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Approximate multi-parametric sensitivity analysis of the constraint matrix in piecewise linear fractional programming

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Abstract

In this paper, we study multi-parametric sensitivity analysis under perturbations in multiple rows or columns of the constraint matrix in programming problems with the piecewise linear fractional objective function using the concept of maximum volume in the tolerance region. The weak maximum volume region is defined for optimal region which can be solved through a maximization problem.A major difficulty may arise under such perturbations from computing the inverse of the perturbed basis matrix. Using an approximation to the inverse of the perturbed basis matrix, we construct critical region for simultaneous and independent perturbations of the basis matrix in the given problem. Necessary and sufficient conditions are derived to classify perturbation parameters as 'focal' and 'non-focal'. Non-focal parameters can be deleted from the analysis, because of their low sensitivity in practice. Theoretical results are illustrated with the help of a numerical example.

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1 Introduction

In practice, numerical results are subject to errors and the exact solution of the problem under consideration is not known. The results obtained by some methods although are approximations of the solution of the problem but they could be considered as the exact

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results of the corresponding perturbed problem and this is the motivation to investigate the sensitivity analysis. We would like to know the effect of data perturbation on the optimal solution. Hence, the study of sensitivity analysis is of great importance. Generally, independent and simultaneous perturbations are investigated. If some of the parameters are more sensitive than the others, then it is better to investigate the tolerance region separately since otherwise the obtained tolerance region may be very small. If the decision maker has the prior knowledge that some parameters can be given unlimited variations without affecting the original solution then we consider those parameters as 'non-focal' and these 'non-focal' parameters can be deleted from the analysis. Wang and Huang [14, 13] introduced the concept of maximum volume in the tolerance region for the multi-parametric sensitivity analysis of a single objective linear programming problem. Their theory allows the more sensitive parameters called as 'focal' to be investigated at their independent levels of sensitivity, simultaneously and independently. In this way the perturbation of the parameters can be investigated with greater flexibility, and thus leads to better improvement comparing to existed approach. Singh et al. [11] extended the results of Wang and Huang to discuss multi-parametric sensitivity analysis in the linear-plus-linear fractional programming problem.

The Piecewise Linear Fractional Programming ($PLFP$) problem has the following form:

$$(PLFP) \qquad \min \quad Z(\mathbf{x}) = \frac{\alpha_0 + \sum_{j=1}^{n} f_j(x_j)}{\beta_0 + \sum_{j=1}^{n} g_j(x_j)}$$
$$s.t: \quad \mathbf{A}\mathbf{x} = \mathbf{b}$$
$$\mathbf{0} \leq \mathbf{x} \leq \mathbf{u},$$

where $f_j(x_j)$ is a continuous piecewise linear convex function, $g_j(x_j)$ is a continuous piecewise linear concave function such that $\beta_0 + \sum_{j=1}^{n} g_j(x_j) > 0$ for each feasible solution $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$, $\mathbf{A}$ is an $m \times n$ matrix of full row rank, $\mathbf{b}$ is an $m-$vector such that $b_i \geq 0$ for each i, and $\mathbf{u} = (u_1, u_2, \ldots, u_n)^T$ is an $n-$vector. This problem is a general case of three problems such as linear programming [2], Linear Fractional Programming (LFP) [1, 10, 12] and piecewise linear programming [4].

Let $0 = \delta_0^j < \delta_1^j < \ldots < \delta_{\tau_j}^j < \delta_{\tau_j+1}^j = u_j$ be an ascending arrangement of the breakpoints of both $f_j(x_j)$ and $g_j(x_j)$. Therefore, within each subinterval $[\delta_i^j, \delta_{i+1}^j], i = 0, 1, \ldots, \tau_j$, both $f_j(x_j)$ and $g_j(x_j)$ are linear functions. Thus $f_j(x_j)$ and $g_j(x_j)$ can be represented as

$$f_j(x_j) = c_i^j x_j + \alpha_i^j, \quad \delta_i^j \leq x_j \leq \delta_{i+1}^j; \quad i = 0, 1, 2, \ldots, \tau_j, \tag{1}$$

and

$$g_j(x_j) = d_i^j x_j + \beta_i^j, \quad \delta_i^j \leq x_j \leq \delta_{i+1}^j; \quad i = 0, 1, 2, \ldots, \tau_j, \tag{2}$$

for some real numbers c_i^j, α_i^j, d_i^j and $\beta_i^j, \quad i = 0, 1, \ldots, \tau_j, \quad j = 1, 2, \ldots, n.$

The following lemmas determine the convexity and the concavity conditions for a continuous piecewise linear function [3].

Lemma 1. *A continuous piecewise linear function $f_j(x_j)$ is convex if and only if its slope is nondecreasing with respect to x_j; that is, $c_0^j \leq c_1^j \leq \ldots, \leq c_{\tau_j}^j, \quad j = 1, 2, \ldots, n.$*

Lemma 2. *A continuous piecewise linear function $g_j(x_j)$ is concave if and only if its slope is non-increasing with respect to x_j; that is, $d_0^j \geq d_1^j \geq \ldots, \geq d_{\tau_j}^j, \quad j = 1, 2, \ldots, n.$*

Let $\mathbf{x}^0$ be an optimal solution to $PLFP$. For each $j = 1, 2, \ldots, n$, choose an index j_i such that $\delta^j_{j_i} \le x^0_j \le \delta^j_{j_i+1}$. Then any optimal solution to the LFP problem:

$$(LFP) \qquad \min \qquad \frac{\alpha^* + \sum_{j=1}^n c^j_{j_i} x_j}{\beta^* + \sum_{j=1}^n d^j_{j_i} x_j}$$

$$s.t: \qquad \mathbf{Ax} = \mathbf{b}$$

$$\delta^j_{j_i} \le x_j \le \delta^j_{j_i+1}, \quad j = 1, 2, \ldots, n,$$

is also an optimal solution to the $PLFP$ where $\alpha^* = \alpha_0 + \sum_{j=1}^n \alpha^j_{j_i}, \beta^* = \beta_0 + \sum_{j=1}^n \beta^j_{j_i}$. Definition of a basic feasible solution (BFS) for $PLFP$ is introduced as follows:

Let $\mathbf{A} = [\mathbf{A}_{.1}, \ldots, \mathbf{A}_{.n}]$ be the coefficients matrix and $B = \{B_1, \ldots, B_m\} \subset \{1, \ldots, n\}$ be a subset of the indices of the columns of matrix $\mathbf{A}$, such that $\mathbf{B} = [\mathbf{A}_{.B_1}, \ldots, \mathbf{A}_{.B_m}]$ is a non-singular matrix with inverse $\mathbf{B}^{-1} = [\beta_{ij}]$. Let $N = \{1, 2, \ldots, n\} \setminus B$. The variables $x_{B_i},\ i = 1, \ldots, m$, are called basic variables and $x_j,\ j \in N$, are referred to as nonbasic variables. These vectors are denoted by $\mathbf{x}_B$ and $\mathbf{x}_N$, respectively. Consequently, the solution $\mathbf{x} = (\mathbf{x}_B, \mathbf{x}_N)$, such that

$$x_j = \delta^j_{\nu_j}, \quad j \in N, \qquad \nu_j \in \{0, 1, \ldots, \tau_j + 1\},$$

$$\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b} - \sum_{j \in N} \mathbf{B}^{-1}\mathbf{A}_{.j} x_j, \tag{3}$$

is called a basic solution. If, in addition $0 \le \mathbf{x}_B \le \mathbf{u}_B$, then $\mathbf{x}$ is a basic feasible solution (BFS). Moreover, if $x_{B_i} \in \{\delta^{B_i}_0, \delta^{B_i}_1, \ldots, \delta^{B_i}_{\tau_{B_i+1}}\}$ for some i, then $\mathbf{x}$ is a degenerate BFS. If $x_{B_i} \notin \{\delta^{B_i}_0, \delta^{B_i}_1, \ldots, \delta^{B_i}_{\tau_{B_i+1}}\}$ for any i, then it is a nondegenerate BFS.

