow this central path numerically. This is typically done by applying Newton method to the equations within the central path conditions (4), whereas a suitable stepsize takes care of the strict inequality constraints.

Smoothing -type methods follow a different approach. The general idea of these methods is to reformulate the optimality conditions (3) as a system of equation (not involving any inequalities):

$$\Phi(x, y, z) = 0 \tag{5}$$

Since system (5) is nonsmooth in general, it then gets approximated by a smooth system of equations to which Newton's method can be applied, see [Chen and Chen, 1999], [Chen and Xiu, 1999], [Hotta and Yoshise, 1999], [Tseng, 1998] and references therein for a couple of examples following this pattern.

Recently, by borrowing some idea from interior point methods and smoothing-type methods, [Engelke and Kanzow, 1999] proposed a predictor-corrector method for the solution of linear programming. Its main idea is to reformulate the optimal conditions (3) as the following nonlinear and nonsmooth equation:

$$\Phi_\tau(\omega) = \Phi(x, y, z) = \begin{pmatrix} c - A^T y - z \\ Ax - b \\ \phi_\tau(x, z) \end{pmatrix} = 0$$

where

$$\phi_\tau(x, z) = (\varphi_\tau(x_1, z_1), \varphi_\tau(x_2, z_2), \cdots, \varphi_\tau(x_n, z_n))$$

and $\varphi_\tau : R^2 \to R$ denotes the smoothed minimum function:

$$\varphi_\tau(a, b) = a + b - \sqrt{(a - b)^2 + 4\tau^2}$$

The method enjoy some merit of smoothing-type methods and interior point methods and has stronger convergence properties under weak assumption. However, it has to solve two full dimension linear systems at each iterate. Moreover, it requires the matrix $A$ have full rank and the starting point $\omega^0 = (x^0, y^0, z^0)$ satisfy the linear equations $A^T y + z = c$ and $Ax = b$. Although the authors stressed that the components $x^0$ and $z^0$ do not have to be positive (like in interior point methods) and it is relatively easy to find such a starting point, it still needs solving two linear system or two simple programs to obtain such a solution, and therefore increase the computation of the algorithm. In fact, these conditions are often used in many smoothing-type algorithm [Engelke and Kanzow, 2001, 2002].

To overcome these drawbacks, we propose a reduced dimension method for the solution of linear programming. The method is motivated by the alternating direction methods for variational inequalities[Wang, Yang and He, 2000] and semidefinite programming problem [Yu, 2004]. Its main difference from interior point methods and smoothing-type methods is that we reformulate the complementarity condition in (3) to a linear projection equation. By using this reformulation, we only have to make one projection and solve a linear system with reduced dimension at each iterate. This can be a significant advantage for large-scale problems since reduced dimension problems can be solved much more efficiently than full dimension ones. Moreover, without requiring full rank of the matrix $A$, we establish the global convergence from any starting point.

This paper is organized as follows: In Section 2, we develop our algorithm and give some preliminaries. In Section 3, we analyze the convergence properties of the algorithm. We conclude the paper with some remarks in the final section.

The notation used in this paper is standard: $R^n$ denotes the n-dimension real vector space. All vectors used in this paper are column vectors, $^T$ denotes transpose, if $\omega = (x^T, y^T, z^T)^T$, we often simply our notation and write $\omega = (x, y, z)$, $\|\cdot\|$ denotes 2-norm. $I$ represents the identity matrix with a consistent dimension.

## 2. Algorithm

In this section, we develop our algorithm for the solution of linear programming problem.

For a given vector $z \in R^n$, define the projection from $z$ onto $R^n_+$ by

$$P(z) = argmin\{\|z - z^+\| \mid z^+ \geq 0\}.$$

The following result is well known about the projection operator $P(\cdot)$, (see [Calamai and More, 1987])

**Lemma1** For any vector $x^{'}, y^{'} \in R^n,\ z^{'} \geq 0$,we have

$$(P(x^{'}) - x^{'})^T (z^{'} - P(x^{'})) \geq 0. \qquad (a)$$

$$\|P(x^{'}) - P(y^{'})\| \leq \|x^{'} - y^{'}\|. \qquad (b)$$

Using the equivalent relationship between complementarity problems and projection equation, the complementarity conditions in (3) is equivalent to the following projection equation:

$$z = P(z - x) \qquad (6)$$

By the above equivalent relationship, let $\omega = (x, y, z)$, the conditions (3) can be rewritten as

$$\Phi(\omega) = \Phi(x, y, z) = \begin{pmatrix} c - A^T y - z \\ Ax - b \\ z - P(z - x) \end{pmatrix} = 0 \qquad (7)$$

Noting that for a given vector $(x_k, y_k)$, if $z_k \geq 0$ satisfying the equality

$$z_k = P(c - A^T y_k - x_k) \qquad (8)$$

at the same time, $(x_k, y_k, z_k)$ satisfies $c - A^T y_k - z_k,\ Ax_k - b = 0$, then $\omega_k = (x_k, y_k, z_k)$ is a solution of equation (7), and therefore $x_k$ is a solution of (1). Hence our main work is how to obtain $x_{k+1}, y_{k+1}$ for the obtained $(x_k, y_k, z_k)$. Before describe the algorithm model, we first give the following Lemma.

