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SOLVING NON-LINEAR FRACTIONAL PROGRAMMING PROBLEM WITH A SPECIAL STRUCTURE USING APPROXIMATION

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Abstract

This paper presents a technique for solving a special class of non-linear fractional programs where the numerator and denominator both are separable functions. Using the concept of piecewise linear approximation, the numerator and denominator both are linearized to form a linear fractional program. The problem is then solved using Charnes and Cooper transformation method. It is proved that the optimal solution of this problem is also the solution of the given problem. It is illustrated by a numerical example.

Keywords: Fractional programming; piecewise linear approximation; separable functions; strictly convex functions; grid points.

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1. INTRODUCTION

Consider the single ratio fractional program

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$$\text{(P)} \qquad \min \frac{f(x)}{h(x)} = \min \frac{\sum_{j=1}^{n} f_j(x_j) + \alpha_0}{\sum_{j=1}^{n} h_j(x_j) + \beta_0}$$

$$= \frac{\sum_{j=1}^{n} k_j x_j^2 + \sum_{j=1}^{n} \ell_j x_j + \alpha_0}{\sum_{j=1}^{n} m_j x_j^2 + \sum_{j=1}^{n} n_j x_j + \beta_0}$$

subject to

$$\sum_{j=1}^{n} g_{ij}(x_j) \le b_i \qquad \text{for } i = 1, 2, \ldots m$$

where f(x) and h(x) are separable nonlinear functions of x, either both k_j and m_j are zero or non-zero, g_{ij}'s are linear ($1 \le i \le m$; $1 \le j \le n$), $b_i \in \mathbb{R}^m$, h(x) is assumed to be non-negative on $X = \{x \in \mathbb{R}^n : g_{ij}(x_j) \le b_i,\ 1 \le i \le m,\ 1 \le j \le n\}$

$x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$.

It is assumed that S is a non-empty bounded convex polyhedron.

These problems are interesting from both theoretical and practical points of view as they arise in some mathematical programming problems and in various practical problems.

Fractional programming problems have been a subject of wide interest since they arise in many fields like agricultural planning, financial analysis of a firm, location theory, capital budgeting problem, portfolio selection problem, cutting stock problem, stochastic processes problem. From time to time survey papers on applications and algorithms on fractional programming have been presented by various authors.

Systematic studies and applications of single-ratio fractional programs generally began to appear in the literature in the early 1960s. Since then a rich body of work has been accomplished on the classification, theory, applications and solutions of these problems. An overview of this work is contained in the articles by Schaible [13], the monographs by Craven [6]. Chang [4] proposed a model which required auxilliary constraints to linearize the mixed 0–1 fractional programming problem. Practical examples of Quadratic Fractional Programs occur in many decision problems, where the criteria is expressed as the ratio of two quadratics. Although the problem (P) can be solved in its present form, but we use the approximation technique to solve it as the approximated problems are easy to handle as compared to the non-linear problems. In fact the approximation technique is more efficient for solving large scale problems.

Single ratio fractional programs appeared in the literature systematically in the early 1960s. The problem (P) presented in this paper is a non-concave fractional program. There are a number of approaches available for globally solving concave fractional programs. In fact, by transforming a non-concave fractional program into a concave program, a great number of the methods of concave programming become available for solution of the problem. Nonconcave fractional programs arise in certain important applications, like portfolio selection problems [9, 11] and stochastic decision making problems [15]. Nonconcave fractional programs fall into the domain of global optimization since a local minimum need not be a global minimum.

In this article we are concerned with a special class of nonconcave fractional programs (P). The article is organized as follows:

Section 2 presents theoretical properties of the problem and reduction of the given problem into a linear fractional program. Section 3 presents a numerical example as an illustration.

