Approximate multi-parametric sensitivity analysis of the constraint matrix in piecewise linear fractional programming

B. Kheirfam∗ K. Mirnia2
1Department of Mathematics Azarbijn University of Tarbiat Moallem, Tabriz, Iran
2Department of Mathematics Tabriz University, Tabriz, Iran
∗b.kheirfam@azaruniv.edu

Abstract

In this paper, we study multi-parametric sensitivity analysis under perturbations in multiple rows or columns of the constraint matrix in programming problems with the piecewise linear fractional objective function using the concept of maximum volume in the tolerance region. The weak maximum volume region is defined for optimal region which can be solved through a maximization problem. A major difficulty may arise under such perturbations from computing the inverse of the perturbed basis matrix. Using an approximation to the inverse of the perturbed basis matrix, we construct critical region for simultaneous and independent perturbations of the basis matrix in the given problem. Necessary and sufficient conditions are derived to classify perturbation parameters as ‘focal’ and ‘non-focal’. Non-focal parameters can be deleted from the analysis, because of their low sensitivity in practice. Theoretical results are illustrated with the help of a numerical example.

AMS: 90C31

keywords: Piecewise linear fractional programming, Fractional programming, Piecewise linear programming, Approximate multi-parametric sensitivity analysis, Tolerance approach.

1 Introduction

In practice, numerical results are subject to errors and the exact solution of the problem under consideration is not known. The results obtained by some methods although are approximations of the solution of the problem but they could be considered as the exact
results of the corresponding perturbed problem and this is the motivation to investigate the sensitivity analysis. We would like to know the effect of data perturbation on the optimal solution. Hence, the study of sensitivity analysis is of great importance. Generally, independent and simultaneous perturbations are investigated. If some of the parameters are more sensitive than the others, then it is better to investigate the tolerance region separately since otherwise the obtained tolerance region may be very small. If the decision maker has the prior knowledge that some parameters can be given unlimited variations without affecting the original solution then we consider those parameters as ‘non-focal’ and these ‘non-focal’ parameters can be deleted from the analysis. Wang and Huang [14, 13] introduced the concept of maximum volume in the tolerance region for the multi-parametric sensitivity analysis of a single objective linear programming problem. Their theory allows the more sensitive parameters called as ‘focal’ to be investigated at their independent levels of sensitivity, simultaneously and independently. In this way the perturbation of the parameters can be investigated with greater flexibility, and thus leads to better improvement comparing to existed approach. Singh et al. [11] extended the results of Wang and Huang to discuss multi-parametric sensitivity analysis in the linear-plus-linear fractional programming problem.

The Piecewise Linear Fractional Programming (PLFP) problem has the following form:

\[
(PLFP) \quad \min \ Z(x) = \frac{\alpha_0 + \sum_{j=1}^{n} f_j(x_j)}{\beta_0 + \sum_{j=1}^{n} g_j(x_j)} \\
\text{s.t.} \quad Ax = b \\
0 \leq x \leq u,
\]

where \( f_j(x_j) \) is a continuous piecewise linear convex function, \( g_j(x_j) \) is a continuous piecewise linear concave function such that \( \beta_0 + \sum_{j=1}^{n} g_j(x_j) > 0 \) for each feasible solution \( x = (x_1, x_2, \ldots, x_n)^T \), \( A \) is an \( m \times n \) matrix of full row rank, \( b \) is an \( m \)-vector such that \( b_i \geq 0 \) for each \( i \), and \( u = (u_1, u_2, \ldots, u_n)^T \) is an \( n \)-vector. This problem is a general case of three problems such as linear programming [2], Linear Fractional Programming (LFP) [1, 10, 12] and piecewise linear programming [4].

Let \( 0 = \delta^i_0 < \delta^i_1 < \ldots < \delta^i_{\tau_j} < \delta^i_{\tau_j+1} = u_j \) be an ascending arrangement of the breakpoints of both \( f_j(x_j) \) and \( g_j(x_j) \). Therefore, within each subinterval \( [\delta^i_j, \delta^i_{j+1}) \), \( i = 0, 1, \ldots, \tau_j \), both \( f_j(x_j) \) and \( g_j(x_j) \) are linear functions. Thus \( f_j(x_j) \) and \( g_j(x_j) \) can be represented as

\[
f_j(x_j) = c^j_i x_j + \alpha^j_i, \quad \delta^i_j \leq x_j \leq \delta^i_{j+1}; \quad i = 0, 1, 2, \ldots, \tau_j, \tag{1}
\]

and

\[
g_j(x_j) = d^j_i x_j + \beta^j_i, \quad \delta^i_j \leq x_j \leq \delta^i_{j+1}; \quad i = 0, 1, 2, \ldots, \tau_j, \tag{2}
\]

for some real numbers \( c^j_i, \alpha^j_i, d^j_i \) and \( \beta^j_i \), \( i = 0, 1, \ldots, \tau_j \), \( j = 1, 2, \ldots, n \).

The following lemmas determine the convexity and the concavity conditions for a continuous piecewise linear function [3].

**Lemma 1.** A continuous piecewise linear function \( f_j(x_j) \) is convex if and only if its slope is nondecreasing with respect to \( x_j \); that is, \( c^j_0 \leq c^j_1 \leq \ldots \leq c^j_j \), \( j = 1, 2, \ldots, n \).

