AMO - Advanced Modeling and Optimization, Volume 11, Number 2, 2009 [1]

A FIRST ORDER PREDICTOR-CORRECTOR INFEASIBLE INTERIOR POINT METHOD FOR SUFFICIENT LINEAR COMPLEMENTARITY PROBLEMS IN A WIDE AND SYMMETRIC NEIGHBORHOOD OF THE CENTRAL PATH [†]

ADRIAN VANCEA [‡]

Abstract. In this paper, a new predictor-corrector method is proposed for solving sufficient linear complementarity problems (LCP) with an infeasible starting point. The method generates a sequence of iterates in a wide and symmetric neighborhood of the infeasible central path of the LCP. If the starting point is feasible or close to being feasible, then an ε-approximate solution is obtained in at most $O((1+\kappa)nL)$ iterations. For a large infeasible starting point, the iteration complexity is $O((1+\kappa)^2 n^{3/2} L)$. The algorithm also converges Q-quadratically to zero for nondegenerate problems. We also present a variant of the original algorithm which does not depend on κ.

Key words. sufficient linear complementarity problem, interior-point, predictor-corrector, superlinear convergence, polynomial complexity.

About the author: Adrian Vancea graduated at the Babes-Bolyai University of Cluj-Napoca in 2002. Currently he is a doctoral student at the Applied Mathematics department of the University of Maryland, Baltimore County(UMBC). His research interests are in the field of optimization, focusing on the study of the interior point methods for linear complementarity problems. He is a member of the Society for Industrial and Applied Mathematics (SIAM) and the American Mathematical Society (AMS).

1. Introduction. Interior-point methods play an important role in modern mathematical programming. These methods have been used to obtain strong theoretical results and they have been successfully implemented in software packages for solving linear (LP), quadratic (QP), semidefinite (SDP), and many other problems. Today, interior point methods not only have polynomial iteration complexity but are also the most effective methods for solving large scale optimization problems. These methods have provided the first polynomial-time algorithms for solving linear programming and other classes of convex optimization problems. Polynomiality is proved by showing that the duality gap converges to zero with the global linear rate of at most $(1 - c/n^p)$. This implies that the duality gap can be reduced to less than 2^{-L} in at most $O(n^p L)$ iterations. The best complexity result to date has $p = \frac{1}{2}$. However the practical performance of interior-point methods is better than the one indicated by these complexity results. This is explained in part by the superlinear convergence.

Zhang, Tapia and Dennis [31] gave sufficient conditions for a class of interior-point methods, in order to produce a sequence of iterates with duality gap converging superlinearly to zero but their algorithm did not have polynomial-time complexity. The results of [31] were generalized for LCP in [32]. The first interior-point method having both polynomial-time complexity and superlinear convergence was the predictor-corrector method of Mizuno, Todd and Ye (MTY) [12]. This algorithm is for LP and has $O(\sqrt{n}L)$ iteration complexity. After that, Ye et al. [30], and independently Mehrotra [10], proved that the duality gap of the iterates produced by

[1]**AMO - Advanced Modeling and Optimization. ISSN: 1841-4311**

[†]Dedicated to Dr. Neculai Andrei for his 60th birthday.

[‡]Department of Mathematics and Statistics, University of Maryland Baltimore County, 1000 Hilltop Circle, Baltimore, MD 22150, USA (`va1@math.umbc.edu`).

MTY converges quadratically to zero. MTY was generalized to monotone linear complementarity problems (LCP) in [6], and the resulting algorithm was proved to have $O(\sqrt{n}L)$ iteration complexity under general conditions, and superlinear convergence under the assumption that the LCP has a (perhaps not unique) strictly complementary solution (LCP is nondegenerate) and the iteration sequence converges. From [2] it follows that the latter assumption always holds. Ye and Anstreicher [29] proved that MTY converges quadratically assuming only that the LCP is nondegenerate. The nondegeneracy assumption is not restrictive, since according to [13] a large class of interior point methods, which contains MTY, can have only linear convergence if this assumption is violated.

Feasible interior point methods start with a strictly feasible interior point and keep feasibility during the algorithm. The problem with this approach is that finding a feasible starting point turns out to be a computational expensive process. The big advantage of infeasible interior point methods is that they don't require any special starting point. They start with an arbitrary positive point and feasibility is reached as optimality is approached. In [12, 30, 10, 6, 29] one assumes that the starting point for the MTY algorithm is strictly feasible. A generalization of the MTY algorithm for infeasible starting points was proposed in [14, 15] for LP, and in [16, 23] for monotone LCP. The methods from [16, 23] are both predictor-corrector algorithms and they use the small neighborhood of the central path. The algorithm proposed in [16] requires two matrix factorizations and at most three backsolves per iteration. Its computational complexity depends on the quality of the starting point. If the starting point is large enough, then the algorithm has $O(nL)$ iteration complexity and if a certain measure of feasibility at the starting point is small enough, then the algorithm has $O(\sqrt{n}L)$ iteration complexity. At each iteration, both feasibility and optimality are reduced at the same rate. Moreover, the algorithm is quadratically convergent for nondegenerate problems. The MTY predictor-corrector algorithm was extended for feasible $P_*(\kappa)$ linear complementarity problems in 1995 by Miao [11]. His algorithm depends on κ, uses the small neighborhood of the central path, has $O((1+\kappa)\sqrt{n}L)$ iteration complexity and is quadratically convergent. For $P_*(\kappa)$ linear complementarity problems, MTY was extended for the infeasible case, in [7]. This algorithm depends on κ, uses the small neighborhood of the central path, requires two matrix factorizations and only two backsolves per iteration. Of course, its complexity depends on the quality of the starting point. If the starting point is "large enough", then the algorithm has $O((1+\kappa)^2 nL)$ iteration complexity and if the starting point is feasible or close to being feasible, then the algorithm has $O((1+\kappa)\sqrt{n}L)$ iteration complexity. Moreover, the algorithm is quadratically convergent for nondegenerate problems. Using a special type of large neighborhood which is still contained in $\mathcal{N}_\infty^-$, Potra and Sheng [20] where able to improve the results from [7] in the following way. They obtained an infeasible algorithm which is independent of κ and when it approaches the solution, it requires only one matrix factorization per iteration. All the other complexity and superlinear convergence properties of [7] are also satisfied. We also mention that in a recent paper [22], Salahi, Peyghami and Terlaky study the complexity of an infeasible interior point algorithm only for (LP), using a special type of large neighborhood based on a specific self-regular proximity function. Their algorithm has $O(n^{3/2}\log n\ \log(n/\varepsilon))$ iteration complexity which is slightly worse than the one we obtain in this paper. Using the large neighborhood, Potra and Liu [18] developed a feasible interior point algorithm which generalizes [17] for $P_*(\kappa)$ problems. The first order version of this algorithm has $O((1+\kappa)nL)$ complexity for general $P_*(\kappa)$

problems and is quadratically convergent for nondegenerate problems. This algorithm depends explicitly on κ, but they also gave a variant which does not depend on κ and has the same properties.