2. DEFINITIONS AND THEORETICAL DEVELOPMENT

In the process of theoretical development, we make use of the following definitions:

Definition 1. Let f be a real valued function defined on a convex set X in $\mathbb{R}^n$. The function f is said to be separable if it can be expressed as the sum of single variable functions.

i.e. $$f(x) = f(x_1, x_2, ..., x_n) = \sum_{j=1}^{n} f_j(x_j) \qquad \forall x \in X$$

Definition 2. Let f be a real valued function defined on a convex set X in $\mathbb{R}^n$. The function f is said to be strictly convex on S if

$$f(\lambda x_1 + (1-\lambda)x_2) < \lambda f(x_1) + (1-\lambda) f(x_2) \qquad \forall x_1, x_2 \in S,$$

$x_1 \neq x_2$ and for each $\lambda \in (0, 1)$.

During the process, we have made use of a property of linear functions that a linear function is both convex and concave in nature.

Reduction to Linear Programming Problem (LPP)

Given problem (P) is

$$\text{(P)} \qquad \min \frac{\sum_{j=1}^{n} f_j(x_j) + \alpha_0}{\sum_{j=1}^{n} h_j(x_j) + \beta_0} = \frac{\sum_{j=1}^{n} k_j x_j^2 + \sum_{j=1}^{n} \ell_j x_j + \alpha_0}{\sum_{j=1}^{n} m_j x_j^2 + \sum_{j=1}^{n} n_j x_j + \beta_0}$$

subject to

$$\sum_{j=1}^{n} g_{ij}(x_j) \leq b_i, \text{ for } i = 1, ...m,$$

$$x_j \geq 0, \ j = 1, ...n$$

where either both k_j and m_j are zero or non-zero.

Let $L = \{j : f_j \text{ and } h_j \text{ are linear}\}$

Suppose that for $j \notin L$, f_j and h_j are strictly convex and that g_{ij} is linear for i = 1, ...m.

Suppose that for each $j \notin L$, f_j, h_j and g_{ij}, for i = 1, ..., m are replaced by their piecewise linear approximations via the grid points $x_{\nu j}$ for $\nu = 1, ...p_j$, yielding the linear fractional program below

$$\text{(P1)} \quad \min \frac{\sum_{j\in L} f_j(x_j) + \sum_{j\notin L} \sum_{\nu=1}^{p_j} \lambda_{\nu j} f_j(x_{\nu j}) + \alpha_0}{\sum_{j\in L} h_j(x_j) + \sum_{j\notin L} \sum_{\nu=1}^{p_j} \lambda_{\nu j} h_j(x_{\nu j}) + \beta_0}$$

subject to

$$\sum_{j\in L} g_{ij}(x_j) + \sum_{j\notin L} \sum_{\nu=1}^{p_j} \lambda_{\nu j} g_{ij}(x_{\nu j}) \le b_i \qquad \text{for } i = 1, ..., m$$

$$\sum_{\nu=1}^{p_j} \lambda_{\nu j} = 1 \qquad \text{for } j \notin L$$

$\lambda_{\nu j} \ge 0$, for $\nu = 1, ...p_j$; $j \notin L$

$x_j \ge 0$ for $j \in L$

where atmost two adjacent $\lambda_{\nu j}$'s are positive for $j \notin L$

With the exception of the constraint that, atmost, two adjacent $\lambda_{\nu j}$'s are positive for $j \notin L$, the above problem is a linear fractional program

Using the well known Charnes and Cooper transformation [5]

$$t = \frac{1}{\sum_{j\in L} h_j(x_j) + \sum_{j\notin L} \sum_{\nu=1}^{p_j} \lambda_{\nu j} h_j(x_{\nu j}) + \beta_0}$$

$y_j = tx_j, \quad j \in L,$

this fractional program can be reduced to the linear program

$$\min \sum_{j \in L} f_j(y_j) + \sum_{j \notin L} \sum_{\nu=1}^{p_j} \lambda'_{\nu j} f_j(x_{\nu j}) + \alpha_0 t$$

subject to

$$\text{(P2)} \qquad \sum_{j \in L} g_{ij}(y_j) + \sum_{j \notin L} \sum_{\nu=1}^{p_j} \lambda'_{\nu j} g_{ij}(x_{\nu j}) - tb_i \le 0, \quad i = 1, 2, \ldots, m$$

$$\sum_{\nu=1}^{p_j} \lambda'_{\nu j} - t = 0$$

$$\sum_{j \in L} h_j(y_j) + \sum_{j \notin L} \sum_{\nu=1}^{p_j} \lambda'_{\nu j} h_j(x_{\nu j}) + \beta_0 t = 1$$

$$y_j \ge 0 \quad \text{for } j \in L$$

$$\lambda'_{\nu j} \ge 0 \qquad \text{for } \nu = 1, \ldots, p_j; \; j \notin L$$