**Lemma 2.** A continuous piecewise linear function \( g_j(x_j) \) is concave if and only if its slope is non-increasing with respect to \( x_j \); that is, \( d^j_0 \geq d^j_1 \geq \ldots \geq d^j_j \), \( j = 1, 2, \ldots, n \).
Let $x^0$ be an optimal solution to PLFP. For each $j = 1, 2, \ldots, n$, choose an index $j_i$ such that $\delta^j_{j_i} \leq x^0_j \leq \delta^j_{j_i+1}$. Then any optimal solution to the LFP problem:

\[
\text{(LFP)} \quad \min \quad \frac{\alpha^* + \sum_{j=1}^n c^j_i x_j}{\beta^* + \sum_{j=1}^n d^j_i x_j}
\]

\[
s.t.: \quad A x = b \\
\delta^j_{j_i} \leq x_j \leq \delta^j_{j_i+1}, \quad j = 1, 2, \ldots, n,
\]

is also an optimal solution to the PLFP where $\alpha^* = \alpha_0 + \sum_{j=1}^n \alpha^j_{j_i}$, $\beta^* = \beta_0 + \sum_{j=1}^n \beta^j_{j_i}$.

Definition of a basic feasible solution (BFS) for PLFP is introduced as follows:

Let $A = [A_1, \ldots, A_n]$ be the coefficients matrix and $B = \{B_1, \ldots, B_m\} \subset \{1, \ldots, n\}$ be a subset of the indices of the columns of matrix $A$, such that $B = [A_{B_1}, \ldots, A_{B_m}]$ is a non-singular matrix with inverse $B^{-1} = [\beta_{ij}]$. Let $N = \{1, 2, \ldots, n\} \setminus B$. The variables $x_{B_i}$, $i = 1, \ldots, m$, are called basic variables and $x_j$, $j \in N$, are referred to as nonbasic variables. These vectors are denoted by $x_B$ and $x_N$, respectively. Consequently, the solution $x = (x_B, x_N)$, such that

\[
x_j = \delta^j_{\mu(j)}, \quad j \in N, \quad \nu_j \in \{0, 1, \ldots, \tau_j + 1\},
\]

\[
x_B = B^{-1}b - \sum_{j \in N} B^{-1}A_j x_j,
\]

is called a basic solution. If, in addition $0 \leq x_B \leq u_B$, then $x$ is a basic feasible solution (BFS). Moreover, if $x_{B_i} \in \{\delta^B_{0i}, \delta^B_{1i}, \ldots, \delta^B_{rB_i+1}\}$ for some $i$, then $x$ is a degenerate BFS. If $x_{B_i} \notin \{\delta^B_{0i}, \delta^B_{1i}, \ldots, \delta^B_{rB_i+1}\}$ for any $i$, then it is a nondegenerate BFS.

It is showed [8] that there exists an optimal solution PLFP which is a BFS. The optimality criteria given by Punnen and Pandy [8] for the problem (PLFP) using the simplex algorithm is stated as follows:

Let $B$ denote the optimal basis matrix and let $x^* = (x^*_B, x^*_N)$ be the corresponding nondegenerate basic feasible solution for the problem (PLFP). This solution will be an optimal solution if

\[
\eta^-_j(x^*) = (c^j_{\nu_j-1} - c_B B^{-1}A_j) - Z(x^*)(d^j_{\nu_j-1} - d_B B^{-1}A_j) \leq 0,
\]

and

\[
\eta^+_j(x^*) = (c^j_{\nu_j} - c_B B^{-1}A_j) - Z(x^*)(d^j_{\nu_j} - d_B B^{-1}A_j) \geq 0,
\]

for $j = 1, 2, \ldots, n$, where $Z(x^*)$ is the objective function value at the optimal solution $x^*$, $c_B$ and $d_B$ are the subvectors of $c$ and $d$ such that their $i$th coordinates corresponding to $B$ are $c^B_{\mu(B_i)}$ and $d^B_{\mu(B_i)}$, respectively. If $\nu_j = \tau_j + 1$ then $\eta^-_j$ is defined as 0. Similarly when $\nu_j = 0$ then $\eta^+_j$ is defined as 0. Note that $\mu(B_i)$ denotes the index for which $\delta_{\mu(B_i)} \leq x^*_i \leq \delta_{\mu(B_i)+1}$. Kheirfam et al. [7] extended the results of Wang and Huang to discuss multi-parametric sensitivity analysis for changes in the objective function coefficients and the right-hand-side of the problem (PLFP).

In this study, we discuss multi-parametric sensitivity analysis for simultaneous and independent perturbations in multiple rows or columns of the constraint matrix in the problem (PLFP) using the concept of maximum volume within the tolerance region.
2 Problem formulation and sensitivity analysis

To proceed sensitivity analysis in the basis matrix of the problem (PLFP), we consider the following perturbed problem:

\[
(PPLFP) \quad \min \quad \frac{\alpha^* + \sum_{j=1}^{n} c^*_j x_j}{\beta^* + \sum_{j=1}^{n} d^*_j x_j} \\
\text{s.t.} \quad (A + \Delta A)x = b \\
\delta_j^j \leq x_j \leq \delta_j^{j+1}, \quad j = 1, 2, \ldots, n,
\]

where

\[
\Delta A = \begin{bmatrix}
\xi_{11} & \xi_{12} & \cdots & \xi_{1m} & 0 & \cdots & 0 \\
\xi_{21} & \xi_{22} & \cdots & \xi_{2m} & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\xi_{m1} & \xi_{m2} & \cdots & \xi_{mm} & 0 & \cdots & 0
\end{bmatrix}_{m \times n}, \tag{4}
\]

\[
\xi_{ir} = \sum_{h=1}^{H} p_{ih} \gamma_h, \quad i = 1, 2, \ldots, m,
\]

are the multi-parametric perturbations defined by the perturbation parameter

\[
\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_H)^T,
\]

and \( p_{jh} \) is constant.

Generally speaking sensitivity analysis means characterizing the sets as critical region in which the entries of the constraint matrix in PLFP may vary simultaneously and independently without changing the optimal basis \( B \) [5]. Let \( R \) be a general notation for a critical region. We now construct the critical region for changes in the entries of the constraint matrix of the problem (PLFP) when perturbation occurs as in the form of (4).