**Lemma 2** Assume $\omega_\star = (x_\star, y_\star, z_\star) \in \Omega^\star$, then we have

$$\begin{pmatrix} x_k - x_\star \\ y_k - y_\star \end{pmatrix}^T \begin{pmatrix} I_n & A^T \\ -A & I_m \end{pmatrix} \begin{pmatrix} c - A^T y_k - z_k \\ Ax_k - b \end{pmatrix} \geq \|c - A^T y_k - z_k\|^2 + \|Ax_k - b\|^2.$$

**Proof.** Since $\omega_\star \in \Omega^\star$, we have

$$\begin{cases} c - A^T y_\star - z_\star = 0 \\ Ax_\star - b = 0 \end{cases} \qquad (9)$$

and $z_\star \geq 0,\ x_\star \geq 0,\ x_\star^T z_\star = 0.$
Since $z_k \geq 0$, it follows that

$$(z_k - z_\star)^T x_\star \geq 0$$

and therefore

$$\begin{pmatrix} z_k - z_\star \\ 0_m \end{pmatrix}^T \begin{pmatrix} x_\star \\ y_\star \end{pmatrix} \geq 0 \qquad (10)$$

On the other hand, from (8) and Lemma 1(a), we get

$$(z_\star - z_k)^T (x_k - (c - A^T y_k - z_k)) \geq 0$$

and therefore

$$\begin{pmatrix} z_\star - z_k \\ 0_m \end{pmatrix}^T \begin{pmatrix} x_k - (c - A^T y_k - z_k) \\ y_k - (Ax_k - b) \end{pmatrix} \geq 0 \qquad (11)$$

From inequalities (10) and (11), we obtain that

$$\begin{pmatrix} z_\star - z_k \\ 0_m \end{pmatrix}^T \begin{pmatrix} x_k - x_\star - (c - A^T y_k - z_k) \\ y_k - y_\star - (Ax_k - b) \end{pmatrix} \geq 0 \qquad (12)$$

According to (9), (12) can be rewritten as

$$\begin{pmatrix} c - A^T y_\star - z_k \\ Ax_\star - b \end{pmatrix}^T \begin{pmatrix} x_k - x_\star - (c - A^T y_k - z_k) \\ y_k - y_\star - (Ax_k - b) \end{pmatrix} \geq 0$$

Rearranging the above inequality, we get the desired result.

This result shows that

$$-\begin{pmatrix} c - A^T y_k - z_k \\ Ax_k - b \end{pmatrix}$$

provides a descent direction for the function

$$\frac{1}{2}\left\| \begin{pmatrix} I_n & A^T \\ -A & I_m \end{pmatrix} \begin{pmatrix} x_k - x_\star \\ y_k - y_\star \end{pmatrix} \right\|^2$$

Thus we can establish the algorithm as follows:

**Algorithm 1 (A Reduced Dimension Algorithm for LP)**

Step 0 Choose $(x_0, y_0) \in R^n \times R^m,\ \gamma \in (0, 2),\ \varepsilon > 0,\ k := 0$

Step 1 Compute $z_k = P(c - A^T y_k - x_k)$, if $\|c - A^T y_k - z_k\|^2 + \|Ax_k - b\|^2 \leq \varepsilon$, stop. Otherwise, go to Step 2.

Step 2 Compute $d_k = (d_k^x, d_k^y) \in R^n \times R^m$ by solving the following linear equation

$$\begin{pmatrix} I_n & -A^T \\ A & I_m \end{pmatrix} \begin{pmatrix} d_k^x \\ d_k^y \end{pmatrix} = -\gamma \begin{pmatrix} c - A^T y_k - z_k \\ Ax_k - b \end{pmatrix} \tag{13}$$

Set $(x_{k+1}, y_{k+1}) = (x_k, y_k) + (d_k^x, d_k^y),\ k = k + 1$, go to Step 1.