Let $\hat{x}_j$ for $j \in L$ and $\hat{\lambda}_{\nu j}$ for $\nu = 1, \ldots, p_j$ and $j \notin L$ solve the above problem. Then we prove the following two theorems:

Theorem 2.1: For each $j \notin L$, atmost two $\lambda_{\nu j}$'s are positive then they must be adjacent.

Proof: To prove this, it suffices to show that for each $j \notin L$, if $\hat{\lambda}_{ij}$ and $\hat{\lambda}_{\rho j}$ are positive, then the grid points x_{ij} and $x_{\rho j}$ must be adjacent.

Let, if possible, there exist $\hat{\lambda}_{ij} > 0$ where x_{ij} and $x_{\rho j}$ be not adjacent.

Then there exists a grid point $x_{\gamma j} \in (x_{ij}, x_{\rho j})$ that can be expressed as

$$x_{\gamma j} = \alpha_1 x_{ij} + \alpha_2 x_{\rho j}$$

where $\alpha_1, \alpha_2 > 0$ s.t. $\alpha_1 + \alpha_2 = 1$

Now, consider the optimal solution to the problem defined above,

Let $u_i \geq 0$ for i = 1, ...m be the optimum Lagrangian multipliers associated with the first m constraints, v_j (for each $j \notin L$) be the optimum Lagrangian multiplier associated with the constraint $\sum_{\nu=1}^{p_j} \lambda'_{\nu j} = 1$ and w_j be the optimum Lagrangian multiplier associated with the constraint

$$\sum_{j \in L} h_j(y_j) + \sum_{j \notin L} \sum_{\nu=1}^{p_j} \lambda'_{\nu j} h_j(x_{\nu j}) + \beta_0 t = 1$$

Then the following subset of the Kuhn-Tucker necessary conditions are satisfied

$$\sum_{j \notin L} f_j(x_{ij}) + \sum_{i=1}^{m} u_i g_{ij}(x_{ij}) + v_j + \sum_{j \notin L} w_j h_j(x_{ij}) = 0 \qquad \text{(a)}$$

$$\sum_{j \notin L} f_j(x_{\rho j}) + \sum_{i=1}^{m} u_i x_{\rho j}(x_{ij}) + v_j + \sum_{j \notin L} w_j h_j(x_{\rho j}) = 0 \qquad \text{(b)}$$

$$\sum_{j \notin L} f_j(x_{\nu j}) + \sum_{i=1}^{m} u_i g_{ij}(x_{\nu j}) + v_j + \sum_{j \notin L} w_j h_j(x_{\nu j}) \geq 0 \qquad \text{(c)}$$

Claim: Condition (c) is violated for $v = \gamma$ since f_j and h_j are strictly convex and g_{ij}'s are convex, and using (a) and (b), we have

$$\sum_{\nu \notin L} f_j(x_{\gamma j}) + \sum_{i=1}^{m} u_i g_{ij}(x_{\gamma j}) + v_j + \sum_{\nu \notin L} w_j h_j(x_{\gamma j}) <$$
$$\sum_{j \notin L} (\alpha_1 f_j(x_{ij}) + \alpha_2 f_j(x_{\rho j})) + \sum_{i=1}^{m} u_i (\alpha_1 g_{ij}(x_{ij}) + \alpha_2 g_{ij}(x_{\rho j}))$$
$$+ \sum_{j \notin L} w_j (\alpha_1 h_j(x_{ij}) + \alpha_2 h_j(x_{\rho j})) + v_j = 0$$

which contradicts (c) for $\nu = \gamma$ and hence x_{ij} and $x_{\rho j}$ must be adjacent.

This completes the proof of the theorem.