In the following theorem, we construct critical region for simultaneous and independent perturbations of a basis matrix of the problem (PLFP).

**Theorem 3.** Assume that rows or columns of the basis matrix are perturbed simultaneously and independently in the form (4), then the critical region \( R \) of the problem (PPLFP) is given by

\[
R = \left\{ \gamma \mid \beta_i, \sum_{h=1}^{H} \left( \sum_{k=1}^{m} p_{kh} x_{B_k} \right) \gamma_h / \left[ 1 + \sum_{h=1}^{H} \left( \sum_{k=1}^{m} \beta_k p_{kh} \right) \gamma_h \right] \leq x_{B_i} - \delta_{\mu(B_i)}^B \\
\beta_i, \sum_{h=1}^{H} \left( \sum_{k=1}^{m} p_{kh} x_{B_k} \right) \gamma_h / \left[ 1 + \sum_{h=1}^{H} \left( \sum_{k=1}^{m} \beta_k p_{kh} \right) \gamma_h \right] \geq x_{B_i} - \delta_{\mu(B_i)+1}^B, \right. \\
\eta^+_j + \frac{\sum_{h=1}^{H} \sum_{k=1}^{m} \sum_{i=1}^{m} \left( (e_{B_i}^B) - Z d_{\mu(B_i)}^B \beta_i \right) (p_{kh} y_{kj}) \gamma_h}{1 + \sum_{h=1}^{H} \left( \sum_{k=1}^{m} \beta_k p_{kh} \right) \gamma_h} \geq 0, \\
\eta^-_j + \frac{\sum_{h=1}^{H} \sum_{k=1}^{m} \sum_{i=1}^{m} \left( (e_{B_i}^B) - Z d_{\mu(B_i)}^B \beta_i \right) (p_{kh} y_{kj}) \gamma_h}{1 + \sum_{h=1}^{H} \left( \sum_{k=1}^{m} \beta_k p_{kh} \right) \gamma_h} \leq 0, \\
1 + \sum_{h=1}^{H} \left( \sum_{k=1}^{m} \beta_k p_{kh} \right) \gamma_h \neq 0 \quad j = 1, 2, \ldots, n, \quad i = 1, 2, \ldots, m \right\}.
\]
where $B^{-1} = [\beta_{ij}], y_j = B^{-1}A_j$ and $p_{kh}$ is the coefficients vector of the parameter $\gamma_h$ in the $j$th column of the perturbed basis matrix.

Proof. Since perturbation of a basis matrix may violate both feasibility and optimality conditions, therefore, we compute $\hat{\eta}_j^+, \hat{\eta}_j^-$ and $\hat{x}_B$, new values of $\eta_j^+, \eta_j^-$ and $x_B$ as follows:

$$\hat{x}_B = (B + \Delta B)^{-1}b - \sum_{j \in N} (B + \Delta B)^{-1}A_j \delta_{ij}^j.$$  

For small perturbations $\Delta B$ in $B$ [9], we get

$$(B + \Delta B)^{-1} = B^{-1} - \frac{B^{-1} \Delta B B^{-1}}{1 + \text{tr}(B^{-1} \Delta B)} ,$$

where

$$1 + \text{tr}(B^{-1} \Delta B) \neq 0.$$  

Therefore,

$$\hat{x}_B = \left( B^{-1} - \frac{B^{-1} \Delta B B^{-1}}{1 + \text{tr}(B^{-1} \Delta B)} \right) b - \sum_{j \in N} \left( B^{-1} - \frac{B^{-1} \Delta B B^{-1}}{1 + \text{tr}(B^{-1} \Delta B)} \right) A_j \delta_{ij}^j$$

$$= B^{-1}b - \frac{B^{-1} \Delta B B^{-1}}{1 + \text{tr}(B^{-1} \Delta B)} b - \sum_{j \in N} B^{-1} A_j \delta_{ij}^j + \sum_{j \in N} \frac{B^{-1} \Delta B B^{-1}}{1 + \text{tr}(B^{-1} \Delta B)} A_j \delta_{ij}^j$$

$$= x_B - \frac{B^{-1} \Delta B b}{1 + \text{tr}(B^{-1} \Delta B)} - \sum_{j \in N} B^{-1} A_j \delta_{ij}^j$$

$$= x_B - \frac{B^{-1} \Delta B x_B}{1 + \text{tr}(B^{-1} \Delta B)}$$

$$= x_B - \frac{[\beta_i \sum_{h=1}^H (\sum_{k=1}^m p_{kh} x_{B_k}) \gamma_h, ..., \beta_m \sum_{h=1}^H (\sum_{k=1}^m p_{kh} x_{B_k}) \gamma_h]^T}{1 + \sum_{h=1}^H (\sum_{k=1}^m \beta_k p_{kh}) \gamma_h}.$$  

Now ith component of $\hat{x}_B$ is given by

$$\hat{x}_{B_i} = x_{B_i} - \frac{\beta_i \sum_{h=1}^H (\sum_{k=1}^m p_{kh} x_{B_k}) \gamma_h}{1 + \sum_{h=1}^H (\sum_{k=1}^m \beta_k p_{kh}) \gamma_h}, \quad i = 1, 2, \ldots, m.$$  

This new basic solution $\hat{x}_B$ is feasible if

$$\delta_{\mu(B_i)}^{B_i} \leq x_{B_i} - \frac{\beta_i \sum_{h=1}^H (\sum_{k=1}^m p_{kh} x_{B_k}) \gamma_h}{1 + \sum_{h=1}^H (\sum_{k=1}^m \beta_k p_{kh}) \gamma_h} \leq \delta_{\mu(B_i)+1}^{B_i}, \quad i = 1, 2, \ldots, m.$$  