Although theoretical results are better for algorithms which use small neighborhoods, it turns out that in practice those which use wide neighborhoods perform better. This is one of the paradoxes of the interior point methods because algorithms which use large neighborhoods of the central path are usually more difficult to analyse and, in general, their computational complexity is worse than the corresponding one for algorithms using smaller neighborhoods. Recent studies made by Colombo and Gondzio [4] revealed that better practical performance are obtained when the symmetric and wide neighborhoods are used. Practical experience suggests that one of the reasons why these algorithms are so efficient is the way in which the quality of centrality is assessed. By centrality we refer here to the way in which the complementarity products $x_i s_i, i = 1, \ldots, n$ are spread. Large discrepancies within the complementarity pairs (bad centering) create problems for the search directions. An unsuccessful iteration is caused not only by small complementarity products, but also by very large ones. The notion of spread in complementarity products is not well characterized by either the small ($\mathcal{N}_2$) or the wide ($\mathcal{N}_\infty^-$) neighborhoods commonly used in the theoretical development of interior point algorithms. To overcome this disadvantage, Colombo and Gondzio used a variation of the usual wide neighborhood, in which they introduced an upper bound on the complementarity pairs. We will use this type of neighborhood of the central path in this paper. We refer to it as the wide symmetric neighborhood and we denote it by $\hat{\mathcal{N}}_\infty^-$. Infeasible interior point methods for LCP in symmetric neighborhoods were first considered by Bonnans and Potra in [3]. While the wide neighborhood ensures that some products do not approach zero too early, it does not prevent them from becoming too large with respect to the average. On the other hand, the wide symmetric neighborhood ensures the decrease of complementarity pairs which are too large, thus taking better care of centrality. These wide symmetric neighborhoods are bigger than $\mathcal{N}_2$ and smaller than $\mathcal{N}_\infty^-$. So in the present paper we will obtain a complexity which is worse than the one in [16], where the small neighborhood is used, and at least the same as the one in [18], where they use a large neighborhood.

This paper will generalize the results from [18] for the infeasible case, but using the wide and symmetric neighborhood of the central path. We improve the existing results from the following points of view. The algorithm which will be presented is an infeasible one, which means that we can choose any initial positive starting point for it. It uses the wide and symmetric neighborhood which is a big advantage because recent results have proved that algorithms which use this type of neighborhoods have better numerical results than the ones which use the classical wide neighborhood of the central path. The algorithm has $O((1+\kappa)nL)$ iteration complexity if the starting point is feasible or close to being feasible. For large infeasible starting points the algorithm has $O((1+\kappa)^2 n^{3/2} L)$ iteration complexity. Moreover, the algorithm converges quadratically for nondegenerate problems. This algorithm depends explicitly on κ, but we will also present a variant of it which does not depend on κ and has the same properties. In this paper we work on sufficient horizontal linear complementarity problems, $P_*(\kappa)$ (HLCP), basically because of their symmetry. These problems are slight generalizations of the standard linear complementarity problem. Equivalence results for different variants of linear complementarity problems can be found in [1].

Conventions. We denote by $I\!N$ the set of all nonnegative integers, and $I\!R$, $I\!R_+$, $I\!R_{++}$ denote the set of real, nonnegative real, and positive real numbers respectively. Given a vector x, the corresponding upper case symbol X denotes the diagonal matrix X defined by the vector x. The symbol e represents the vector of all ones with appropriate dimension.

We denote component-wise operations on vectors by the usual notations for real numbers. Thus, given two vectors u and v of the same dimension, uv, u/v, etc. will denote the vectors with components u_iv_i, u_i/v_i, etc. This notation is consistent as long as component-wise operations always have precedence in relation to matrix operations. Note that $uv \equiv Uv$ and if A is a matrix, then $Auv \equiv AUv$, but in general $A\,(uv) \neq (Au)\,v$. Also, if f is a scalar function and v is a vector, then $f(v)$ denotes the vector with components $f(v_i)$. For example, if $v \in I\!R^n_+$ and $\lambda \in I\!R$, then $\sqrt{v}$ denotes the vector with components $\sqrt{v_i}$, and $\lambda - v$ denotes the vector with components $\lambda - v_i$. Traditionally the vector $\lambda - v$ is written as $\lambda e - v$. If $\|.\|$ is a vector norm on $I\!R^n$ and A is a matrix, then the operator norm induced by $\|.\|$ is defined by $\|A\| := \max\{\|Ax\| : \|x\| = 1\}$. As a particular case we note that if U is the diagonal matrix defined by the vector u, then $\| U \|_2 = \| u \|_\infty$.

If x and s are vectors in $I\!R^n$ and τ is a scalar in $I\!R$, then the vector $z \in I\!R^{2n}$ obtained by concatenating x and s is denoted by $z = \lceil x,\, s \rfloor = \left[x^T,\, s^T \right]^T$, the mean value of xs is denoted by $\mu(z) := (x^T s)/n$, and $\lceil x,\, s,\, \tau \rfloor := \left[x^T,\, s^T,\, \tau \right]^T$.

2. The sufficient homogeneous linear complementarity problem. Given two matrices Q and R in $I\!R^{n\times n}$, and a vector b in $I\!R^n$, the horizontal linear complementarity problem (HLCP) is finding a pair of vectors $z = \lceil x,\, s \rfloor$ such that

$$\begin{aligned} xs &= 0 \\ Qx + Rs &= b \\ x, s &\geq 0. \end{aligned} \tag{2.1}$$

The standard (monotone) linear complementarity problem (SLCP or simply LCP) corresponds to the case where $R = -I$, and Q is positive semidefinite. Let $\kappa \geq 0$ be a given constant. We say that (2.1) is a $P_*(\kappa)$ HLCP if

$$Qu + Rv = 0 \text{ implies } (1 + 4\kappa) \sum_{i\in\mathcal{I}^+} u_iv_i + \sum_{i\in\mathcal{I}^-} u_iv_i \geq 0, \text{ for any } u, v \in I\!R^n\,, \tag{2.2}$$

where $\mathcal{I}^+ = \{i : u_iv_i > 0\}$ and $\mathcal{I}^- = \{i : u_iv_i < 0\}$. If the above condition is satisfied, then we say (Q, R) is a $P_*(\kappa)$ pair and write $(Q, R) \in P_*(\kappa)$. In the case $R = -I$, $(Q, -I)$ is a $P_*(\kappa)$ pair if and only if Q is a $P_*(\kappa)$ matrix, that is,

$$(1 + 4\kappa) \sum_{i\in\hat{\mathcal{I}}^+} x_i[Qx]_i + \sum_{i\in\hat{\mathcal{I}}^-} x_i[Qx]_i \geq 0, \quad \forall x \in I\!R^n\,,$$

where $\hat{\mathcal{I}}^+ = \{i : x_i[Qx]_i > 0\}$ and $\hat{\mathcal{I}}^- = \{i : x_i[Qx]_i < 0\}$. Problem (2.1) is then called a $P_*(\kappa)$ LCP and it is extensively discussed in [8]. If (Q, R) belongs to the class

$$P_* = \bigcup_{\kappa\geq 0} P_*(\kappa)\,,$$

then we say that (Q, R) is a P_* pair and (2.1) is a P_* HLCP. The handicap of a sufficient pair (Q, R) is defined by $\chi(Q, R) = \min\{\kappa : \kappa \geq 0, (Q, R) \in P_*(\kappa)\}$.