Theorem 2.2: Let $\hat{x}_j = \sum_{\nu=1}^{p_j} \hat{\lambda}_{\nu j} x_{\nu j}$ for $j \notin L$. Then the vector $\hat{x}$ whose jth component is $\hat{x}_j$ for $j \notin L$ and y_j for $j \in L$ is feasible to problem (P).

Proof: For proving this theorem, making use of the facts that g_{ij} is convex for $j \notin L$ and for each i = 1, ...,m and $\hat{y}_j$ for $j \in L$ and $\hat{\lambda}_{\nu j}$ for ν = 1, ..., p_j and $j \notin L$ satisfy the constraints in (1), we have

$$g_i(\hat{x}) = \sum_{j \in L} g_{ij}(y_j) + \sum_{j \notin L} g_{ij}(\hat{x}_j)$$

$$= \sum_{j \in L} g_{ij}(y_j) + \sum_{j \notin L} g_{ij}\left(\sum_{\nu=1}^{k_j} \hat{\lambda}_{\nu j} x_{\nu j} \right)$$

$$\leq \sum_{j \in L} g_{ij}(y_j) + \sum_{j \notin L} \sum_{\nu=1}^{k_j} \hat{\lambda}_{\nu j} g_{ij}(x_{\nu j})$$

$$\leq b_i \qquad \text{for } i = 1, ...m$$

Further $y_j \geq 0$ for $j \in L$ and $\hat{x}_j = \sum_{\nu=1}^{p_j} \lambda_{\nu j} x_{\nu j} \geq 0$ for $j \notin L$, since $\hat{\lambda}_{\nu j}$, $x_{\nu j}$ are non-negative. For ν = 1, ..., p_j and $j \notin L$.

Hence $\hat{x}$ is a feasible solution of problem (P). This completes the proof of the theorem.

Note: Since it is assumed that for each $j \notin L$, f_j and h_j are strictly convex and that g_{ij} is linear for i = 1, …, m, we arrive at an optimal solution to the original problem (P).

3. SOLUTION TECHNIQUE

In the course of theoretical development, a single ratio fractional program is considered with a special structure. At first, the numerator and denominator both, which are separable non-linear functions are replaced by their piecewise linear approximations via grid points [2], thereby reducing the problem (P) to a linear fractional program with the exception of the constraint that, atmost, two adjacent λ_{vj}'s are positive for j $\notin$ L. By using Charnes and Cooper transformation, the linear fractional program is further reduced to a linear program and an optimal solution to the problem is obtained.

4. TECHNICAL REPRESENTATION OF THE ALGORITHM

The procedure proposed above is summarized in the following algorithm:

Step 0: Initialization

Consider the problem

$$(P) \qquad \min \frac{\sum_{j=1}^{n} f(x_j) + \alpha_0}{\sum_{j=1}^{n} h_j(x_j) + \beta_0}$$

s.t. $g_{ij}(x_j) \le b_i$ $\quad i = 1, ..., m; \quad j = 1, ..., n$

$x_j \ge 0$ for $j = 1, 2, ..., n$

satisfying the assumptions of the problem

Step 1: Define the set L.

Step 2: Replace f_j, h_j and g_{ij} by their piecewise linear approximations for $j \notin L$, $i = 1,...,m$ thereby reducing the given problem to a linear fractional program (P1).

Step 3: Reduce the linear fractional program (P1) to an equivalent linear program (P2) using Charnes and Cooper transformation.

Step 4: Solve the problem (P2) which determines the optimal solution of problem (P)

5. NUMERICAL EXAMPLE

Consider the following separable quadratic fractional programming problem.

$$\min \frac{2x_1^2 + x_2^2 + 2x_3^2 - 6x_1 - 5x_3 + x_4 + 11}{x_1^2 + 2x_2^2 + x_3^2 + 2x_2 + x_3 + 35}$$

subject to

$$\begin{aligned}
x_1 + x_2 + x_3 + x_4 &\leq 4 \\
x_1 + x_3 &\leq 6 \\
x_1 + x_2 &\leq 6 \\
x_1 - 2x_2 &\leq 0 \\
-2x_3 + x_4 &\leq 0 \\
x_1, x_2, x_3, x_4 &\geq 0
\end{aligned}$$

This problem satisfies the assumptions of the problem considered in this paper.