It is showed [8] that there exists an optimal solution $PLFP$ which is a BFS. The optimality criteria given by Punnen and Pandy [8] for the problem ($PLFP$) using the simplex algorithm is stated as follows:

Let $\mathbf{B}$ denote the optimal basis matrix and let $\mathbf{x}^* = (\mathbf{x}^*_B, \mathbf{x}^*_N)$ be the corresponding nondegenerate basic feasible solution for the problem ($PLFP$). This solution will be an optimal solution if

$$\eta^-_j(\mathbf{x}^*) = (c^j_{\nu_j-1} - \mathbf{c}_B\mathbf{B}^{-1}\mathbf{A}_{.j}) - Z(\mathbf{x}^*)(d^j_{\nu_j-1} - \mathbf{d}_B\mathbf{B}^{-1}\mathbf{A}_{.j}) \le 0,$$

and

$$\eta^+_j(\mathbf{x}^*) = (c^j_{\nu_j} - \mathbf{c}_B\mathbf{B}^{-1}\mathbf{A}_{.j}) - Z(\mathbf{x}^*)(d^j_{\nu_j} - \mathbf{d}_B\mathbf{B}^{-1}\mathbf{A}_{.j}) \ge 0,$$

for $j = 1, 2, \ldots, n$, where $Z(\mathbf{x}^*)$ is the objective function value at the optimal solution $\mathbf{x}^*$, $\mathbf{c}_B$ and $\mathbf{d}_B$ are the subvectors of $\mathbf{c}$ and $\mathbf{d}$ such that their ith coordinates corresponding to $\mathbf{B}$ are $c^{B_i}_{\mu(B_i)}$ and $d^{B_i}_{\mu(B_i)}$, respectively. If $\nu_j = \tau_j + 1$ then η^+_j is defined as 0. Similarly when $\nu_j = 0$ then η^-_j is defined as 0. Note that $\mu(B_i)$ denotes the index for which $\delta^{B_i}_{\mu(B_i)} \le x^*_{B_i} \le \delta^{B_i}_{\mu(B_i)+1}$. Kheirfam et al. [7] extended the results of Wang and Huang to discuss multi-parametric sensitivity analysis for changes in the objective function coefficients and the right-hand-side of the problem(PLFP).

In this study, we discuss multi-parametric sensitivity analysis for simultaneous and independent perturbations in multiple rows or columns of the constraint matrix in the problem (PLFP) using the concept of maximum volume within the tolerance region.

2 Problem formulation and sensitivity analysis

To proceed sensitivity analysis in the basis matrix of the problem (PLFP), we consider the following perturbed problem:

$$
\begin{aligned}
(PPLFP) \qquad & \min \quad \frac{\alpha^* + \sum_{j=1}^{n} c_{j_i}^{j} x_j}{\beta^* + \sum_{j=1}^{n} d_{j_i}^{j} x_j} \\
& s.t: \quad (\mathbf{A} + \triangle\mathbf{A})\mathbf{x} = \mathbf{b} \\
& \qquad\quad \delta_{j_i}^{j} \leq x_j \leq \delta_{j_i+1}^{j}, \quad j = 1, 2, \ldots, n,
\end{aligned}
$$

where

$$
\triangle\mathbf{A} = \begin{bmatrix} \xi_{11} & \xi_{12} & \ldots & \xi_{1m} & 0 & \ldots & 0 \\ \xi_{21} & \xi_{22} & \ldots & \xi_{2m} & 0 & \ldots & 0 \\ \vdots & \vdots & & & & & \vdots \\ \xi_{m1} & \xi_{m2} & \ldots & \xi_{mm} & 0 & \ldots & 0 \end{bmatrix}_{m\times n}, \tag{4}
$$

$$
\xi_{ir} = \sum_{h=1}^{H} p_{ih}\gamma_h, \quad i = 1, 2, \ldots, m,
$$

are the multi-parametric perturbations defined by the perturbation parameter

$$
\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_H)^T,
$$

and p_{jh} is constant.

Generally speaking sensitivity analysis means characterizing the sets as critical region in which the entries of the constraint matrix in $PLFP$ may vary simultaneously and independently without changing the optimal basis $\mathbf{B}$ [5]. Let R be a general notation for a critical region. We now construct the critical region for changes in the entries of the constraint matrix of the problem (PLFP) when perturbation occurs as in the form of (4).

In the following theorem, we construct critical region for simultaneous and independent perturbations of a basis matrix of the problem (PLFP).

Theorem 3. *Assume that rows or columns of the basis matrix are perturbed simultaneously and independently in the form (4), then the critical region R of the problem (PPLFP) is given by*

$$
\begin{aligned}
R = \Big\{\gamma \Big| \; & \beta_{i.} \textstyle\sum_{h=1}^{H} \big(\sum_{k=1}^{m} \mathbf{p}_{.kh} x_{B_k}\big)\gamma_h \big/ \big[1 + \sum_{h=1}^{H} \big(\sum_{k=1}^{m} \beta_{k.}\mathbf{p}_{.kh}\big)\gamma_h\big] \leq x_{B_i} - \delta_{\mu(B_i)}^{B_i}, \\
& \beta_{i.} \textstyle\sum_{h=1}^{H} \big(\sum_{k=1}^{m} \mathbf{p}_{.kh} x_{B_k}\big)\gamma_h \big/ \big[1 + \sum_{h=1}^{H} \big(\sum_{k=1}^{m} \beta_{k.}\mathbf{p}_{.kh}\big)\gamma_h\big] \geq x_{B_i} - \delta_{\mu(B_i)+1}^{B_i}, \\
& \eta_j^{+} + \frac{\sum_{h=1}^{H}\sum_{k=1}^{m}\sum_{i=1}^{m}\Big((c_{\mu(B_i)}^{B_i} - Zd_{\mu(B_i)}^{B_i})\beta_{i.}\Big)(\mathbf{p}_{.kh} y_{kj})\gamma_h}{1 + \sum_{h=1}^{H}\big(\sum_{k=1}^{m}\beta_{k.}\mathbf{p}_{.kh}\big)\gamma_h} \geq 0, \\
& \eta_j^{-} + \frac{\sum_{h=1}^{H}\sum_{k=1}^{m}\sum_{i=1}^{m}\Big((c_{\mu(B_i)}^{B_i} - Zd_{\mu(B_i)}^{B_i})\beta_{i.}\Big)(\mathbf{p}_{.kh} y_{kj})\gamma_h}{1 + \sum_{h=1}^{H}\big(\sum_{k=1}^{m}\beta_{k.}\mathbf{p}_{.kh}\big)\gamma_h} \leq 0, \\
& 1 + \textstyle\sum_{h=1}^{H}\big(\sum_{k=1}^{m}\beta_{k.}\mathbf{p}_{.kh}\big)\gamma_h \neq 0 \quad j = 1, 2, \ldots, n, \quad i = 1, 2, \ldots, m\Big\},
\end{aligned}
$$

where $\boldsymbol{B}^{-1} = [\beta_{ij}], \mathbf{y}_{.j} = \boldsymbol{B}^{-1}\mathbf{A}_{.j}$ *and* $\mathbf{p}_{.kh}$ *is the coefficients vector of the parameter* γ_h *in the* j*th column of the perturbed basis matrix.*