The class of sufficient matrices was defined by Cottle et al. in [5]. The appropriate generalization to sufficient pair [25, 26] is in terms of the null space of the matrix $[Q\ R] \in I\!R^{n\times 2n}$

$$\Phi := \mathcal{N}([Q\ R]) = \{\lceil u,\, v \rfloor :\ Qu + Rv = 0\} \tag{2.3}$$

and its orthogonal space

$$\Phi^{\perp} = \left\{\lceil u,\, v \rfloor :\ u = Q^T x,\, v = R^T x,\ \text{ for some } x \in I\!R^n\right\}\,. \tag{2.4}$$

The pair (Q, R) is called column sufficient if

$$\lceil u,\, v \rfloor \in \Phi, \quad uv \le 0 \text{ implies } uv = 0\,,$$

and row sufficient if

$$\lceil u,\, v \rfloor \in \Phi^{\perp}, \quad uv \ge 0 \text{ implies } uv = 0\,.$$

(Q, R) is a sufficient pair if it is both column and row sufficient. The corresponding results of row and column sufficient matrices in [5] can be extended to row and column sufficient pairs: (Q, R) is a sufficient pair if and only if for any b, the HLCP (2.1) has a convex (perhaps empty) solution set and every KKT point of

$$\begin{array}{cc} \min & x^T s \\ \text{s.t.} & Qx + Rs = b \\ & x, s \ge 0 \end{array}$$

is a solution of (2.1).

Väliaho's result [27] states that a matrix is sufficient if and only if it is a $P_*(\kappa)$ matrix for some $\kappa \ge 0$. The result can be extended to sufficient pairs by using the equivalence results from [1] (see also [24]): (Q, R) is a sufficient pair if and only if there is a finite $\kappa \ge 0$ so that (Q, R) is a $P_*(\kappa)$ pair. By extension, a P_* HLCP will be called a sufficient HLCP and a P_* pair will be called a sufficient pair.

Let us note that if (Q, R) is a sufficient pair, then the matrix $[Q\ R]$ is full rank. In fact, we have the following slightly stronger result.

THEOREM 2.1. *([9]) If Q and R are two $n \times n$ matrices such that the pair (Q, R) is column sufficient, then the matrix $[Q\ R]$ is full rank.*

The above theorem is important for obtaining analyticity of weighted central paths defined by the following system

$$\begin{aligned} xs &= \tau p, \\ Qx + Rs &= b - \tau\bar{b}, \end{aligned}$$

where $p \in \mathbb{R}^n_{++}$ and $\bar{b} \in \mathbb{R}^n$ is a suitable perturbation vector. For HLCP, we define the set of feasible points by

$$\mathcal{F} := \{z = \lceil x,\, s \rfloor \in I\!R^{2n}_{+}\ :\ Qx + Rs = b\}. \tag{2.5}$$

The relative interior of $\mathcal{F}$, which is also known as the set of strictly feasible points or the set of interior points, is given by

$$\mathcal{F}^0 = \mathcal{F} \bigcap I\!R^{2n}_{++}\,.$$

Also, related to the weighted central path, we define the set

$$\mathcal{F}_{\bar{b}} := \{\lceil z,\, \tau \rfloor = \lceil x,\, s,\, \tau \rfloor \in R^{2n+1}_{++} \,:\, Qx + Rs = b - \tau\bar{b}\}, \tag{2.6}$$

which we assume nonempty.

The set of solutions (or the optimal face) of HLCP is defined by

$$\mathcal{F}^* := \{z^* = \lceil x^*,\, s^* \rfloor \in \mathcal{F} \,:\, x^* s^* = 0\}. \tag{2.7}$$

A solution $\lceil x^*,\, s^* \rfloor \in \mathcal{F}^*$ of HLCP is called *strictly complementary* if $x^* + s^* > 0$. The set of all strictly complementary solutions is denoted by

$$\mathcal{F}^c = \{z^* = \lceil x^*,\, s^* \rfloor \in \mathcal{F}^* :\; x^* + s^* > 0\}.$$

Not every HLCP has a strictly complementary solution. HLCP is called *nondegenerate* if it has a strictly complementary solution. Otherwise, it is called *degenerate*.
We introduce the following parameter

$$\sigma := \begin{cases} 0, & \text{if the HLCP is nondegenerate} \\ 1, & \text{if the HLCP is degenerate.} \end{cases}$$

to present the result of Stoer and Wechs [25] which will be used in the analysis of our algorithm.

THEOREM 2.2. *([25]) Let HLCP be sufficient and let σ be defined as above. Assume that $\mathcal{F}^* \neq \emptyset$ and that there exists z such that $\lceil z,\, \tilde{\tau} \rfloor \in \mathcal{F}_{\bar{b}}$ for some $\tilde{\tau} > 0$ and $\bar{b}$ in $\mathbb{R}^n$. Then the system*

$$\begin{aligned} x(t,p)s(t,p) &= tp, \\ Qx(t,p) + Rs(t,p) &= b - t\bar{b} \end{aligned} \tag{2.8}$$

has a unique positive solution $z(t,p) = \lceil x(t,p),\, s(t,p) \rfloor$ for any $t \in (0,\, \tilde{\tau}]$ and any $p \in \mathbb{R}^n_{++}$. Moreover, the function $\tilde{z}(\rho,p) := z(\rho^{1+\sigma},p)$ is an analytic function in $\rho = t^{1/(1+\sigma)}$ and p, that can be extended analytically to an open neighborhood of $[0,\tilde{\rho}] \times \mathbb{R}^n_{++}$, where $\tilde{\rho} = \tilde{\tau}^{1/(1+\sigma)}$. For any compact set $\mathcal{K} \subset I\!R^n_{++}$ and any integer $i \in I\!N$ there are constants $\bar{c}(\mathcal{K}, i)$ such that

$$\left\| \frac{\partial^i \tilde{z}(\rho,p)}{\partial \rho^i} \right\|_2 \leq \bar{c}(\mathcal{K}, i),\ \forall \rho \in [0,\tilde{\rho}], \forall p \in \mathcal{K},\ i = 0,1,2,\ldots \quad .$$

Note that due to Theorem 2.1, the assumption $\text{rank}[Q\ R] = n$ originally imposed in [25], is now omitted.