Note that L = {4}, since there are no non-linear terms involving x_4 and hence we will not take any grid points for x_4.

It is clear from the constraint that x_1, x_2 and x_3 must lie in the interval [0, 6]. Although we are making use of equally spaced grid points, but it need not always be so.

For the variables x_1, x_2 and x_3, we use the grid points 0, 2, 4 and 6, so that

$x_{11} = 0, \quad x_{21} = 2, \quad x_{31} = 4, \quad x_{41} = 6;$

$x_{12} = 0, \quad x_{22} = 2, \quad x_{32} = 4, \quad x_{42} = 6;$

$x_{13} = 0, \quad x_{23} = 2, \quad x_{33} = 4, \quad x_{43} = 6.$

Therefore, $x_1 = 0\lambda_{11} + 2\lambda_{21} + 4\lambda_{31} + 6\lambda_{41}$

$= 2\lambda_{21} + 4\lambda_{31} + 6\lambda_{41}$

$x_2 = 0\lambda_{12} + 2\lambda_{22} + 4\lambda_{32} + 6\lambda_{42}$

$= 2\lambda_{22} + 4\lambda_{32} + 6\lambda_{42}$

$x_3 = 0\lambda_{13} + 2\lambda_{23} + 4\lambda_{33} + 6\lambda_{43}$

$= 2\lambda_{23} + 4\lambda_{33} + 6\lambda_{43}$

$\lambda_{11} + \lambda_{21} + \lambda_{31} + \lambda_{41} = 1$

$\lambda_{12} + \lambda_{22} + \lambda_{32} + \lambda_{42} = 1$

$\lambda_{13} + \lambda_{23} + \lambda_{33} + \lambda_{43} = 1$

$\lambda_{\nu 1}, \lambda_{\nu 2}, \lambda_{\nu 3} \geq 0 \quad \text{for } \nu = 1, 2, 3, 4$

The piecewise linear approximation of

$$f(x) = (-4\lambda_{21} + 8\lambda_{31} + 36\lambda_{41}) + (4\lambda_{22} + 16\lambda_{32} + 36\lambda_{42}) + (-2\lambda_{23} + 12\lambda_{33} + 42\lambda_{43}) + x_4 + 11$$

and the piecewise linear approximation of

$$h(x) = (4\lambda_{21} + 16\lambda_{31} + 36\lambda_{41}) + (4\lambda_{22} + 24\lambda_{32} + 60\lambda_{42}) + (6\lambda_{23} + 20\lambda_{33} + 42\lambda_{43}) + 3x_4 + 35$$

Hence, the problem reduces to the following linear fractional programming problem

$$\min \frac{(-4\lambda_{21}+8\lambda_{31}+36\lambda_{41})+(4\lambda_{22}+16\lambda_{32}+36\lambda_{42})+(-2\lambda_{23}+12\lambda_{33}+42\lambda_{43})+x_4+11}{(4\lambda_{21}+16\lambda_{31}+36\lambda_{41})+(4\lambda_{22}+24\lambda_{32}+60\lambda_{42})+(6\lambda_{23}+20\lambda_{33}+42\lambda_{43})+3x_4+35}$$

subject to

$$(2\lambda_{21} + 4\lambda_{31} + 6\lambda_{41}) + (2\lambda_{22} + 4\lambda_{32} + 6\lambda_{42}) + (2\lambda_{23} + 4\lambda_{33} + 6\lambda_{43}) + x_4 \leq 4$$

$$(2\lambda_{21} + 4\lambda_{31} + 6\lambda_{41}) + (2\lambda_{23} + 4\lambda_{33} + 6\lambda_{43}) \leq 6$$

$$(2\lambda_{21} + 4\lambda_{31} + 6\lambda_{41}) + (2\lambda_{22} + 4\lambda_{32} + 6\lambda_{42}) \leq 6$$

$$2\lambda_{21} + 4\lambda_{31} + 6\lambda_{41} - 4\lambda_{22} - 8\lambda_{32} - 12\lambda_{42} \leq 0$$

$$-4\lambda_{23} - 8\lambda_{33} - 12\lambda_{43} + x_4 \leq 0$$

$$\lambda_{11} + \lambda_{21} + \lambda_{31} + \lambda_{41} = 1$$

$$\lambda_{12} + \lambda_{22} + \lambda_{32} + \lambda_{42} = 1$$

$$\lambda_{13} + \lambda_{23} + \lambda_{33} + \lambda_{43} = 1$$

$$\lambda_{\nu j} \geq 0 \quad \text{for } \nu = 1, 2, 3, 4;\ j = 1, 2, 3$$

$$x_4 \geq 0$$

Atmost, two $\lambda_{\nu j}$'s are positive for $j \notin L$.