Proof. Since perturbation of a basis matrix may violate both feasibility and optimality conditions, therefore, we compute $\hat{\eta}_j^+$, $\hat{\eta}_j^-$ and $\hat{\mathbf{x}}_B$, new values of η_j^+, η_j^- and $\mathbf{x}_B$ as follows:

$$\hat{\mathbf{x}}_B = (\mathbf{B} + \triangle\mathbf{B})^{-1}\mathbf{b} - \textstyle\sum_{j\in N}(\mathbf{B} + \triangle\mathbf{B})^{-1}\mathbf{A}_{.j}\delta^j_{\nu_j}.$$

For small perturbations $\triangle\mathbf{B}$ in $\mathbf{B}$ [9], we get

$$(\mathbf{B} + \triangle\mathbf{B})^{-1} = \mathbf{B}^{-1} - \frac{\mathbf{B}^{-1}\triangle\mathbf{B}\mathbf{B}^{-1}}{1 + \mathrm{tr}(\mathbf{B}^{-1}\triangle\mathbf{B})},$$

where

$$1 + \mathrm{tr}(\mathbf{B}^{-1}\triangle\mathbf{B}) \neq 0.$$

Therefore,

$$\begin{aligned}
\hat{\mathbf{x}}_B &= \Big(\mathbf{B}^{-1} - \frac{\mathbf{B}^{-1}\triangle\mathbf{B}\mathbf{B}^{-1}}{1 + \mathrm{tr}(\mathbf{B}^{-1}\triangle\mathbf{B})}\Big)\mathbf{b} - \textstyle\sum_{j\in N}\Big(\mathbf{B}^{-1} - \dfrac{\mathbf{B}^{-1}\triangle\mathbf{B}\mathbf{B}^{-1}}{1 + \mathrm{tr}(\mathbf{B}^{-1}\triangle\mathbf{B})}\Big)\mathbf{A}_{.j}\delta^j_{\nu_j} \\
&= \mathbf{B}^{-1}\mathbf{b} - \frac{\mathbf{B}^{-1}\triangle\mathbf{B}\mathbf{B}^{-1}}{1 + \mathrm{tr}(\mathbf{B}^{-1}\triangle\mathbf{B})}\mathbf{b} - \textstyle\sum_{j\in N}\mathbf{B}^{-1}\mathbf{A}_{.j}\delta^j_{\nu_j} + \textstyle\sum_{j\in N}\dfrac{\mathbf{B}^{-1}\triangle\mathbf{B}\mathbf{B}^{-1}}{1 + \mathrm{tr}(\mathbf{B}^{-1}\triangle\mathbf{B})}\mathbf{A}_{.j}\delta^j_{\nu_j} \\
&= \mathbf{x}_B - \frac{\mathbf{B}^{-1}\triangle\mathbf{B}}{1 + \mathrm{tr}(\mathbf{B}^{-1}\triangle\mathbf{B})}\big(\mathbf{B}^{-1}\mathbf{b} - \textstyle\sum_{j\in N}\mathbf{B}^{-1}\mathbf{A}_{.j}\delta^j_{\nu_j}\big) \\
&= \mathbf{x}_B - \frac{\mathbf{B}^{-1}\triangle\mathbf{B}\mathbf{x}_B}{1 + \mathrm{tr}(\mathbf{B}^{-1}\triangle\mathbf{B})} \\
&= \mathbf{x}_B - \frac{\Big[\beta_{1.}\sum_{h=1}^{H}\big(\sum_{k=1}^{m}\mathbf{p}_{.kh}x_{B_k}\big)\gamma_h, \ldots, \beta_{m.}\sum_{h=1}^{H}\big(\sum_{k=1}^{m}\mathbf{p}_{.kh}x_{B_k}\big)\gamma_h\Big]^T}{1 + \sum_{h=1}^{H}\big(\sum_{k=1}^{m}\beta_{k.}\mathbf{p}_{.kh}\big)\gamma_h}.
\end{aligned}$$

Now ith component of $\hat{\mathbf{x}}_B$ is given by

$$\hat{x}_{B_i} = x_{B_i} - \frac{\beta_{i.}\sum_{h=1}^{H}\big(\sum_{k=1}^{m}\mathbf{p}_{.kh}x_{B_k}\big)\gamma_h}{1 + \sum_{h=1}^{H}\big(\sum_{k=1}^{m}\beta_{k.}\mathbf{p}_{.kh}\big)\gamma_h}, \quad i = 1, 2, \ldots, m.$$

This new basic solution $\hat{\mathbf{x}}_B$ is feasible if

$$\delta^{B_i}_{\mu(B_i)} \leq x_{B_i} - \frac{\beta_{i.}\sum_{h=1}^{H}\big(\sum_{k=1}^{m}\mathbf{p}_{.kh}x_{B_k}\big)\gamma_h}{1 + \sum_{h=1}^{H}\big(\sum_{k=1}^{m}\beta_{k.}\mathbf{p}_{.kh}\big)\gamma_h} \leq \delta^{B_i}_{\mu(B_i)+1}, \quad i = 1, 2, \ldots, m.$$

Thus,we have

$$\frac{\beta_{i.}\sum_{h=1}^{H}\big(\sum_{k=1}^{m}\mathbf{p}_{.kh}x_{B_k}\big)\gamma_h}{1 + \sum_{h=1}^{H}\big(\sum_{k=1}^{m}\beta_{k.}\mathbf{p}_{.kh}\big)\gamma_h} \leq x_{B_i} - \delta^{B_i}_{\mu(B_i)},$$

and

$$\frac{\beta_{i.}\sum_{h=1}^{H}\big(\sum_{k=1}^{m}\mathbf{p}_{.kh}x_{B_k}\big)\gamma_h}{1+\sum_{h=1}^{H}\big(\sum_{k=1}^{m}\beta_{k.}\mathbf{p}_{.kh}\big)\gamma_h} \geq x_{B_i}-\delta^{B_i}_{\mu(B_i)+1}, \quad i=1,2,\dots,m.$$