3. Infeasible First Order Predictor-Corrector Algorithm. We are interested in algorithms for solving HLCP by following approximately the *infeasible central path pinned on* $\bar{b}$ denoted by $\mathcal{C}_{\bar{b}}$ defined as the set of vectors $\lceil x,\, s,\, \tau \rfloor$ satisfying

$$\begin{aligned} xs &= \tau e, \\ Qx + Rs &= b - \tau\bar{b}, \end{aligned} \tag{3.1}$$

or equivalently

$$F_\tau(z) := \begin{bmatrix} xs - \tau e \\ Qx + Rs - b + \tau\bar{b} \end{bmatrix} = 0. \tag{3.2}$$

where $\overline{b} := -r^0/\tau_0$ is a constant vector with

$$r^0 = Qx^0 + Rs^0 - b \tag{3.3}$$

being the initial residual at the starting point $\lceil z^0, \tau_0 \rfloor = \lceil x^0, s^0, \tau_0 \rfloor \in I\!R^{2n+1}_{++}$. It follows from the definition of $\overline{b}$ that $\lceil z^0, \tau_0 \rfloor$ satisfies the second equation in (3.1). Furthermore, for arbitrary $s^0 > 0$ and $\tau_0 > 0$, the first equation in (3.1) is also satisfied by picking $x^0 = \tau_0/s^0$. This means that the starting point chosen in this way belongs to the infeasible central path pinned on $\overline{b}$.

The iterations of the infeasible interior-point method in this paper follow $\mathcal{C}_{\overline{b}}$ by generating points in

$$\hat{\mathcal{N}}^-_\infty(\alpha) := \{\lceil z, \tau \rfloor = \lceil x, s, \tau \rfloor \in I\!R^{2n+1}_{++} \; : \; \hat{\delta}^-_\infty(z,\tau) \le \alpha \}, \tag{3.4}$$

where $0 < \alpha < 1$ is a given parameter and

$$\hat{\delta}^-_\infty(z,\tau) := \max\{\left\| \left[\frac{xs}{\tau} - e\right]^- \right\|_\infty , \left\| \left[\frac{\tau}{xs} - e\right]^- \right\|_\infty \}$$

is a proximity measure of z to the central path. Alternatively, if we denote

$$\mathcal{D}_s(\beta) = \{\lceil z, \tau \rfloor = \lceil x, s, \tau \rfloor \in I\!R^{2n+1}_{++} : \beta\tau \le xs \le \frac{\tau}{\beta}\},$$

then the neighborhood $\hat{\mathcal{N}}^-_\infty(\beta)$ can also be written as

$$\hat{\mathcal{N}}^-_\infty(\beta) = \mathcal{D}_s(1-\beta).$$

Also, related to the infeasibility of z, we define the residual at z by

$$r := Qx + Rs - b. \tag{3.5}$$

In the predictor step we are given a point $\lceil z, \tau \rfloor = \lceil x, s, \tau \rfloor \in \mathcal{D}_s(\beta)$, where β is a given parameter in the interval $(1/2, 1)$, and we compute the affine scaling direction at z:

$$w = \lceil u, v \rfloor = -F_0'(z)^{-1}F_0(z)\,. \tag{3.6}$$

We want to move along that direction as far as possible while preserving the condition $\lceil z(\theta), \tau(\theta) \rfloor \in \mathcal{D}_s\left((1-\gamma)\beta\right)$. The predictor step length is defined as

$$\overline{\theta} = \sup\left\{0 < \tilde{\theta} < 1 \; : \; \lceil z(\theta), \tau(\theta) \rfloor \in \mathcal{D}_s((1-\gamma)\beta)\;, \forall\, \theta \in [0, \tilde{\theta}]\right\}, \tag{3.7}$$

where

$$z(\theta) = z + \theta w\;,\; \tau(\theta) = (1-\theta)\tau\,,$$

and

$$\gamma = \frac{(1-\beta)^2}{32(1+\kappa)n}\,. \tag{3.8}$$

When we will analyse the corrector step we will see why we chose this particular value for γ.
The output of the predictor step is the point

$$\lceil \overline{x}, \overline{s}, \overline{\tau} \rfloor = \lceil \overline{z}, \overline{\tau} \rfloor = \lceil z(\overline{\theta}), \tau(\overline{\theta}) \rfloor \in \mathcal{D}_s((1-\gamma)\beta). \tag{3.9}$$

Here we also define $\tau_0 := \mu_0 := \mu(z_0)$ and $\overline{\tau} := \tau(\overline{\theta})$.
If $\lceil \overline{z}, \overline{\tau} \rfloor$ is not in $\mathcal{D}_s(\beta)$ then we perform a corrector step.
In the corrector step we are given a point $\lceil \overline{z}, \overline{\tau} \rfloor \in \mathcal{D}_s((1-\gamma)\beta)$ and we compute the Newton direction of $F_{\overline{\tau}}$ at $\overline{z}$:

$$\overline{w} = \lceil \overline{u}\,, \overline{v}\, \rfloor = -F'_{\overline{\tau}}(\overline{z}\,)^{-1} F_{\overline{\tau}}(\overline{z}\,)\,, \tag{3.10}$$

which is also known as the centering direction at $\overline{z}$. We denote

$$\begin{array}{rcl} \overline{x}\,(\theta) & = & \overline{x}\, + \theta\overline{u}\,,\ \overline{s}\,(\theta) = \overline{s}\, + \theta\overline{v}\,,\ \overline{z}\,(\theta) = \lceil \overline{x}\,(\theta),\, \overline{s}\,(\theta) \rfloor, \\ \overline{\mu} & = & \mu(\overline{z}\,),\ \overline{\mu}\,(\theta) = \mu(\overline{z}\,(\theta)) \end{array} \tag{3.11}$$

and we determine the corrector step length as

$$\theta_+ = \operatorname{argmin}\, \{\overline{\mu}(\theta) : \lceil \overline{z}(\theta),\, \overline{\tau} \rfloor \in \mathcal{D}_s(\beta)\}. \tag{3.12}$$

The output of the corrector step is the point

$$\lceil z^+,\, \tau^+ \rfloor = \lceil x^+,\, s^+,\, \tau^+ \rfloor = \lceil \overline{z}(\theta_+),\, \overline{\tau} \rfloor \in \mathcal{D}_s(\beta). \tag{3.13}$$

Since $\lceil z^+,\, \tau^+ \rfloor \in \mathcal{D}_s(\beta)$ we can set $z \leftarrow z^+$ and start another predictor-corrector iteration. This leads to the following algorithm.