Relaxing the condition that, atmost two $\lambda_{\nu j}$'s are positive for $j \notin L$, the above problem is a linear fractional program which is reduced to the following linear programming problem by using Charnes and Cooper transformation

$$t = \frac{1}{(4\lambda_{21}+16\lambda_{31}+36\lambda_{41})+(4\lambda_{22}+24\lambda_{32}+60\lambda_{42})+(6\lambda_{23}+20\lambda_{33}+42\lambda_{43})+3x_4+35}$$

subject to

$$\min(-4\lambda'_{21}+8\lambda'_{31}+36\lambda'_{41})+(4\lambda'_{22}+16\lambda'_{32}+36\lambda'_{42})+(-2\lambda'_{23}+12\lambda'_{33}+42\lambda'_{43})+y_4+11$$

subject to

$$(2\lambda'_{21}+4\lambda'_{31}+6\lambda'_{41})+(2\lambda'_{22}+4\lambda'_{32}+6\lambda'_{42})+(2\lambda'_{23}+4\lambda'_{33}+6\lambda'_{43})+y_4-4t\le 0$$

$$(2\lambda'_{21}+4\lambda'_{31}+6\lambda'_{41})+(2\lambda'_{23}+4\lambda'_{33}+6\lambda'_{43})-6t\le 0$$

$$(2\lambda'_{21}+4\lambda'_{31}+6\lambda'_{41})+(2\lambda'_{22}+4\lambda'_{32}+6\lambda'_{42})-6t\le 0$$

$$2\lambda'_{21}+4\lambda'_{31}+6\lambda'_{41}-4\lambda'_{22}-8\lambda'_{32}-12\lambda'_{42}\le 0$$

$$-4\lambda'_{23}-8\lambda'_{33}-12\lambda'_{43}+y_4\le 0$$

$$\lambda'_{11}+\lambda'_{21}+\lambda'_{31}+\lambda'_{41}-t=0$$

$$\lambda'_{21}+\lambda'_{22}+\lambda'_{32}+\lambda'_{42}-t=0$$

$$\lambda'_{31}+\lambda'_{32}+\lambda'_{33}+\lambda'_{43}-t=0$$

$$\lambda'_{\nu j}\ge 0 \quad \text{for } \nu = 1, 2, 3, 4;\ j = 1, 2, 3$$

$$y_4\ge 0$$

The optimal solution is $(\lambda'_{11}, \lambda'_{21}, \lambda'_{31}, \lambda'_{41}, \lambda'_{12}, \lambda'_{22}, \lambda'_{32}, \lambda'_{42}, \lambda'_{13}, \lambda'_{23}, \lambda'_{33}, \lambda'_{43}, \lambda_4, t) = (0.0074, 0.0148, 0, 0, 0.0148, 0.0074, 0, 0, 0, 0.0222, 0, 0, 0, 0.0222)$. Hence optimal solution of the original problem is $(x_1, x_2, x_3, x_4) = \left(\frac{4}{3}, \frac{2}{3}, 2, 0\right)$.

6. *SUMMARY AND CONCLUSION*

The single ratio fractional programs play an important role in the formulation of variety of decision problems such as facility location, production planning. In this paper we have proposed an efficient algorithm for solving a special class of non-linear fractional programs where the numerator and denominator both are separable functions. Here we present a solution technique in which the numerator and denominator both are replaced by their piecewise linear approximations via grid points resulting in a linear fractional program. Using Charnes and Cooper transformation method, the linear fractional program is then solved. Finally, the solution technique is illustrated by a numerical example.

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