For the new solution $\hat{\mathbf{x}}_B$, to satisfy optimality condition, the new values $\hat{\eta}_j^+$ and $\hat{\eta}_j^-$ are computed as follows:

$$\hat{\eta}_j^+ = \Big(c^j_{\nu_j} - \mathbf{c}_B(\mathbf{B}+\triangle\mathbf{B})^{-1}\mathbf{A}_{.j}\Big) - Z\Big(d^j_{\nu_j} - \mathbf{d}_B(\mathbf{B}+\triangle\mathbf{B})^{-1}\mathbf{A}_{.j}\Big)$$

$$= \Big(c^j_{\nu_j} - \mathbf{c}_B(\mathbf{B}^{-1} - \frac{\mathbf{B}^{-1}\triangle\mathbf{B}\mathbf{B}^{-1}}{1+\mathrm{tr}(\mathbf{B}^{-1}\triangle\mathbf{B})})\mathbf{A}_{.j}\Big) - Z\Big(d^j_{\nu_j} - \mathbf{d}_B(\mathbf{B}^{-1} - \frac{\mathbf{B}^{-1}\triangle\mathbf{B}\mathbf{B}^{-1}}{1+\mathrm{tr}(\mathbf{B}^{-1}\triangle\mathbf{B})})\mathbf{A}_{.j}\Big)$$

$$= \eta_j^+ + (\mathbf{c}_B - Z\mathbf{d}_B)\frac{\Big[\beta_{1.}\sum_{h=1}^{H}\big(\sum_{k=1}^{m}\mathbf{p}_{.kh}y_{kj}\big)\gamma_h,\dots,\beta_{m.}\sum_{h=1}^{H}\big(\sum_{k=1}^{m}\mathbf{p}_{.kh}y_{kj}\big)\gamma_h\Big]^T}{1+\sum_{h=1}^{H}\big(\sum_{k=1}^{m}\beta_{k.}\mathbf{p}_{.kh}\big)\gamma_h}$$

$$= \eta_j^+ + \frac{\sum_{h=1}^{H}\sum_{k=1}^{m}\sum_{i=1}^{m}\Big((c^{B_i}_{\mu(B_i)} - Zd^{B_i}_{\mu(B_i)})\beta_{i.}\Big)(\mathbf{p}_{.kh}y_{kj})\gamma_h}{1+\sum_{h=1}^{H}\big(\sum_{k=1}^{m}\beta_{k.}\mathbf{p}_{.kh}\big)\gamma_h},$$

and similarly

$$\hat{\eta}_j^- = \eta_j^- + \frac{\sum_{h=1}^{H}\sum_{k=1}^{m}\sum_{i=1}^{m}\Big((c^{B_i}_{\mu(B_i)} - Zd^{B_i}_{\mu(B_i)})\beta_{i.}\Big)(\mathbf{p}_{.kh}y_{kj})\gamma_h}{1+\sum_{h=1}^{H}\big(\sum_{k=1}^{m}\beta_{k.}\mathbf{p}_{.kh}\big)\gamma_h}.$$

This new solution $\hat{\mathbf{x}}_B$ is optimal if $\hat{\eta}_j^- \leq 0$ and $\hat{\eta}_j^+ \geq 0$, j=1, 2, …,n. Therefore, the proof is complete. □

It can be seen that a parameter γ_h is absent in R if and only if

$$\beta_{k.}\mathbf{p}_{.kh} = 0, \quad \mathbf{p}_{.kh}x_{B_k} = 0, \quad k=1,2,\dots,m,$$

and

$$\mathbf{p}_{.kh}y_{kj} = 0, \quad k=1,2,\dots,m, \quad j \in N.$$

Those parameters which do not appear in R are called 'non-focal' and can be varied unlimitedly.

Remark 4. If $\beta_0 = 1$ and $g_j(x_j) = 0, j=1,2,\dots,n$, then the $PLFP$ reduces to PLP in which case the critical region R is given by

$$R = \Big\{\gamma \Big|\ \ \beta_{i.} \textstyle\sum_{h=1}^{H} \big(\sum_{k=1}^{m} \mathbf{p}_{.kh} x_{B_k} \big) \gamma_h \big/ \big[1 + \sum_{h=1}^{H} \big(\sum_{k=1}^{m} \beta_{k.} \mathbf{p}_{.kh} \big) \gamma_h \big] \leq x_{B_i} - \delta_{\mu(B_i)}^{B_i},$$

$$\beta_{i.} \textstyle\sum_{h=1}^{H} \big(\sum_{k=1}^{m} \mathbf{p}_{.kh} x_{B_k} \big) \gamma_h \big/ \big[1 + \sum_{h=1}^{H} \big(\sum_{k=1}^{m} \beta_{k.} \mathbf{p}_{.kh} \big) \gamma_h \big] \geq x_{B_i} - \delta_{\mu(B_i)+1}^{B_i},$$

$$(c_{\nu_j}^{j} - \mathbf{c}_B \mathbf{B}^{-1} \mathbf{A}_{.j}) + \frac{\sum_{h=1}^{H} \sum_{k=1}^{m} \sum_{i=1}^{m} (c_{\mu(B_i)}^{B_i}) \beta_{i.} (\mathbf{p}_{.kh} y_{kj}) \gamma_h}{1 + \sum_{h=1}^{H} \big(\sum_{k=1}^{m} \beta_{k.} \mathbf{p}_{.kh} \big) \gamma_h} \geq 0,$$

$$(c_{\nu_j - 1}^{j} - \mathbf{c}_B \mathbf{B}^{-1} \mathbf{A}_{.j}) + \frac{\sum_{h=1}^{H} \sum_{k=1}^{m} \sum_{i=1}^{m} (c_{\mu(B_i)}^{B_i}) \beta_{i.} (\mathbf{p}_{.kh} y_{kj}) \gamma_h}{1 + \sum_{h=1}^{H} \big(\sum_{k=1}^{m} \beta_{k.} \mathbf{p}_{.kh} \big) \gamma_h} \leq 0,$$

$$1 + \textstyle\sum_{h=1}^{H} \big(\sum_{k=1}^{m} \beta_{k.} \mathbf{p}_{.kh} \big) \gamma_h \neq 0 \quad j = 1, 2, \ldots, n, \quad i = 1, 2, \ldots, m \Big\}.$$