## 3. Global Convergence

In this section, we discuss the global convergence property of our algorithm. In what follows, we assume that the solution set $\Omega^\star$ of (3) is nonempty.

The next result accounts for the terminate rule used in Step1.

**Lemma 3** Let $\omega_k = (x_k, y_k, z_k)$ be generated by Algorithm 1. Then we have

$$\|\Phi(\omega_k)\|^2 \leq 2(\|c - A^T y_k - z_k\|^2 + \|Ax_k - b\|^2).$$

The following result plays an important role in the convergence analysis.

**Lemma 4** Let $\{\omega_k\}$ be generated by Algorithm 1. Then for any $\omega_\star \in \Omega^\star$, we have

$$\begin{aligned} \left\| \begin{pmatrix} I_n & -A^T \\ A & I_m \end{pmatrix} \begin{pmatrix} x_{k+1} - x_\star \\ y_{k+1} - y_\star \end{pmatrix} \right\|^2 &\leq \left\| \begin{pmatrix} I_n & -A^T \\ A & I_m \end{pmatrix} \begin{pmatrix} x_k - x_\star \\ y_k - y\star \end{pmatrix} \right\|^2 \\ &- \gamma(2 - \gamma)(\|c - A^T y_k - z_k\|^2 + \|Ax_k - b\|^2) \end{aligned}$$

**Proof.** From equation (13), we have

$$\begin{pmatrix} I_n & -A^T \\ A & I_m \end{pmatrix} \begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} = \begin{pmatrix} I_n & -A^T \\ A & I_m \end{pmatrix} \begin{pmatrix} x_k \\ y_k \end{pmatrix} - \gamma \begin{pmatrix} c - A^T y_k - z_k \\ Ax_k - b \end{pmatrix}$$

Adding $\begin{pmatrix} I_n & -A^T \\ A & I_m \end{pmatrix} \begin{pmatrix} -x_\star \\ -y_\star \end{pmatrix}$ from both sides in the above equality, we have

$$\begin{pmatrix} I_n & -A^T \\ A & I_m \end{pmatrix} \begin{pmatrix} x_{k+1} - x_\star \\ y_{k+1} - y_\star \end{pmatrix} = \begin{pmatrix} I_n & -A^T \\ A & I_m \end{pmatrix} \begin{pmatrix} x_k - x_\star \\ y_k - x_\star \end{pmatrix} - \gamma \begin{pmatrix} c - A^T y_k - z_k \\ Ax_k - b \end{pmatrix}$$

Hence by means of lemma 2, we obtain

$$\begin{aligned} \| \begin{pmatrix} I_n & -A^T \\ A & I_m \end{pmatrix} \begin{pmatrix} x_{k+1} - x_\star \\ y_{k+1} - y_\star \end{pmatrix} \|^2 &= \| \begin{pmatrix} I_n & -A^T \\ A & I_m \end{pmatrix} \begin{pmatrix} x_k - x_\star \\ y_k - x_\star \end{pmatrix} - \gamma \begin{pmatrix} c - A^T y_k - z_k \\ Ax_k - b \end{pmatrix} \|^2 \\ &= \| \begin{pmatrix} I_n & -A^T \\ A & I_m \end{pmatrix} \begin{pmatrix} x_k - x_\star \\ y_k - y_\star \end{pmatrix} \|^2 \\ &\quad -2\gamma \begin{pmatrix} x_k - x_\star \\ y_k - y_\star \end{pmatrix}^T \begin{pmatrix} I_n & A^T \\ -A & I_m \end{pmatrix} \begin{pmatrix} c - A^T y_k - z_k \\ Ax_k - b \end{pmatrix} \\ &\quad +\gamma^2 \| \begin{pmatrix} c - A^T y_k - z_k \\ Ax_k - b \end{pmatrix} \|^2 \\ &\leq \| \begin{pmatrix} I_n & A^T \\ -A & I_m \end{pmatrix} \begin{pmatrix} x_k - x_\star \\ y_k - y_\star \end{pmatrix} \|^2 \\ &\quad -\gamma(2-\gamma)(\|Ax_k - b\|^2 + \|c - A^T y_k - z_k\|^2) \end{aligned}$$

This completes the proof.

**Lemma 5** Let $\{\omega_k\}$ be generated by Algorithm 1. Then we have

$$\lim_{k \to \infty} [\|Ax_k - b\|^2 + \|c - A^T y_k - z_k\|^2] = 0$$

and

$$\lim_{k \to \infty} \|\Phi(\omega_k)\| = 0.$$

**Proof.** From lemma 3 and lemma 4 ,the result is easily obtained.

The following result is our main convergence result of algorithm 1.