Thus, we have

$$\frac{\beta_i \sum_{h=1}^H (\sum_{k=1}^m p_{kh} x_{B_k}) \gamma_h}{1 + \sum_{h=1}^H (\sum_{k=1}^m \beta_k p_{kh}) \gamma_h} \leq x_{B_i} - \delta_{\mu(B_i)}^{B_i}.$$
and
\[
\frac{\beta_i \sum_{h=1}^H \left( \sum_{k=1}^m p_{khx_{B_k}} \right) \gamma_h}{1 + \sum_{h=1}^H \left( \sum_{k=1}^m \beta_k p_{kh} \right) \gamma_h} \geq x_{B_i} - \delta_{B_i^{\mu(B_i)+1}} \quad i = 1, 2, \ldots, m.
\]

For the new solution \( \hat{x}_B \), to satisfy optimality condition, the new values \( \hat{\eta}^+ \) and \( \hat{\eta}^- \) are computed as follows:

\[
\hat{\eta}^+_j = \left( c^j - c_B(B + \Delta B)^{-1} A_j \right) - Z \left( d^j - d_B(B + \Delta B)^{-1} A_j \right)
\]

\[
= \left( c^j - c_B - \frac{B^{-1}\Delta BB^{-1}}{1 + \text{tr}(B^{-1}\Delta B)} A_j \right) - Z \left( d^j - d_B(B^{-1} - \frac{B^{-1}\Delta BB^{-1}}{1 + \text{tr}(B^{-1}\Delta B)} A_j \right)
\]

\[
= \eta^+_j + (c_B - Zd_B) \left[ \beta_1 \sum_{h=1}^H \left( \sum_{k=1}^m p_{kh} y_{kj} \right) \gamma_h, \ldots, \beta_m \sum_{h=1}^H \left( \sum_{k=1}^m p_{kh} y_{kj} \right) \gamma_h \right]^T
\]

\[
1 + \sum_{h=1}^H \left( \sum_{k=1}^m \beta_k p_{kh} \right) \gamma_h
\]

\[
= \eta^+_j + \frac{\sum_{h=1}^H \sum_{k=1}^m \sum_{i=1}^m \left( c_{i(B_i)}^j - Zd_{i(B_i)}^j \beta_k \right) (p_{kh} y_{kj}) \gamma_h}{1 + \sum_{h=1}^H \left( \sum_{k=1}^m \beta_k p_{kh} \right) \gamma_h},
\]

and similarly

\[
\hat{\eta}^-_j = \eta^-_j + \frac{\sum_{h=1}^H \sum_{k=1}^m \sum_{i=1}^m \left( c_{i(B_i)}^j - Zd_{i(B_i)}^j \beta_k \right) (p_{kh} y_{kj}) \gamma_h}{1 + \sum_{h=1}^H \left( \sum_{k=1}^m \beta_k p_{kh} \right) \gamma_h}.
\]

This new solution \( \hat{x}_B \) is optimal if \( \hat{\eta}^-_j \leq 0 \) and \( \hat{\eta}^+_j \geq 0 \), \( j = 1, 2, \ldots, n \). Therefore, the proof is complete.

It can be seen that a parameter \( \gamma_h \) is absent in \( R \) if and only if

\[
\beta_k p_{kh} = 0, \quad p_{khx_{B_k}} = 0, \quad k = 1, 2, \ldots, m,
\]

and

\[
p_{kh} y_{kj} = 0, \quad k = 1, 2, \ldots, m, \quad j \in N.
\]

Those parameters which do not appear in \( R \) are called ‘non-focal’ and can be varied unlimitedly.

Remark 4. If \( \beta_0 = 1 \) and \( g_j(x_j) = 0, \quad j = 1, 2, \ldots, n \), then the PLFP reduces to PLP in which case the critical region \( R \) is given by
Remark 1: If \( \beta_i, \sum_{h=1}^{H} (\sum_{k=1}^{m} P_{kh} x_{B_k}) \gamma_h \) reduces to an LP, then the PLFP reduces to an LP with bounded variables and we have \( c^j_{ij} = c_j, \delta^j_{ij} = 0 \) and \( \delta^j_{i,j+1} = u_j, j = 1, 2, \ldots, n \). In this case the critical region \( R \) reduces to

\[
R = \left\{ \gamma \middle| \beta_i, \sum_{h=1}^{H} (\sum_{k=1}^{m} P_{kh} x_{B_k}) \gamma_h \leq x_{B_i}, \right. \\
\left. \beta_i, \sum_{h=1}^{H} (\sum_{k=1}^{m} P_{kh} x_{B_k}) \gamma_h \geq x_{B_i} - u_{B_i}, \right. \\
\left. 1 + \sum_{h=1}^{H} (\sum_{k=1}^{m} \beta_k P_{kh}) \gamma_h \neq 0 \right\}
\]

Moreover, if \( u_j = \infty \) \( \forall j \), in this case PLFP reduces to an LP and we have

\[
R = \left\{ \gamma \middle| \beta_i, \sum_{h=1}^{H} (\sum_{k=1}^{m} P_{kh} x_{B_k}) \gamma_h \leq x_{B_i}, \right. \\
\left. \beta_i, \sum_{h=1}^{H} (\sum_{k=1}^{m} P_{kh} x_{B_k}) \gamma_h \geq x_{B_i} - u_{B_i}, \right. \\
\left. 1 + \sum_{h=1}^{H} (\sum_{k=1}^{m} \beta_k P_{kh}) \gamma_h \neq 0 \right\}
\]