Algorithm 1

Given $\kappa \geq \chi(Q, R)$, $\beta \in (1/2,\, 1)$ and a starting point $\lceil z^0,\, \mu(z^0) \rfloor \in \mathcal{D}_s(\beta)$:
Compute γ from (3.8) ;
Set $\mu_0 \leftarrow \mu(z^0)$, $\tau_0 \leftarrow \mu_0$, $k \leftarrow 0$;
repeat
(predictor step)
Set $z \leftarrow z^k$;
r_1. Compute predictor direction (3.6);
r_2. Compute predictor steplength (3.7);
r_3. Compute $\lceil \overline{z}\,, \overline{\tau} \rfloor$ from (3.9);
If $\mu(\overline{z}\,) = 0$ then STOP: $\overline{z}$ is an optimal solution;
If $\lceil \overline{z}\,, \overline{\tau} \rfloor \in \mathcal{D}_s(\beta)$, then set $z^{k+1} \leftarrow \overline{z}$, $\mu_{k+1} \leftarrow \mu(\overline{z}\,)$, $\tau_{k+1} \leftarrow \overline{\tau}$,
$k \leftarrow k+1$ and RETURN;
(corrector step)
r_4. Compute corrector direction (3.10);
r_5. Compute corrector steplength (3.12);
r_6. Compute $\lceil z^+,\, \tau^+ \rfloor$ from (3.13);
Set $z^{k+1} \leftarrow z^+$, $\mu_{k+1} \leftarrow \mu(z^+)$, $\tau_{k+1} \leftarrow \tau^+$, $k \leftarrow k+1$, and RETURN;
until some stopping criterion is satisfied.

A standard stopping criterion is

$$x^{k\,T} s^k \leq \epsilon\,. \tag{3.14}$$

We will see that if the problem has a solution, then for any $\epsilon > 0$ the algorithm stops in a finite number (say K_ϵ) of iterations. If $\epsilon = 0$ then the problem is likely to generate an infinite sequence. However it may happen that at a certain iteration (let us say at iteration K_0) an exact solution is obtained, and therefore the algorithm terminates at iteration K_0. If this (unlikely) phenomenon does not happen we set $K_0 = \infty$.

In step r_1 of Algorithm 1, the affine scaling direction $w = \lceil u, v \rfloor$ can be computed as the solution of the following linear system:

$$\begin{array}{rcl} su + xv & = & -xs \\ Qu + Rv & = & -r \end{array} \tag{3.15}$$

where

$$r = Qx + Rs - b\,. \tag{3.16}$$

In step r_2, we find the largest θ that satisfies

$$(1-\gamma)\beta\tau(\theta) \le x(\theta)s(\theta) \le \frac{\tau(\theta)}{(1-\gamma)\beta} \tag{3.17}$$

where

$$x(\theta) = x + \theta u,\ s(\theta) = s + \theta v,$$

$$\mu = \mu(z),\ \mu(\theta) = \mu(z(\theta)) = x(\theta)^T s(\theta)/n,\ \tau(\theta) = (1-\theta)\tau,\ \tau_0 = \mu_0.$$

According to (3.15) we have

$$x(\theta)s(\theta) = (1-\theta)xs + \theta^2 uv\,, \quad \mu(\theta) = (1-\theta)\mu + \theta^2 u^T v/n\,. \tag{3.18}$$

Let us clarify the meaning of the first inequality from (3.17).

$$\begin{aligned} x(\theta)s(\theta) \ge (1-\gamma)\beta\tau(\theta) \quad &\Leftrightarrow \quad (1-\theta)xs + \theta^2 uv \ge (1-\gamma)\beta(1-\theta)\tau e \\ &\Leftrightarrow \quad \theta^2 uv + (1-\theta)[xs - \beta\tau e + \gamma\beta\tau e] \ge 0\,. \end{aligned} \tag{3.19}$$

Since $\lceil z, \tau \rfloor \in \mathcal{D}_s(\beta)$ it follows that $xs \ge \beta\tau e$, so (3.19) is true if

$$\theta^2 uv + (1-\theta)\gamma\beta\tau e \ge 0 \Leftrightarrow \theta^2 \frac{uv}{\gamma\beta\tau} + (1-\theta)e \ge 0.$$

Now, since

$$\theta^2 \frac{uv}{\gamma\beta\tau} + (1-\theta)e \ge -\theta^2 \|\frac{uv}{\gamma\beta\tau}\| - \theta + 1$$

we observe that (3.19) is true if

$$-\theta^2 \|\frac{uv}{\gamma\beta\tau}\| - \theta + 1 \ge 0 \Leftrightarrow h(\theta) := \theta^2 \|\frac{uv}{\gamma\beta\tau}\| + \theta - 1 \le 0. \tag{3.20}$$

Since $h(0) = -1 \le 0$ and $a_1 := \frac{\|uv\|}{\gamma\beta\tau} \ge 0$ we see that $h(\theta) \le 0,$ for every $\theta \in [0, \overline{\theta}^1]$ where

$$\overline{\theta}^1 = \frac{-1 + \sqrt{1 + 4a_1}}{2a_1} = \frac{2}{1 + \sqrt{1 + 4a_1}}\,. \tag{3.21}$$

So, (3.19) is true for every $\theta \in [0, \overline{\theta}^1]$.
Let us see the meaning of the second inequality from (3.17)

$$\begin{aligned} x(\theta)s(\theta) \leq \tfrac{\tau(\theta)}{(1-\gamma)\beta} \quad &\Leftrightarrow \quad (1-\theta)xs + \theta^2 uv \leq \tfrac{(1-\theta)\tau}{(1-\gamma)\beta} e \\ &\Leftrightarrow \quad \theta^2 uv + (1-\theta)[xs - \tfrac{\tau}{(1-\gamma)\beta} e] \leq 0 \\ &\Leftrightarrow \quad \theta^2 uv + (1-\theta)[xs - \tfrac{\tau}{\beta} e + \tfrac{\tau}{\beta} e - \tfrac{\tau}{(1-\gamma)\beta} e] \leq 0\,. \end{aligned} \tag{3.22}$$

Since $\lceil z, \tau \rfloor \in \mathcal{D}_s(\beta)$ it follows that (3.22) is true if

$$\begin{aligned} \theta^2 uv + (1-\theta)[\tfrac{\tau}{\beta} e - \tfrac{\tau}{(1-\gamma)\beta} e] &\leq 0 \Leftrightarrow \\ \theta^2 uv - (1-\theta)\tfrac{\gamma}{(1-\gamma)\beta} \tau e &\leq 0 \Leftrightarrow \\ \theta^2 \tfrac{uv(1-\gamma)\beta}{\gamma\tau} + (\theta - 1)e &\leq 0. \end{aligned}$$

Given that

$$\theta^2 \frac{uv(1-\gamma)\beta}{\gamma\tau} + (\theta-1)e \leq \theta^2 \frac{\|uv\|(1-\gamma)\beta}{\gamma\tau} + \theta - 1$$

we observe that (3.22) is true if

$$h(\theta) := \theta^2 \frac{\|uv\|(1-\gamma)\beta}{\gamma\tau} + \theta - 1 \leq 0. \tag{3.23}$$

Since $h(0) = -1 \leq 0$ and $a_2 := \frac{\|uv\|(1-\gamma)\beta}{\gamma\tau} \geq 0$ we have that $h(\theta) \leq 0$, for every $\theta \in [0, \overline{\theta}^2]$, where

$$\overline{\theta}^2 = \frac{-1 + \sqrt{1 + 4a_2}}{2a_2} = \frac{2}{1 + \sqrt{1 + 4a_2}}\,. \tag{3.24}$$