**Theorem 1** Suppose that $\{\omega_k\}$ is generated by Algorithm 1. Then the whole sequence $\{\omega_k\}$ converges to a solution of problem (3).

**Proof.** Suppose that $\overline{\omega} = (\overline{x}, \overline{y}, \overline{z})$ is a solution of (3), since

$$\begin{aligned} \| \begin{pmatrix} I_n & A^T \\ -A & I_m \end{pmatrix} \begin{pmatrix} x_k - \overline{x} \\ y_k - \overline{y} \end{pmatrix} \|^2 &= \| \begin{pmatrix} x_k - \overline{x} \\ y_k - \overline{y} \end{pmatrix} + \begin{pmatrix} 0_n & A^T \\ -A & 0_m \end{pmatrix} \begin{pmatrix} x_k - \overline{x} \\ y_k - \overline{y} \end{pmatrix} \|^2 \\ &= \| \begin{pmatrix} x_k - \overline{x} \\ y_k - \overline{y} \end{pmatrix} \|^2 + \| \begin{pmatrix} 0_n & A^T \\ -A & 0_m \end{pmatrix} \begin{pmatrix} x_k - \overline{x} \\ y_k - \overline{y} \end{pmatrix} \|^2 \end{aligned}$$

hence from lemma 4, we have

$$\| \begin{pmatrix} x_k - \overline{x} \\ y_k - \overline{y} \end{pmatrix} \|^2 \leq \| \begin{pmatrix} I_n & A^T \\ -A & I_m \end{pmatrix} \begin{pmatrix} x_k - \overline{x} \\ y_k - \overline{y} \end{pmatrix} \|^2 \leq \| \begin{pmatrix} I_n & A^T \\ -A & I_m \end{pmatrix} \begin{pmatrix} x_0 - \overline{x} \\ y_0 - \overline{y} \end{pmatrix} \|^2 \tag{14}$$

On the other hand

$$\begin{aligned}\|z_k - \overline{z}\| &= \|z_k - (c - A^T\overline{y})\| \\ &= \|c - A^T y_k - z_k + A^T(y_k - \overline{y})\| \\ &\leq \|c - A^T y_k - z_k\| + \|A^T(y_k - \overline{y})\| \quad (15)\end{aligned}$$

Hence the sequence $\{(x_k, y_k, z_k)\}$ is bounded, as a result, it has at least one accumulation point.
Let $(x_\star, y_\star, z_\star)$ be an accumulation of $\{(x_k, y_k, z_k)\}$ and $\{(x_{ki}, y_{ki}, z_{ki})\}$ converges to $(x_\star, y_\star, z_\star)$. From Lemma 5, we have

$$\Phi(x_\star, y_\star, z_\star) = \lim_{i\to\infty} \Phi(x_{ki}, y_{ki}, z_{ki}) = 0.$$

Hence $(x_\star, y_\star, z_\star)$ is a solution of problem (3). Substituting $(\overline{x}, \overline{y}, \overline{z})$ in (14)(15) by $(x_\star, y_\star, z_\star)$ we have

$$\| \begin{pmatrix} x_k - x_\star \\ y_k - y_\star \end{pmatrix} \|^2 \leq \| \begin{pmatrix} I_n & A^T \\ -A & I_m \end{pmatrix} \begin{pmatrix} x_k - x_\star \\ y_k - y_\star \end{pmatrix} \|^2 \quad (16)$$

and

$$\|z_k - z_\star\| \leq \|c - A^T y_k - z_k\| + \|A^T(y_k - y_\star)\| \quad (17)$$

Since $\{(x_{ki}, y_{ki})\}$ is a subsequence of $\{(x_k, y_k)\}$, the limitation of sequence

$\{\| \begin{pmatrix} I_n & A^T \\ -A & I_m \end{pmatrix} \begin{pmatrix} x_k - x_\star \\ y_k - y_\star \end{pmatrix} \|\}$ exists, hence

$$\lim_{k\to\infty} \| \begin{pmatrix} I_n & A^T \\ -A & I_m \end{pmatrix} \begin{pmatrix} x_k - x_\star \\ y_k - y_\star \end{pmatrix} \| = 0.$$

So it follows from (16)(17) that

$$\lim_{k\to\infty} (x_k, y_k, z_k) = (x_\star, y_\star, z_\star).$$

This completes the proof.

## 4. Conclusion

In this paper, we develop a reduced dimension algorithm for linear programming, compared with the interior point algorithm and smoothing algorithm, our algorithm enjoys some better properties and convergence result. How to obtain the fast convergence property of the algorithm deservers further studying.

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