To perform sensitivity analysis, we decompose the critical region \( R \) as follows:

\[
R_{d+} = \left\{ \gamma \middle| 1 + \sum_{h=1}^{H} (\sum_{k=1}^{m} \beta_k P_{kh}) \gamma_h > 0 \right\}, \\
R_{d-} = \left\{ \gamma \middle| 1 + \sum_{h=1}^{H} (\sum_{k=1}^{m} \beta_k P_{kh}) \gamma_h < 0 \right\},
\]

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\[ R_{e^+} = \left\{ \gamma | \eta_j^+ \left[ 1 + \sum_{h=1}^{H} \left( \sum_{k=1}^{m} \beta_k \mathbf{p}_{kh} \right) \gamma_h \right] + \sum_{h=1}^{H} \sum_{k=1}^{m} \sum_{i=1}^{m} \left( (\epsilon_{\mu(B_i)}^B - Z d_{\mu(B_i)}^B) \beta_i \right) (\mathbf{p}_{kh} y_{kj}) \gamma_h \geq 0 \right\}, \]

\[ R_{e^-} = \left\{ \gamma | \eta_j^- \left[ 1 + \sum_{h=1}^{H} \left( \sum_{k=1}^{m} \beta_k \mathbf{p}_{kh} \right) \gamma_h \right] + \sum_{h=1}^{H} \sum_{k=1}^{m} \sum_{i=1}^{m} \left( (\epsilon_{\mu(B_i)}^B - Z d_{\mu(B_i)}^B) \beta_i \right) (\mathbf{p}_{kh} y_{kj}) \gamma_h \leq 0 \right\}, \]

\[ R_{f^+} = \left\{ \gamma | \eta_j^+ \left[ 1 + \sum_{h=1}^{H} \left( \sum_{k=1}^{m} \beta_k \mathbf{p}_{kh} \right) \gamma_h \right] + \sum_{h=1}^{H} \sum_{k=1}^{m} \sum_{i=1}^{m} \left( (\epsilon_{\mu(B_i)}^B - Z d_{\mu(B_i)}^B) \beta_i \right) (\mathbf{p}_{kh} y_{kj}) \gamma_h \geq 0 \right\}, \]

\[ R_{f^-} = \left\{ \gamma | \eta_j^- \left[ 1 + \sum_{h=1}^{H} \left( \sum_{k=1}^{m} \beta_k \mathbf{p}_{kh} \right) \gamma_h \right] + \sum_{h=1}^{H} \sum_{k=1}^{m} \sum_{i=1}^{m} \left( (\epsilon_{\mu(B_i)}^B - Z d_{\mu(B_i)}^B) \beta_i \right) (\mathbf{p}_{kh} y_{kj}) \gamma_h \leq 0 \right\}, \]

\[ R_{b^+} = \left\{ \gamma | \beta_i \sum_{h=1}^{H} \left( \sum_{k=1}^{m} \mathbf{p}_{kh} x_{B_k} \right) \gamma_h \leq \left[ 1 + \sum_{h=1}^{H} \left( \sum_{k=1}^{m} \beta_k \mathbf{p}_{kh} \right) \gamma_h \right] (x_{B_i} - \delta_{\mu(B_i)}) \right\}, \]

\[ R_{b^-} = \left\{ \gamma | \beta_i \sum_{h=1}^{H} \left( \sum_{k=1}^{m} \mathbf{p}_{kh} x_{B_k} \right) \gamma_h \geq \left[ 1 + \sum_{h=1}^{H} \left( \sum_{k=1}^{m} \beta_k \mathbf{p}_{kh} \right) \gamma_h \right] (x_{B_i} - \delta_{\mu(B_i)}) \right\}, \]

\[ R_{c^+} = \left\{ \gamma | \beta_i \sum_{h=1}^{H} \left( \sum_{k=1}^{m} \mathbf{p}_{kh} x_{B_k} \right) \gamma_h \geq \left[ 1 + \sum_{h=1}^{H} \left( \sum_{k=1}^{m} \beta_k \mathbf{p}_{kh} \right) \gamma_h \right] (x_{B_i} - \delta_{\mu(B_i)+1}) \right\}, \]

\[ R_{c^-} = \left\{ \gamma | \beta_i \sum_{h=1}^{H} \left( \sum_{k=1}^{m} \mathbf{p}_{kh} x_{B_k} \right) \gamma_h \leq \left[ 1 + \sum_{h=1}^{H} \left( \sum_{k=1}^{m} \beta_k \mathbf{p}_{kh} \right) \gamma_h \right] (x_{B_i} - \delta_{\mu(B_i)+1}) \right\}. \]

Then

\[ R = \left\{ \gamma | \gamma \in (R_{d^+} \cap R_{b^+} \cap R_{c^+} \cap R_{f^+}) \quad \text{or} \quad \gamma \in (R_{d^-} \cap R_{b^-} \cap R_{c^-} \cap R_{f^-}) \right\} \]

can be decomposed into two disjoint regions: \( R_1 = \left\{ R_{d^+} \cap R_{b^+} \cap R_{c^+} \cap R_{f^+} \right\} \) and \( R_2 = \left\{ R_{d^-} \cap R_{b^-} \cap R_{c^-} \cap R_{f^-} \right\}. \)

**Definition 6.** [14] The Maximum Volume Region (MVR) \( B_R \) in a critical region \( R \) is given by

\[ B_R = \left\{ \gamma = (\gamma_1, \ldots, \gamma_H)^T | |\gamma_j| \leq k_j^*, \ j = 1, 2, \ldots, H \right\}, \]

where \( k^* = (k_1^*, k_2^*, \ldots, k_H^*) \) is the optimal solution of the following maximization problem:

\[ \max_{k \in K(R)} V(k) = k_1 k_2 \ldots k_H, \tag{5} \]

where

\[ K(R) = \{ k = (k_1, k_2, \ldots, k_H)^T | |\gamma_j| \leq k_j, \ j = 1, 2, \ldots, H \} \]

implying \( \gamma = (\gamma_1, \gamma_2, \ldots, \gamma_H)^T \in R \).