So, (3.19) is true for every $\theta \in [0, \overline{\theta}^2]$. In conclusion, if $\theta \in [0, \min\{\overline{\theta}^1; \overline{\theta}^2\}]$ then $\lceil z(\theta), \tau(\theta) \rfloor \in \mathcal{D}_s((1-\gamma)\beta)$.
Since $a_2 = a_1\beta^2(1-\gamma) \leq a_1$ it follows that $\min\{\overline{\theta}^1; \overline{\theta}^2\} = \overline{\theta}^1$. So if $\theta \in [0, \overline{\theta}^1]$ then $\lceil z(\theta), \tau(\theta) \rfloor \in \mathcal{D}_s((1-\gamma)\beta)$ and from the definition of $\overline{\theta}$ it follows that $\overline{\theta} \geq \overline{\theta}^1$.
In step r_4 of Algorithm 1, the centering direction can be computed as the solution of the following linear system

$$\begin{aligned} \overline{s}\,\overline{u} + \overline{x}\,\overline{v} &= \overline{\tau} e - \overline{x}\,\overline{s} \\ Q\overline{u} + R\overline{v} &= 0\,. \end{aligned} \tag{3.25}$$

From (3.25) it follows that

$$\overline{x}(\theta)\overline{s}(\theta) = (1-\theta)\overline{x}\,\overline{s} + \theta\overline{\tau} e + \theta^2 \overline{u}\,\overline{v}\,, \quad \overline{\mu}(\theta) = (1-\theta)\overline{\mu} + \theta\overline{\tau} + \theta^2 \overline{u}^T \overline{v}/n\,. \tag{3.26}$$

The corrector step is given by

$$\theta_+ = \operatorname{argmin}\{\overline{\mu}(\theta) : \lceil \overline{z}(\theta), \overline{\tau} \rfloor \in \mathcal{D}_s(\beta)\}. \tag{3.27}$$

The output of the corrector is the point

$$\lceil z^+,\, \tau^+ \rfloor = \lceil x^+,\, s^+,\, \tau^+ \rfloor = \lceil \overline{z}(\theta_+),\, \overline{\tau} \rfloor \in \mathcal{D}_s(\beta). \tag{3.28}$$

Now we will study under what conditions over γ the corrector step makes sense. We would like to find for what values of γ there exists θ_+. For this to be true there should exist a θ such that

$$\begin{aligned} \lceil \overline{z}(\theta),\, \overline{\tau} \rfloor \in \mathcal{D}_s(\beta) \quad &\Leftrightarrow \quad \beta\overline{\tau} \le \overline{x}(\theta)\overline{s}(\theta) \le \tfrac{\overline{\tau}}{\beta} \\ &\Leftrightarrow \quad \beta \le f(\theta) := (1-\theta)\tfrac{\overline{xs}}{\overline{\tau}} + \theta e + \theta^2 \tfrac{\overline{uv}}{\overline{\tau}} \le \tfrac{1}{\beta}. \end{aligned} \tag{3.29}$$

Since $\lceil \overline{x},\, \overline{s},\, \overline{\tau} \rfloor \in \mathcal{D}_s((1-\gamma)\beta)$ we have that

$$f(\theta) \le (1-\theta)\frac{\overline{xs}}{\overline{\tau}} + \theta + \theta^2 \frac{\|\overline{uv}\|}{\overline{\tau}} \le (1-\theta)\frac{1}{(1-\gamma)\beta} + \theta + \theta^2 \frac{\|\overline{uv}\|}{\overline{\tau}}.$$

In addition $\lceil \overline{u},\, \overline{v} \rfloor$ is the solution of (3.25) and from Lemma 3.3 we have that

$$\begin{aligned} \|\overline{uv}\| \;&\le\; \left(\tfrac{1}{\sqrt{8}} + \kappa\right) \|\tilde{a}\|^2 = \left(\tfrac{1}{\sqrt{8}} + \kappa\right) \|\overline{\tau}(\overline{xs})^{-1/2} - (\overline{xs})^{1/2}\|^2 \\ &\le\; \left(\tfrac{1}{\sqrt{8}} + \kappa\right) \left(\overline{\tau}\|(\overline{xs})^{-1/2}\| + \|(\overline{xs})^{1/2}\|\right)^2 \\ &\le\; \left(\tfrac{1}{\sqrt{8}} + \kappa\right) \left(\overline{\tau}\sqrt{\tfrac{n}{(1-\gamma)\beta\overline{\tau}}} + \sqrt{\tfrac{n\overline{\tau}}{(1-\gamma)\beta}}\right)^2 \\ &\le\; \left(\tfrac{1}{\sqrt{8}} + \kappa\right) \tfrac{4n}{(1-\gamma)\beta}\overline{\tau}. \end{aligned} \tag{3.30}$$

Now we note that

$$f(\theta) \le \frac{1-\theta}{(1-\gamma)\beta} + \theta + \theta^2 \left(\frac{1}{\sqrt{8}} + \kappa\right) \frac{4n}{(1-\gamma)\beta} := f_1(\theta).$$

We would like to find out for what values of γ we have that $f_1(\theta) \le \frac{1}{\beta}$.
If we denote $a := \left(\frac{1}{\sqrt{8}} + \kappa\right) 4n$ and $b := \frac{1}{1-\gamma}$ we have

$$f_1(\theta) \le \frac{1}{\beta} \Leftrightarrow g(\theta) := \theta^2 ab + \theta(\beta - b) + b - 1 \le 0.$$

Since $ab > 0$ and $b - 1 \ge 0$ the above relation holds if

$$\Delta := (\beta - b)^2 - 4ab(b-1) \ge 0 \Leftrightarrow h(b) := b^2(1-4a) + 2b(2a-\beta) + \beta^2 \ge 0.$$

Since $1 - 4a \le 0$ and $h(0) = \beta^2 \ge 0$, the above relation is satisfied if

$$0 \le b \le \frac{2a - \beta + \sqrt{(2a-\beta)^2 + \beta^2(4a-1)}}{4a-1} := w.$$

Since $b = \frac{1}{1-\gamma}$, if we denote $\Delta := \sqrt{4a^2 - 4a\beta(1-\beta)}$, the above inequality is true if

$$\gamma \le 1 - \frac{1}{w} = \frac{1 - \beta + \Delta - 2a}{2a - \beta + \Delta} = (1-\beta)\frac{1 - \frac{4a\beta}{2a+\Delta}}{2a - \beta + \Delta}. \tag{3.31}$$