Remark 5. If $\beta_0 = 1$, $g_j(x_j) = 0$ and $f_j(x_j), j = 1, 2, \ldots, n$, are linear then the $PLFP$ reduces to an LP with bounded variables and we have $c_{j_i}^{j} = c_j$, $\delta_{j_i}^{j} = 0$ and $\delta_{j_i+1}^{j} = u_j, j = 1, 2, \ldots, n$. In this case the critical region R reduces to

$$R = \Big\{\gamma \Big|\ \ \beta_{i.} \textstyle\sum_{h=1}^{H} \big(\sum_{k=1}^{m} \mathbf{p}_{.kh} x_{B_k} \big) \gamma_h \big/ \big[1 + \sum_{h=1}^{H} \big(\sum_{k=1}^{m} \beta_{k.} \mathbf{p}_{.kh} \big) \gamma_h \big] \leq x_{B_i},$$

$$\beta_{i.} \textstyle\sum_{h=1}^{H} \big(\sum_{k=1}^{m} \mathbf{p}_{.kh} x_{B_k} \big) \gamma_h \big/ \big[1 + \sum_{h=1}^{H} \big(\sum_{k=1}^{m} \beta_{k.} \mathbf{p}_{.kh} \big) \gamma_h \big] \geq x_{B_i} - u_{B_i},$$

$$(c_j - \mathbf{c}_B \mathbf{B}^{-1} \mathbf{A}_{.j}) + \frac{\sum_{h=1}^{H} \sum_{k=1}^{m} \sum_{i=1}^{m} (c_{B_i} \beta_{i.}) (\mathbf{p}_{.kh} y_{kj}) \gamma_h}{1 + \sum_{h=1}^{H} \big(\sum_{k=1}^{m} \beta_{k.} \mathbf{p}_{.kh} \big) \gamma_h} \geq 0, \quad \text{for } j \in N, x_j = 0,$$

$$(c_j - \mathbf{c}_B \mathbf{B}^{-1} \mathbf{A}_{.j}) + \frac{\sum_{h=1}^{H} \sum_{k=1}^{m} \sum_{i=1}^{m} (c_{B_i} \beta_{i.}) (\mathbf{p}_{.kh} y_{kj}) \gamma_h}{1 + \sum_{h=1}^{H} \big(\sum_{k=1}^{m} \beta_{k.} \mathbf{p}_{.kh} \big) \gamma_h} \leq 0, \quad \text{for } j \in N, x_j = u_j,$$

$$1 + \textstyle\sum_{h=1}^{H} \big(\sum_{k=1}^{m} \beta_{k.} \mathbf{p}_{.kh} \big) \gamma_h \neq 0 \quad i = 1, 2, \ldots, m \Big\}.$$

Moreover, if $u_j = \infty \ \forall j$, in this case $PLFP$ reduces to an LP and we have

$$R = \Big\{\gamma \Big|\ \ \beta_{i.} \textstyle\sum_{h=1}^{H} \big(\sum_{k=1}^{m} \mathbf{p}_{.kh} x_{B_k} \big) \gamma_h \big/ \big[1 + \sum_{h=1}^{H} \big(\sum_{k=1}^{m} \beta_{k.} \mathbf{p}_{.kh} \big) \gamma_h \big] \leq x_{B_i},$$

$$(c_j - \mathbf{c}_B \mathbf{B}^{-1} \mathbf{A}_{.j}) + \frac{\sum_{h=1}^{H} \sum_{k=1}^{m} \sum_{i=1}^{m} (c_{B_i} \beta_{i.}) (\mathbf{p}_{.kh} y_{kj}) \gamma_h}{1 + \sum_{h=1}^{H} \big(\sum_{k=1}^{m} \beta_{k.} \mathbf{p}_{.kh} \big) \gamma_h} \geq 0,$$

$$j = 1, 2, \ldots, n, \quad i = 1, 2, \ldots, m \Big\}.$$

To perform sensitivity analysis, we decompose the critical region R as follows:

$$R_{d^+} = \Big\{\gamma \Big|\ 1 + \textstyle\sum_{h=1}^{H} \big(\sum_{k=1}^{m} \beta_{k.} \mathbf{p}_{.kh} \big) \gamma_h > 0 \Big\},$$

$$R_{d^-} = \Big\{\gamma \Big|\ 1 + \textstyle\sum_{h=1}^{H} \big(\sum_{k=1}^{m} \beta_{k.} \mathbf{p}_{.kh} \big) \gamma_h < 0 \Big\},$$

$$R_{e^+} = \Big\{\gamma \big| \ \eta_j^+ \big[1 + \textstyle\sum_{h=1}^{H} (\sum_{k=1}^{m} \beta_{k.}\mathbf{p}_{.kh})\gamma_h\big] + $$
$$\textstyle\sum_{h=1}^{H}\sum_{k=1}^{m}\sum_{i=1}^{m} \Big((c_{\mu(B_i)}^{B_i} - Z d_{\mu(B_i)}^{B_i})\beta_{i.}\Big)(\mathbf{p}_{.kh} y_{kj})\gamma_h \geq 0\Big\},$$

$$R_{e^-} = \Big\{\gamma \big| \ \eta_j^+ \big[1 + \textstyle\sum_{h=1}^{H} (\sum_{k=1}^{m} \beta_{k.}\mathbf{p}_{.kh})\gamma_h\big] + $$
$$\textstyle\sum_{h=1}^{H}\sum_{k=1}^{m}\sum_{i=1}^{m} \Big((c_{\mu(B_i)}^{B_i} - Z d_{\mu(B_i)}^{B_i})\beta_{i.}\Big)(\mathbf{p}_{.kh} y_{kj})\gamma_h \leq 0\Big\},$$

$$R_{f^+} = \Big\{\gamma \big| \ \eta_j^- \big[1 + \textstyle\sum_{h=1}^{H} (\sum_{k=1}^{m} \beta_{k.}\mathbf{p}_{.kh})\gamma_h\big] + $$
$$\textstyle\sum_{h=1}^{H}\sum_{k=1}^{m}\sum_{i=1}^{m} \Big((c_{\mu(B_i)}^{B_i} - Z d_{\mu(B_i)}^{B_i})\beta_{i.}\Big)(\mathbf{p}_{.kh} y_{kj})\gamma_h \geq 0\Big\},$$

$$R_{f^-} = \Big\{\gamma \big| \ \eta_j^- \big[1 + \textstyle\sum_{h=1}^{H} (\sum_{k=1}^{m} \beta_{k.}\mathbf{p}_{.kh})\gamma_h\big] + $$
$$\textstyle\sum_{h=1}^{H}\sum_{k=1}^{m}\sum_{i=1}^{m} \Big((c_{\mu(B_i)}^{B_i} - Z d_{\mu(B_i)}^{B_i})\beta_{i.}\Big)(\mathbf{p}_{.kh} y_{kj})\gamma_h \leq 0\Big\},$$

$$R_{b^+} = \Big\{\gamma \big| \ \beta_{i.} \textstyle\sum_{h=1}^{H} (\sum_{k=1}^{m} \mathbf{p}_{.kh} x_{B_k})\gamma_h \leq \big[1 + \sum_{h=1}^{H} (\sum_{k=1}^{m} \beta_{k.}\mathbf{p}_{.kh})\gamma_h\big](x_{B_i} - \delta_{\mu(B_i)}^{B_i})\Big\},$$

$$R_{b^-} = \Big\{\gamma \big| \ \beta_{i.} \textstyle\sum_{h=1}^{H} (\sum_{k=1}^{m} \mathbf{p}_{.kh} x_{B_k})\gamma_h \geq \big[1 + \sum_{h=1}^{H} (\sum_{k=1}^{m} \beta_{k.}\mathbf{p}_{.kh})\gamma_h\big](x_{B_i} - \delta_{\mu(B_i)}^{B_i})\Big\},$$

$$R_{c^+} = \Big\{\gamma \big| \ \beta_{i.} \textstyle\sum_{h=1}^{H} (\sum_{k=1}^{m} \mathbf{p}_{.kh} x_{B_k})\gamma_h \geq \big[1 + \sum_{h=1}^{H} (\sum_{k=1}^{m} \beta_{k.}\mathbf{p}_{.kh})\gamma_h\big](x_{B_i} - \delta_{\mu(B_i)+1}^{B_i})\Big\},$$

$$R_{c^-} = \Big\{\gamma \big| \ \beta_{i.} \textstyle\sum_{h=1}^{H} (\sum_{k=1}^{m} \mathbf{p}_{.kh} x_{B_k})\gamma_h \leq \big[1 + \sum_{h=1}^{H} (\sum_{k=1}^{m} \beta_{k.}\mathbf{p}_{.kh})\gamma_h\big](x_{B_i} - \delta_{\mu(B_i)+1}^{B_i})\Big\}.$$