The volume of \( B_R \) is \( \text{Vol}(B_R) = 2^H k_1^* k_2^* \ldots k_H^* \).
As we want to determine the tolerance region with the largest volume in the problem, we present the following definition [11].

**Definition 7.** A symmetrically rectangular parallelepiped $B_\psi = \{ \gamma = (\gamma_1, \ldots, \gamma_H)^T | \gamma_j \leq \bar{\lambda}_j, j = 1, 2, \ldots, H \}$ of a region $\psi$ is the Weak Maximal Volume Region (WMVR) of $\psi$ if $\bar{\nu} = \bar{\lambda}_1 \bar{\lambda}_2 \ldots \bar{\lambda}_H$ is the least upper bound of the set $\{ \nu = \lambda_1 \lambda_2 \ldots \lambda_H | \gamma_j \leq \lambda_j, j = 1, 2, \ldots, H \}$. Furthermore, if $B_\psi$ is a subset of $\psi$, that is, $\bar{\nu}$ is also the maximum, we say $B_\psi$ is a Strong Maximal Volume Region (SMVR) of $\psi$, and denote it by $B_\psi$.

**Theorem 8.** [13] Let $B_\psi$ be a WMVR of a region $\psi$, then a necessary and sufficient condition for $B_\psi$ to be a SMVR of $\psi$ is that $\{ \gamma = (\gamma_1, \ldots, \gamma_H)^T | \gamma_j | \leq \bar{\lambda}_j, j = 1, 2, \ldots, H \} \subseteq R_{d^+} = \{ \gamma | 1 + \sum_{h=1}^{H} \left( \sum_{k=1}^{m} \beta_k p_{kh} \right) \gamma_h > 0 \}$.

Since the critical region $R$ is a polyhedral set, therefore there exists $L = [\ell_{ij}] \in \mathbb{R}^{I \times J}$, $d = \{d_i\} \in \mathbb{R}^I$, $I, J \in \mathbb{N}$ where $I$ and $J$ are the numbers of constraints and variables of $R$, respectively, such that $R = \{ \gamma = (\gamma_1, \ldots, \gamma_J)^T | L \gamma \leq d \}$. While deleting all non-focal parameters of $R$, we can assume that $\ell_{ij} \neq 0$, $j = 1, 2, \ldots, J$.

**Remark 9.** It follows from Theorem 3 that $\gamma = 0$ belongs to $R$ and thus we have $d \geq 0$.

We consider the case $d > 0$. Then, the problem (5) can be restated as follows:

$$\max_{k \in K(R)} V(k) = k_1 \cdot k_2 \ldots k_J,$$

where

$$K(R) = \{ k = (k_1, k_2, \ldots, k_J)^T | |\gamma_j| \leq k_j, j = 1, 2, \ldots, J \} \implies L \gamma \leq d \}.$$

We have the following Theorem:

**Theorem 10.** [14] The MVR $B_R$ of a polyhedral set $R = \{ \gamma = (\gamma_1, \ldots, \gamma_J)^T | L \gamma \leq d \}$ with $d > 0$ and $\ell_{ij} \neq 0$ is bounded and defined by

$$B_R = \{ \gamma = (\gamma_1, \ldots, \gamma_J)^T | \gamma_j | \leq k^*_j, j = 1, 2, \ldots, J \},$$

where $k^* = (k^*_1, k^*_2, \ldots, k^*_J)$ is the unique optimal solution of the following maximization problem:

$$\max V(k) = k_1 \cdot k_2 \ldots k_J$$

s.t.: $|L|k \leq d$

$k \geq 0$,

where $|L|$ is obtained by changing the negative elements of matrix $L$ to be positive.

3 Illustrative example

In this section, an example is presented to illustrate the obtained results in identifying the critical regions.
Example 1. Consider the problem (PLFP) as:

$$\min_{j=1}^{4} f_j(x_j)$$

subject to:

$$\begin{align*}
3x_1 + 4x_2 + x_3 + 2x_4 &= 21 \\
x_1 + 3x_2 + x_3 + 3x_4 &= 13 \\
2x_1 + x_2 + 2x_3 + 3x_4 &= 14 \\
0 \leq x_1 \leq 5, & \quad 0 \leq x_2 \leq 3, & \quad 0 \leq x_3 \leq 5, & \quad 0 \leq x_4 \leq 5
\end{align*}$$

where

$$f_1(x_1) = \begin{cases}
3x_1, & 0 \leq x_1 \leq 1, \\
4x_1 - 1, & 1 \leq x_1 \leq 5,
\end{cases}$$

$$g_1(x_1) = \begin{cases}
4x_1 + 1, & 0 \leq x_1 \leq 1, \\
3x_1 + 2, & 1 \leq x_1 \leq 5,
\end{cases}$$

$$f_2(x_2) = \begin{cases}
2x_2 + 1, & 0 \leq x_2 \leq 1, \\
3x_2, & 1 \leq x_2 \leq 3,
\end{cases}$$

$$g_2(x_2) = \begin{cases}
3x_2 + 1, & 0 \leq x_2 \leq 1, \\
2x_2 + 2, & 1 \leq x_2 \leq 3,
\end{cases}$$

$$f_3(x_3) = \begin{cases}
x_3 + 3, & 0 \leq x_3 \leq 2, \\
x_3 + 1, & 2 \leq x_3 \leq 3,
\end{cases}$$

$$g_3(x_3) = \begin{cases}
x_3 + 3, & 0 \leq x_3 \leq 2, \\
x_3 + 3, & 2 \leq x_3 \leq 3,
\end{cases}$$

$$f_4(x_4) = \begin{cases}
x_4 + 1, & 0 \leq x_4 \leq 1, \\
2x_4, & 1 \leq x_4 \leq 3,
\end{cases}$$

$$g_4(x_4) = \begin{cases}
x_4 + 4, & 0 \leq x_4 \leq 1, \\
x_4 + 4, & 1 \leq x_4 \leq 3,
\end{cases}$$