We have

$$\begin{aligned}(1-\beta)\frac{1-\frac{4a\beta}{2a+\Delta}}{2a-\beta+\Delta} &\geq \frac{1-\beta}{4a-\beta}\left(1-\frac{4a\beta}{2a+\Delta}\right)=\frac{1-\beta}{4a-\beta}\frac{\Delta-(4a\beta-2a)}{2a+\Delta}\\ &= \frac{1-\beta}{4a-\beta}\frac{\Delta^2-(4a\beta-2a)^2}{(2a+\Delta)(4a\beta-2a+\Delta)}=\frac{1-\beta}{4a-\beta}\frac{4a\beta(4a-4a\beta+\beta-1)}{(2a+\Delta)(4a\beta-2a+\Delta)} \qquad (3.32)\\ &\geq \frac{1-\beta}{4a-\beta}\frac{\beta(4a-4a\beta+\beta-1)}{4a\beta-2a+\Delta}\geq\frac{(1-\beta)^2}{2(4a-\beta)}.\end{aligned}$$

Therefore, if

$$\gamma \leq \frac{(1-\beta)^2}{2(4a-\beta)}=\frac{(1-\beta)^2}{2}\frac{1}{16n\left(\frac{1}{\sqrt{8}}+\kappa\right)-\beta} \tag{3.33}$$

the second inequality from (3.29) is satisfied.
Now, let us see under what conditions on γ will the first inequality from (3.29) be satisfied.
Since $\lceil\, \overline{x},\, \overline{s},\, \overline{\tau}\, \rfloor \in \mathcal{D}_s((1-\gamma)\beta)$ we have that

$$f(\theta)\geq(1-\theta)\beta(1-\gamma)+\theta-\theta^2\frac{\|\overline{uv}\|_2}{\overline{\tau}}\geq(1-\theta)\beta(1-\gamma)+\theta-\theta^2\frac{4n\left(\frac{1}{\sqrt{8}}+\kappa\right)}{(1-\gamma)\beta}:=f_2(\theta).$$

We would like to find out for what values of γ is $f_2(\theta)\geq\beta$.
If we denote $a:=\left(\frac{1}{\sqrt{8}}+\kappa\right)4n$ and $b:=1-\gamma$ we have that

$$f_2(\theta)\geq\beta \Leftrightarrow g_2(\theta):=\theta^2a+\theta b\beta(b\beta-1)-\beta^2b(b-1)\leq 0.$$

Since $a>0$ the above relation holds if

$$\Delta:=b^2\beta^2(b\beta-1)^2+4a\beta^2b(b-1)\geq 0\Leftrightarrow[\beta^2b^3-2\beta b^2]+[b(1+4a)-4a]\geq 0. \tag{3.34}$$

Since $\frac{4a+2\beta-\beta^2}{4a+1}=1-\frac{(1-\beta)^2}{4a+1}\leq 1-\frac{(1-\beta)^2}{2(4a-\beta)}=b<1$, we have

$$[b(1+4a)-4a]\geq 2\beta-\beta^2.$$

Now we see that (3.34) will be true if

$$\beta^2b^3-2\beta b^2+2\beta-\beta^2=(1-b)(-\beta^2b^2+\beta(2-\beta)b+\beta(2-\beta))\geq 0.$$

Since $b=1-\gamma\leq 1$, this is true for those b which satisfy

$$h(b):=-\beta^2b^2+\beta(2-\beta)b+\beta(2-\beta)\geq 0. \tag{3.35}$$

Since $1/2<\beta<1$ we have that $h(0)\geq 0$ and also $h(1)\geq 0$. From here we see that $h(b)\geq 0$, for all b such that $\frac{4a+2\beta-\beta^2}{4a+1}\leq b<1$.
So, if

$$b=1-\gamma\geq\frac{4a+2\beta-\beta^2}{4a+1}\Leftrightarrow\gamma\leq\frac{(1-\beta)^2}{4a+1}=\frac{(1-\beta)^2}{16n\left(\frac{1}{\sqrt{8}}+\kappa\right)+1}, \tag{3.36}$$

the second inequality from (3.29) is satisfied.
In conclusion, from (3.33) and (3.36) we see that inequality (3.29) is satisfied for

$$0<\gamma\leq\frac{(1-\beta)^2}{32(1+\kappa)n}. \tag{3.37}$$

The above inequality motivates the definition (3.8) of γ.

3.1. Technical Results. We start this section by stating several basic results. Most of them have been known in one form or another in the interior point literature. We will be using the formulations from [21].

LEMMA 3.1. *Assume that HLCP (2.1) is $P_*(\kappa)$, and let $w = \lceil u, v \rfloor$ be the solution of the following linear system*

$$\begin{array}{rcl} su + xv & = & a \\ Qu + Rv & = & 0\,, \end{array}$$

where $z = \lceil x, s \rfloor \in \mathbb{R}^{2n}_{++}$ and $a \in \mathbb{R}^n$ are given vectors, and consider the index sets:

$$\mathcal{I}^+ = \{i : u_i v_i > 0\}, \quad \mathcal{I}^- = \{i : u_i v_i < 0\}.$$

Then the following inequalities are satisfied:

$$\frac{1}{1+4\kappa} \| u\, v \|_\infty \le \sum_{i \in \mathcal{I}_+} u_i v_i \le \frac{1}{4} \left\| (xs)^{-1/2}\, a \right\|_2^2 .$$

LEMMA 3.2. *Assume that HLCP (2.1) is $P_*(\kappa)$, and let $w = \lceil u, v \rfloor$ be the solution of the following linear system*

$$\begin{array}{rcl} su + xv & = & a \\ Qu + Rv & = & 0\,, \end{array}$$

where $z = \lceil x, s \rfloor \in \mathbb{R}^{2n}_{++}$ and $a \in \mathbb{R}^n$ are given vectors. Then the following inequality holds:

$$u^T v \ge -\kappa \left\| (xs)^{-1/2}\, a \right\|_2^2 . \tag{3.38}$$

Next, we find bounds for the solution of a linear system of the form

$$\begin{aligned} su + xv &= a \\ Qu + Rv &= 0. \end{aligned} \tag{3.39}$$

We use the following notations:

$$D := X^{-1/2} S^{1/2}, \tag{3.40}$$

$$\| w \|_z^2 = \| \lceil u, v \rfloor \|_z^2 := \|Du\|^2 + \left\|D^{-1} v\right\|^2 , \tag{3.41}$$

$$\tilde{a} := (XS)^{-1/2} a. \tag{3.42}$$

LEMMA 3.3. *Let HLCP be sufficient, and $z = \lceil x, s \rfloor$ and a be vectors in $\mathbb{R}^{2n}_{++}$ and $\mathbb{R}^n$, respectively. The linear system* (3.39) *has a unique solution $w = \lceil u, v \rfloor$ satisfying*

$$\| w \|_z^2 \le (1 + 2\kappa)\ \|\tilde{a}\|^2 , \quad \|uv\| \le \left(\frac{1}{\sqrt{8}} + \kappa \right) \|\tilde{a}\|^2 .$$

Proof. The proof uses similar techniques as the ones from Lemma 3.1 in [19]. ∎

The following result is an extension of Lemma 3.3 to the system

$$\begin{aligned} su + xv &= a \\ Qu + Rv &= \tilde{b}\ . \end{aligned} \tag{3.43}$$