Then
$R = \Big\{\gamma \big| \ \gamma \in (R_{d^+} \cap R_{b^+} \cap R_{c^+} \cap R_{e^+} \cap R_{f^-}) \quad \text{or} \quad \gamma \in (R_{d^-} \cap R_{b^-} \cap R_{c^-} \cap R_{e^-} \cap R_{f^+})\Big\}$
can be decomposed into two disjoint regions: $R_1 = \Big\{R_{d^+} \cap R_{b^+} \cap R_{c^+} \cap R_{e^+} \cap R_{f^-}\Big\}$ and $R_2 = \Big\{R_{d^-} \cap R_{b^-} \cap R_{c^-} \cap R_{e^-} \cap R_{f^+}\Big\}$.

Definition 6. [14] The Maximum Volume Region (MVR) B_R in a critical region R is given by
$$B_R = \{\gamma = (\gamma_1, \ldots, \gamma_H)^T \,|\, |\gamma_j| \leq k_j^*, \ j = 1, 2, \ldots, H\},$$
where $k^* = (k_1^*, k_2^*, \ldots, k_H^*)$ is the optimal solution of the following maximization problem:

$$\max_{k \in K(R)} V(k) = k_1.k_2.\ldots k_H, \tag{5}$$

where

$K(R) = \{k = (k_1, k_2, \ldots, k_H)^T \,|\, |\gamma_j| \leq k_j, \ j = 1, 2, \ldots, H \ \text{ implies } \gamma = (\gamma_1, \gamma_2, \ldots, \gamma_H)^T \in R\}$.
The volume of B_R is $\text{Vol}(B_R) = 2^H k_1^* k_2^* \ldots k_H^*$.

As we want to determine the tolerance region with the largest volume in the problem, we present the following definition [11].

Definition 7. A symmetrically rectangular parallelepiped $B_{\hat{\psi}} = \{\gamma = (\gamma_1, \ldots, \gamma_H)^T |\ |\gamma_j| \leq \bar{\lambda}_j,\ j = 1, 2, \ldots, H\}$ of a region ψ is the Weak Maximal Volume Region (WMVR) of ψ if $\bar{\upsilon} = \bar{\lambda}_1\bar{\lambda}_2 \ldots \bar{\lambda}_H$ is the least upper bound of the set $\{\upsilon = \lambda_1\lambda_2 \ldots \lambda_H |\ |\gamma_j| \leq \lambda_j, j = 1, 2, \ldots, H$ implies $\gamma = (\gamma_1, \ldots, \gamma_H)^T \in \psi\}$. Furthermore, if $B_{\hat{\psi}}$ is a subset of ψ, that is, $\bar{\upsilon}$ is also the maximum, we say $B_{\hat{\psi}}$ is a Strong Maximal Volume Region (SMVR) of ψ, and denote it by $B_{\bar{\psi}}$.

Theorem 8. *[13] Let $B_{\hat{\psi}}$ be a WMVR of a region ψ, then a necessary and sufficient condition for $B_{\hat{\psi}}$ to be a SMVR of ψ is that $\{\gamma = (\gamma_1, \ldots, \gamma_H)^T |\ |\gamma_j| = \bar{\lambda}_j,\ j = 1, 2, \ldots, H\} \subseteq R_{d^+} = \{\gamma \big|\ 1 + \sum_{h=1}^{H} \big(\sum_{k=1}^{m} \beta_{k.}\mathbf{p}_{.kh}\big)\gamma_h > 0\}$.*

Since the critical region R is a polyhedral set, therefore there exists $L = [\ell_{ij}] \in \mathbb{R}^{I \times J}$, $d = \{d_i\} \in \mathbb{R}^I$, $I, J \in \mathbb{N}$ where I and J are the numbers of constraints and variables of R, respectively, such that $R = \{\gamma = (\gamma_1, \ldots, \gamma_J)^T |\ L\gamma \leq d\}$. While deleting all non-focal parameters of R, we can assume that $\ell_{.j} \neq 0, j = 1, 2, \ldots, J$.

Remark 9. It follows from Theorem 3 that $\gamma = 0$ belongs to R and thus we have $d \geq 0$.

We consider the case $d > 0$. Then, the problem (5) can be restated as follows:

$$\max_{k \in K(R)} V(k) = k_1.k_2.\ldots.k_J, \tag{6}$$

where

$$K(R) = \{k = (k_1, k_2, \ldots, k_J)^T |\ |\gamma_j| \leq k_j,\ j = 1, 2, \ldots, J \text{ implies } L\gamma \leq d\}.$$

We have the following Theorem:

Theorem 10. *[14] The MVR B_R of a polyhedral set $R = \{\gamma = (\gamma_1, \ldots, \gamma_J)^T |\ L\gamma \leq d\}$ with $d > 0$ and $\ell_{.j} \neq 0$ is bounded and defined by*

$$B_R = \{\gamma = (\gamma_1, \ldots, \gamma_J)^T |\ |\gamma_j| \leq k_j^*,\ j = 1, 2, \ldots, J\},$$

where $k^* = (k_1^*, k_2^*, \ldots, k_J^*)$ is the unique optimal solution of the following maximization problem:

$$\begin{aligned} &\max V(k) = k_1.k_2.\ldots.k_J \\ &\qquad s.t: \ |L|k \leq d \\ &\qquad\qquad k \geq 0, \end{aligned}$$

where $|L|$ is obtained by changing the negative elements of matrix L to be positive.

3 Illustrative example

In this section, an example is presented to illustrate the obtained results in identifying the critical regions.