Using simplex algorithm of Punnen and Pandey [8], the initial and the final simplex tables are given as follows:

**Tableau 1**

<table>
<thead>
<tr>
<th>(c_B)</th>
<th>(d_B)</th>
<th>(x_B)</th>
<th>(x_1)</th>
<th>(x_2)</th>
<th>(x_3)</th>
<th>(x_4)</th>
<th>(x_5)</th>
<th>(x_6)</th>
<th>(x_7)</th>
<th>(b)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(M)</td>
<td>0 (x_5)</td>
<td>3</td>
<td>4</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>21</td>
<td></td>
</tr>
<tr>
<td>(M)</td>
<td>0 (x_6)</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>3</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>13</td>
<td></td>
</tr>
<tr>
<td>(M)</td>
<td>0 (x_7)</td>
<td>2</td>
<td>1</td>
<td>3</td>
<td>3</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>14</td>
<td></td>
</tr>
<tr>
<td>(\eta_j^+)</td>
<td>2 - 54M</td>
<td>(-\frac{7 - 176M}{4})</td>
<td>(-\frac{11 - 186M}{4})</td>
<td>4 - 56M</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>(z = \frac{5 + 48M}{4})</td>
<td></td>
</tr>
<tr>
<td>(\eta_j)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>21</td>
<td>13</td>
<td>14</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(x_j)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Final**

<table>
<thead>
<tr>
<th>(c_B)</th>
<th>(d_B)</th>
<th>(x_B)</th>
<th>(x_1)</th>
<th>(x_2)</th>
<th>(x_3)</th>
<th>(x_4)</th>
<th>(x_5)</th>
<th>(x_6)</th>
<th>(x_7)</th>
<th>(b)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>2</td>
<td>(x_2)</td>
<td>0</td>
<td>1</td>
<td>(-\frac{3}{10})</td>
<td>0</td>
<td>(\frac{3}{20})</td>
<td>(\frac{1}{4})</td>
<td>(-\frac{7}{20})</td>
<td>(3/2)</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>(x_4)</td>
<td>0</td>
<td>0</td>
<td>(\frac{1}{2})</td>
<td>1</td>
<td>(-\frac{1}{4})</td>
<td>(\frac{1}{4})</td>
<td>(\frac{1}{4})</td>
<td>(3/2)</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>(x_1)</td>
<td>1</td>
<td>0</td>
<td>(\frac{2}{5})</td>
<td>0</td>
<td>(\frac{3}{10})</td>
<td>(-\frac{1}{2})</td>
<td>(3/10)</td>
<td>4</td>
</tr>
<tr>
<td>(\eta_j^+)</td>
<td>0</td>
<td>0</td>
<td>1.031</td>
<td>0</td>
<td>(M - 2.623)</td>
<td>(M + 1)</td>
<td>(M - 1.438)</td>
<td>(z = .885)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\eta_j)</td>
<td>0</td>
<td>0</td>
<td>(-.854)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(x_j)</td>
<td>32/10</td>
<td>21/10</td>
<td>2</td>
<td>1/2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The optimal solution is \(x^* = (32/10, 21/10, 2, 1/2, 0, 0, 0)^T\). Here \(B = \{2, 4, 1\}\) and the submatrix of the optimal basis is \(\begin{pmatrix} 4 & 2 & 3 \\ 3 & 3 & 1 \\ 1 & 3 & 2 \end{pmatrix}\) and its inverse.
Let the perturbation matrix $\Delta A$ be given as follows:

$$\Delta A = \begin{bmatrix}
2\gamma_1 - \gamma_2 + 3\gamma_3 & 0 & 0 & 0 \\
\gamma_1 + 2\gamma_2 + 5\gamma_3 & \gamma_1 + 3\gamma_2 - \gamma_3 & 0 & 0 \\
0 & 4\gamma_1 + 2\gamma_2 - 2\gamma_3 & 0 & \gamma_1 - 2\gamma_2 + \gamma_3
\end{bmatrix}.$$

Therefore,

$$p_{11} = (2, 1, 0)^T, \quad p_{12} = (-1, 2, 0)^T, \quad p_{13} = (3, 5, 0)^T,$$

$$p_{21} = (0, 1, 4)^T, \quad p_{22} = (0, 3, 1)^T, \quad p_{23} = (0, -1, -2)^T,$$

$$p_{31} = (0, 0, 1)^T, \quad p_{32} = (0, 0, -1)^T, \quad p_{33} = (0, 0, 1)^T,$$

$$R_{d+} = \left\{ \gamma \mid 1 + 2.1\gamma_1 + 2.05\gamma_2 + 1.25\gamma_3 > 0 \right\},$$

$$R_{d-} = \left\{ \gamma \mid 1 + 2.1\gamma_1 + 2.05\gamma_2 + 1.25\gamma_3 < 0 \right\},$$

$$R_{b+} = \left\{ \gamma \mid -7.71\gamma_1 + 0.9\gamma_2 + 1.3\gamma_3 \leq 1.1, \ -0.15\gamma_1 - 0.3\gamma_2 + 0.85\gamma_3 \leq 0.5, \ 0.47\gamma_1 + 4.82\gamma_2 - 3.08\gamma_3 \leq 2.2 \right\},$$