Example 1. *Consider the problem (PLFP) as:*

$$\begin{array}{lllll}
\min & \dfrac{\sum_{j=1}^{4} f_j(x_j)}{\sum_{j=1}^{4} g_j(x_j)} & & & \\
s.t: & 3x_1+ & 4x_2+ & x_3+ & 2x_4 = & 21 \\
 & x_1+ & 3x_2+ & x_3+ & 3x_4 = & 13 \\
 & 2x_1+ & x_2+ & 2x_3+ & 3x_4 = & 14 \\
 & 0 \le x_1 \le 5, & 0 \le x_2 \le 3, & 0 \le x_3 \le 5, & 0 \le x_4 \le 5
\end{array}$$

where

$$f_1(x_1) = \begin{cases} 3x_1, & 0 \le x_1 \le 1, \\ 4x_1 - 1, & 1 \le x_1 \le 5, \end{cases} \qquad g_1(x_1) = \begin{cases} 4x_1 + 1, & 0 \le x_1 \le 1, \\ 3x_1 + 2, & 1 \le x_1 \le 5, \end{cases}$$

$$f_2(x_2) = \begin{cases} 2x_2 + 1, & 0 \le x_2 \le 1, \\ 3x_2, & 1 \le x_2 \le 3, \end{cases} \qquad g_2(x_2) = \begin{cases} 3x_2 + 1, & 0 \le x_2 \le 1, \\ 2x_2 + 2, & 1 \le x_2 \le 3, \end{cases}$$

$$f_3(x_3) = \begin{cases} x_3 + 3, & 0 \le x_3 \le 2, \\ 2x_3 + 1, & 2 \le x_3 \le 3, \\ 3x_3 - 2, & 3 \le x_3 \le 5, \end{cases} \qquad g_3(x_3) = \begin{cases} 3x_3 + 1, & 0 \le x_3 \le 2, \\ 2x_3 + 3, & 2 \le x_3 \le 3, \\ x_3 + 6, & 3 \le x_3 \le 5, \end{cases}$$

$$f_4(x_4) = \begin{cases} x_4 + 1, & 0 \le x_4 \le 1, \\ 2x_4, & 1 \le x_4 \le 3, \\ 3x_4 - 3, & 3 \le x_4 \le 5, \end{cases} \qquad g_4(x_4) = \begin{cases} 4x_4 + 1, & 0 \le x_4 \le 1, \\ 2x_4 + 3, & 1 \le x_4 \le 3, \\ x_4 + 6, & 3 \le x_4 \le 5. \end{cases}$$

Using simplex algorithm of Punnen and Pandey [8], the initial and the final simplex tables are given as follows

Tableau 1

c_B	d_B	x_B	x_1	x_2	x_3	x_4	x_5	x_6	x_7	$\bar{b}$
M	0	x_5	3	4	1	2	1	0	0	21
M	0	x_6	1	3	1	3	0	1	0	13
M	0	x_7	2	1	2	3	0	0	1	14
η_j^+			$2-54M$	$\frac{-7-176M}{4}$	$\frac{-11-160M}{4}$	$4-56M$	0	0	0	$z = \frac{5+48M}{4}$
η_j^-			0	0	0	0	0	0	0	
x_j			0	0	0	0	21	13	14	

Final

c_B	d_B	x_B	x_1	x_2	x_3	x_4	x_5	x_6	x_7	$\bar{b}$
3	2	x_2	0	1	$-3/10$	0	$3/20$	$1/4$	$-7/20$	$3/2$
1	4	x_4	0	0	$1/2$	1	$-1/4$	$1/4$	$1/4$	$3/2$
4	3	x_1	1	0	$2/5$	0	$3/10$	$-1/2$	$3/10$	4
η_j^+			0	0	1.031	0	$M-2.623$	$M+1$	$M-1.438$	$z = .885$
η_j^-			0	0	$-.854$	0	0	0	0	
x_j			$32/10$	$21/10$	2	$1/2$	0	0	0	

The optimal solution is $x^* = (32/10, 21/10, 2, 1/2, 0, 0, 0)^T$. Here $B = \{2, 4, 1)\}$ and the submatrix of the optimal basis is $\begin{pmatrix} 4 & 2 & 3 \\ 3 & 3 & 1 \\ 1 & 3 & 2 \end{pmatrix}$ and its inverse

$$\beta = \mathbf{B}^{-1} = \begin{pmatrix} 3/20 & 1/4 & -7/20 \\ -1/4 & 1/4 & 1/4 \\ 3/10 & -1/2 & 3/10 \end{pmatrix}.$$

Let the perturbation matrix $\triangle\mathbf{A}$ be given as follows:

$$\triangle\mathbf{A} = \begin{bmatrix} 2\gamma_1 - \gamma_2 + 3\gamma_3 & 0 & 0 & 0 \\ \gamma_1 + 2\gamma_2 + 5\gamma_3 & \gamma_1 + 3\gamma_2 - \gamma_3 & 0 & 0 \\ 0 & 4\gamma_1 + \gamma_2 - 2\gamma_3 & 0 & \gamma_1 - \gamma_2 + \gamma_3 \end{bmatrix}.$$

Therefore,

$$\mathbf{p}_{.11} = (2,1,0)^T, \quad \mathbf{p}_{.12} = (-1,2,0)^T, \quad \mathbf{p}_{.13} = (3,5,0)^T,$$

$$\mathbf{p}_{.21} = (0,1,4)^T, \quad \mathbf{p}_{.22} = (0,3,1)^T, \quad \mathbf{p}_{.23} = (0,-1,-2)^T,$$

$$\mathbf{p}_{.31} = (0,0,1)^T, \quad \mathbf{p}_{.32} = (0,0,-1)^T, \quad \mathbf{p}_{.33} = (0,0,1)^T,$$

$$R_{d^+} = \Big\{\gamma \Big|\ 1 + 2.1\gamma_1 + 2.05\gamma_2 + 1.25\gamma_3 > 0\Big\},$$

$$R_{d^-} = \Big\{\gamma \Big|\ 1 + 2.1\gamma_1 + 2.05\gamma_2 + 1.25\gamma_3 < 0\Big\},$$

$$R_{b^+} = \Big\{\gamma \Big|\ -7.71\gamma_1 + 0.9\gamma_2 + 1.3\gamma_3 \leq 1.1, \quad -0.15\gamma_1 - 0.3\gamma_2 + 0.85\gamma_3 \leq 0.5, \\ 0.47\gamma_1 - 4.82\gamma_2 - 3.08\gamma_3 \leq 2.2\Big\},$$

$$R_{b^-} = \Big\{\gamma \Big|\ -7.71\gamma_1 + 0.9\gamma_2 + 1.3\gamma_3 \geq 1.1, \quad -0.15\gamma_1 - 0.3\gamma_2 + 0.85\gamma_3 \geq 0.5, \\ 0.47\gamma_1 - 4.82\gamma_2 - 3.08\gamma_3 \geq 2.2\Big\},$$

$$R_{c^+} = \Big\{\gamma \Big|\ -3.51\gamma_1 + 3\gamma_2 + 3.8\gamma_3 \geq -1.89, \quad 1.95\gamma_1 + 0.75\gamma_2 + 2.1\gamma_3 \geq -0.5, \\ 5.3\gamma_1 - 2.4\gamma_2 - 0.2\gamma_3 \geq -1.8\Big\},$$

$$R_{c^-} = \Big\{\gamma \Big|\ -3.51\gamma_1 + 3\gamma_2 + 3.8\gamma_3 \leq -1.89, \quad 1.95\gamma_1 + 0.75\gamma_2 + 2.1\gamma_3 \leq -0.5, \\ 5.3\gamma_1 - 2.4\gamma_2 - 0.2\gamma_3 \leq -1.8\Big\},$$

$$R_{e^+} = \Big\{\gamma \Big|\ 0.27\gamma_1 + 0.8\gamma_2 + 2.17\gamma_3 \geq -1.031\Big\},$$

$$R_{e^-} = \Big\{\gamma \Big|\ 0.27\gamma_1 + 0.8\gamma_2 + 2.17\gamma_3 \leq -1.031\Big\},$$

$$R_{f^+} = \Big\{\gamma \Big|\ -3.67\gamma_1 - 1.18\gamma_2 - 0.19\gamma_3 \geq 0.854\Big\},$$

$$R_{f^-} = \Big\{\gamma \Big|\ -3.67\gamma_1 - 1.18\gamma_2 - 0.19\gamma_3 \leq 0.854\Big\}.$$