$$R_{b-} = \left\{ \gamma \mid -7.71\gamma_1 + 0.9\gamma_2 + 1.3\gamma_3 \geq 1.1, \ -0.15\gamma_1 - 0.3\gamma_2 + 0.85\gamma_3 \geq 0.5, \ 0.47\gamma_1 + 4.82\gamma_2 - 3.08\gamma_3 \geq 2.2 \right\},$$

$$R_{c+} = \left\{ \gamma \mid -3.51\gamma_1 + 3\gamma_2 + 3.8\gamma_3 \geq -1.89, \ 1.95\gamma_1 + 0.75\gamma_2 + 2.1\gamma_3 \geq -0.5, \ 5.3\gamma_1 - 2.4\gamma_2 - 0.2\gamma_3 \geq -1.8 \right\},$$

$$R_{c-} = \left\{ \gamma \mid -3.51\gamma_1 + 3\gamma_2 + 3.8\gamma_3 \leq -1.89, \ 1.95\gamma_1 + 0.75\gamma_2 + 2.1\gamma_3 \leq -0.5, \ 5.3\gamma_1 - 2.4\gamma_2 - 0.2\gamma_3 \leq -1.8 \right\},$$

$$R_{e+} = \left\{ \gamma \mid 0.27\gamma_1 + 0.8\gamma_2 + 2.17\gamma_3 \geq -1.031 \right\},$$

$$R_{e-} = \left\{ \gamma \mid 0.27\gamma_1 + 0.8\gamma_2 + 2.17\gamma_3 \leq -1.031 \right\},$$

$$R_{f+} = \left\{ \gamma \mid -3.67\gamma_1 - 1.18\gamma_2 - 0.19\gamma_3 \geq 0.854 \right\},$$

$$R_{f-} = \left\{ \gamma \mid -3.67\gamma_1 - 1.18\gamma_2 - 0.19\gamma_3 \leq 0.854 \right\}.$$

Hence,

$$R_1 = \left\{ \gamma \mid 1 + 2.1\gamma_1 + 2.05\gamma_2 + 1.25\gamma_3 > 0, \ -7.71\gamma_1 + 0.9\gamma_2 + 1.3\gamma_3 \leq 1.1, \right\}.$$
by solving the following problem:

\[ R \text{ and } \text{Volume Region (WMVR with Vol(complementarity problem.} \]

We expect that this approach is applicable for conic programming and linear as “focal” and “non-focal”. This approach reduces the number of parameters in the final matrix perturbations in the problem (PLFP) by classifying the perturbation parameters

In this paper, we discuss approximate multi-parametric sensitivity analysis under basis conclusion

\[ R_2 = \left\{ \gamma \mid 1 + 2.1 \gamma_1 + 2.05 \gamma_2 + 1.25 \gamma_3 < 0, \; -7.71 \gamma_1 + 0.9 \gamma_2 + 1.3 \gamma_3 \leq 1.1, \; -0.15 \gamma_1 - 0.3 \gamma_2 + 0.85 \gamma_3 \geq 0.5, \; 0.47 \gamma_1 - 4.82 \gamma_2 - 3.08 \gamma_3 \geq 2.2, \right\} \]

where \( \gamma \) belongs to only one of the two disjoint regions \( R_1 = \left\{ R_{d+} \cap R_{b+} \cap R_{c+} \cap R_{e+} \cap R_{f+} \right\} \)

and \( R_2 = \left\{ R_{d-} \cap R_{b-} \cap R_{c-} \cap R_{e+} \cap R_{f+} \right\} \).

In this example all the parameters are focal parameters. The MVR of \( S_R \) is obtained by solving the following problem:

\[
\begin{align*}
\text{max} & \quad V(k) = k_1.k_2.k_3 \\
\text{s.t} & \quad 2.1k_1 + 2.05k_2 + 1.25k_3 \leq 1 \\
& \quad 7.71k_1 + 0.9k_2 + 1.3k_3 \leq 1.1 \\
& \quad 0.15k_1 + 0.3k_2 + 0.85k_3 \leq 0.5 \\
& \quad 0.47k_1 + 4.82k_2 + 3.08k_3 \leq 2.2 \\
& \quad 3.51k_1 + 3k_2 + 3.8k_3 \leq 1.89 \\
& \quad 1.95k_1 + 0.75k_2 + 2.1k_3 \leq 0.5 \\
& \quad 5.3k_1 + 2.4k_2 + 0.2k_3 \leq 1.8 \\
& \quad 0.27k_1 + 0.8k_2 + 2.17k_3 \leq 1.031 \\
& \quad 3.67k_1 + 1.18k_2 + 0.19k_3 \leq 0.854 \\
& \quad k_1, k_2, k_3 \geq 0.
\end{align*}
\]

The optimal solution of the problem is \( k^* = (0.0828, 0.2231, 0.0815) \). The Weak Maximum Volume Region (WMVR) \( B_R \) of \( R \) is \( \{ \gamma \mid |\gamma_1| \leq 0.0828, |\gamma_2| \leq 0.2231, |\gamma_3| \leq 0.0815 \} \) with \( \text{Vol}(B_R) = 2^3(0.0828)(0.2231)(0.0815) \). Since \( \{ \gamma \mid |\gamma_1| = 0.0828, |\gamma_2| = 0.2231, |\gamma_3| = 0.0815 \} \) \( \subseteq S_{d+} \) therefore the WMVR of \( R \) is also a SMVR.

4 conclusion

In this paper, we discuss approximate multi-parametric sensitivity analysis under basis matrix perturbations in the problem (PLFP) by classifying the perturbation parameters as “focal” and “non-focal”. This approach reduces the number of parameters in the final analysis. We expect that this approach is applicable for conic programming and linear complementarity problem.
References