Hence,

$$R_1 = \Big\{\gamma \Big|\ 1 + 2.1\gamma_1 + 2.05\gamma_2 + 1.25\gamma_3 > 0, \ -7.71\gamma_1 + 0.9\gamma_2 + 1.3\gamma_3 \leq 1.1,$$

$$
\begin{aligned}
&-0.15\gamma_1-0.3\gamma_2+0.85\gamma_3\le 0.5,\quad 0.47\gamma_1-4.82\gamma_2-3.08\gamma_3\le 2.2,\\
&-3.51\gamma_1+3\gamma_2+3.8\gamma_3\ge -1.89,\quad 1.95\gamma_1+0.75\gamma_2+2.1\gamma_3\ge -0.5,\\
&5.3\gamma_1-2.4\gamma_2-0.2\gamma_3\ge -1.8,\quad 0.27\gamma_1+0.8\gamma_2+2.17\gamma_3\ge -1.031,\\
&\qquad -3.67\gamma_1-1.18\gamma_2-0.19\gamma_3\le 0.854\Big\},
\end{aligned}
$$

$$
\begin{aligned}
R_2=\Big\{\gamma\Big|\ &1+2.1\gamma_1+2.05\gamma_2+1.25\gamma_3<0,\ -7.71\gamma_1+0.9\gamma_2+1.3\gamma_3\ge 1.1,\\
&-0.15\gamma_1-0.3\gamma_2+0.85\gamma_3\ge 0.5,\quad 0.47\gamma_1-4.82\gamma_2-3.08\gamma_3\ge 2.2,\\
&-3.51\gamma_1+3\gamma_2+3.8\gamma_3\le -1.89,\quad 1.95\gamma_1+0.75\gamma_2+2.1\gamma_3\le -0.5,\\
&5.3\gamma_1-2.4\gamma_2-0.2\gamma_3\le -1.8,\quad 0.27\gamma_1+0.8\gamma_2+2.17\gamma_3\le -1.031,\\
&\qquad -3.67\gamma_1-1.18\gamma_2-0.19\gamma_3\ge 0.854\Big\},
\end{aligned}
$$

where γ belongs to only one of the two disjoint regions $R_1=\Big\{R_{d^+}\cap R_{b^+}\cap R_{c^+}\cap R_{e^+}\cap R_{f^-}\Big\}$ and $R_2=\Big\{R_{d^-}\cap R_{b^-}\cap R_{c^-}\cap R_{e^+}\cap R_{f^+}\Big\}$.

In this example all the parameters are focal parameters. The MVR of S_R is obtained by solving the following problem:

$$
\begin{aligned}
\max\quad & V(k)=k_1.k_2.k_3\\
s.t:\quad & 2.1k_1+2.05k_2+1.25k_3\le 1\\
& 7.71k_1+0.9k_2+1.3k_3\le 1.1\\
& 0.15k_1+0.3k_2+0.85k_3\le 0.5\\
& 0.47k_1+4.82k_2+3.08k_3\le 2.2\\
& 3.51k_1+3k_2+3.8k_3\le 1.89\\
& 1.95k_1+0.75k_2+2.1k_3\le 0.5\\
& 5.3k_1+2.4k_2+0.2k_3\le 1.8\\
& 0.27k_1+0.8k_2+2.17k_3\le 1.031\\
& 3.67k_1+1.18k_2+0.19k_3\le 0.854\\
& k_1,k_2,k_3\ge 0.
\end{aligned}
$$

The optimal solution of the problem is $k^*=(0.0828, 0.2231, 0.0815)$. The Weak Maximum Volume Region ($WMVR$) B_R of R is $\{\gamma|\ |\gamma_1|\le 0.0828, |\gamma_2|\le 0.2231, |\gamma_3|\le 0.0815\}$ with $\mathrm{Vol}(B_R)=2^3(0.0828)(0.2231)(0.0815)$. Since $\{\gamma|\ |\gamma_1|=0.0828, |\gamma_2|=0.2231, |\gamma_3|=0.0815\}\subseteq S_{d^+}$ therefore the $WMVR$ of R is also a $SMVR$.

4 conclusion

In this paper, we discuss approximate multi-parametric sensitivity analysis under basis matrix perturbations in the problem (PLFP) by classifying the perturbation parameters as “focal” and “non-focal”. This approach reduces the number of parameters in the final analysis. We expect that this approach is applicable for conic programming and linear complementarity problem.

References

[1] Bajalinov E.B., *Linear Fractional Programming: Theory, Methods, Applications and software*, Kluwer Academic Publishers, 2003.

[2] Chavatal V., *Linear programming*, W.H. Freeman and Company, New York, 1983.

[3] Dantzig G.B., Thapa M.N., *Linear Programming, 1:Introduction*, Springer, 1997.

[4] Fourer R., *A simplex algorithm for piecewise-linear programming I: Derivation and proof*, Mathematical Programming, **13**(1985), 204-233.

[5] Gal T., *Post-optimal Analysis, Parametric Programming and Related Topics*, McGraw-Hill, New York, 1979.

[6] Gal T., Greenberg H.J., Advances in Sensitivity Analysis and Parametric Programming, Kluwer Academic Press, Boston, MA, 1997.

[7] Kheirfam B., Mirnia K., *Multi-parametric sensitivity analysis in piecewise linear fractional programming*, Journal of Industrial and Management Optimization(JIMO), **4**,no. 2(2008), 343-351.

[8] Punnen A., Pandey P., *A simplex algorithm for piecewise-linear fractional programming problems*, European Journal of Operational Research, **178** (2007), 343-358.

[9] Ravi N., Wendell R.E., *The tolerance approach to sensitivity analysis of matrix coefficients in linear programming: general pertubations*, Journal of the Operational Research Society, **36**(10)(1985), 943-950.

[10] Schaible S., Ibaraki T., *Fractional programming*, European journal of operational research, **12**(1983), 247-255.

[11] Singh S., Gupta P., Bhatia D., *Multiparametric sensitivity analysis of the constraint matrix in linear-plus-linear fractional programming problem*, Applied Mathematics and Computation, **170** (2005), 1243-1260.

[12] Swarup K., *Linear fractional programming*, Operations Research, **10**(1965), 380-387.

[13] Wang H.F., Huang C.S., *The maximal tolerance analysis on the constraint matrix in linear programming*, Journal of the Chinese Institute of Engineers , **15**(5)(1992), 507-517.

[14] Wang H.F., Huang C.S., *Multi-parametric analysis of the maximum tolerance in a linear programming problem*, European Journal of Operational Research, **67**(1993), 75-87.