LEMMA 3.4. *Let HLCP be sufficient, and $z = \lceil x, s \rfloor$ and $\lceil a, \tilde{b} \rfloor$ be vectors in $I\!R^{2n}_{++}$ and $I\!R^{2n}$, respectively. The linear system* (3.43) *has a unique solution $w = \lceil u, v \rfloor$ that satisfies the following properties:*

$$\| w \|_z = \sqrt{1+2\kappa} \left(\|\tilde{a}\| + \frac{1+\sqrt{2+4\kappa}}{\sqrt{1+2\kappa}} \zeta(z,\tilde{b}) \right),$$

$$\|uv\| \leq \left(\frac{1}{\sqrt{8}} + \kappa \right) \left(\|\tilde{a}\| + 2\sqrt{2}\zeta(z,\tilde{b}) \right)^2,$$

where $\tilde{a}$ is defined in (3.42) *and*

$$\zeta(z,\tilde{b})^2 = \| \tilde{w} \|_z^2 := \min \{ \| \lceil u, v \rfloor \|_z^2 \, : \, Qu + Rv = \tilde{b} \}. \tag{3.44}$$

Proof. First, observe that from the definition of $\|.\|_z$ in (3.41) and Theorem 2.1 it follows that there is a unique solution $w = \lceil u, v \rfloor$ to (3.43). Then $\overline{w} := w - \tilde{w}$ satisfies

$$\begin{aligned} s\overline{u} + x\overline{v} &= a - (s\tilde{u} + x\tilde{v}) \\ Q\overline{u} + R\overline{v} &= 0\,, \end{aligned}$$

and from Lemma 3.3 it follows that

$$\begin{aligned} \| w \|_z &\leq \| \overline{w} \|_z + \| \tilde{w} \|_z \leq \sqrt{1+2\kappa}\, \left\| \tilde{a} - (D\tilde{u} + D^{-1}\tilde{v}) \right\| + \| \tilde{w} \|_z \\ &\leq \sqrt{1+2\kappa}\, \left(\|\tilde{a}\| + \left\| (D\tilde{u} + D^{-1}\tilde{v}) \right\| \right) + \| \tilde{w} \|_z \\ &\leq \sqrt{1+2\kappa} \left(\|\tilde{a}\| + \sqrt{\|D\tilde{u}\|^2 + \|D^{-1}\tilde{v}\|^2 + 2\,\|D\tilde{u}\|\,\|D^{-1}\tilde{v}\|} \right) + \| \tilde{w} \|_z \\ &\leq \sqrt{1+2\kappa} \left(\|\tilde{a}\| + \sqrt{2}\, \| \tilde{w} \|_z \right) + \| \tilde{w} \|_z \\ &= \sqrt{1+2\kappa} \left(\|\tilde{a}\| + \frac{1+\sqrt{2+4\kappa}}{\sqrt{1+2\kappa}} \| \tilde{w} \|_z \right). \end{aligned}$$

This proves the first inequality in the lemma. Now, we define $\overline{a} := \tilde{a} - (D\tilde{u} + D^{-1}\tilde{v})$. In a similar way as above, we get $\|\overline{a}\| \leq \|\tilde{a}\| + \left\| D\tilde{u} + D^{-1}\tilde{v} \right\| \leq \|\tilde{a}\| + \sqrt{2}\, \| \tilde{w} \|_z$. Using the previous bound for $\| w \|_z$ and Lemma 3.3 we obtain

$$\begin{aligned} \|uv\| &= \|(\overline{u} + \tilde{u})(\overline{v} + \tilde{v})\| \\ &\leq \|\overline{uv}\| + \|D\,\overline{u}\|\,\left\|D^{-1}\tilde{v}\right\| + \|D\tilde{u}\|\,\left\|D^{-1}\overline{v}\right\| + \|D\tilde{u}\|\,\left\|D^{-1}\tilde{v}\right\| \\ &\leq \|\overline{uv}\| + \| \overline{w} \|_z \| \tilde{w} \|_z + \frac{1}{2} \| \tilde{w} \|_z^2 \\ &\leq \frac{1+\sqrt{8}\kappa}{\sqrt{8}} \|\overline{a}\|^2 + \sqrt{1+2\kappa}\, \|\overline{a}\|\, \| \tilde{w} \|_z + \frac{1}{2} \| \tilde{w} \|_z^2 \\ &\leq \frac{1+\sqrt{8}\kappa}{\sqrt{8}} \left(\|\overline{a}\| + \frac{\sqrt{2+4\kappa}}{1+\sqrt{8}\kappa} \| \tilde{w} \|_z \right)^2 \\ &\leq \frac{1+\sqrt{8}\kappa}{\sqrt{8}} \left(\|\tilde{a}\| + \left(\sqrt{2} + \frac{\sqrt{2+4\kappa}}{1+\sqrt{8}\kappa} \right) \| \tilde{w} \|_z \right)^2 \\ &\leq \frac{1+\sqrt{8}\kappa}{\sqrt{8}} \left(\|\tilde{a}\| + 2\sqrt{2}\, \| \tilde{w} \|_z \right)^2. \end{aligned}$$

∎

The next lemma bounds the quantity $\zeta(z,\tilde{b})$ that appears in Lemma 3.4.

LEMMA 3.5. *Let HLCP be sufficient, and z, z^0 be vectors in $I\!R^{2n}_{++}$. The following inequality holds:*

$$\zeta(z,\tilde{b}) \le \|(xsx^0s^0)^{-1/2}\|_\infty (x^Ts^0 + s^Tx^0)\zeta(z^0,\tilde{b}),$$

where ζ is defined in (3.44).

LEMMA 3.6. *Let HLCP be sufficient and assume that $\mathcal{F}^*$ in* (2.7) *is nonempty. For all $z^* = \lceil x^*, s^* \rfloor \in \mathcal{F}^*$ and $\lceil z, \tau \rfloor = \lceil x, s, \tau \rfloor \in \mathcal{F}_{\overline{b}}$ with $0 < \tau < \tau_0$, we have*

$$x^Ts^0 + s^Tx^0 \le (1+4\kappa)\left(max\,\{1,\zeta^*\} + \frac{\mu(z)}{\tau}\right) n\tau_0, \tag{3.45}$$

where $\zeta^* := ((x^0)^Ts^* + (s^0)^Tx^*)/((x^0)^Ts^0)$ and $\lceil z^0, \tau_0 \rfloor = \lceil x^0, s^0, \tau_0 \rfloor$ is a starting point in $I\!R^{2n}_{++}$. Moreover, if the starting point is large enough in the sense that

$$z^0 \ge z^* \text{ for some } z^* \in \mathcal{F}^*, \tag{3.46}$$

then

$$\left(x^Ts^0 + s^Tx^0\right) \le (1+4\kappa)\left(2 + \frac{\mu(z)}{\tau}\right) n\tau_0. \tag{3.47}$$

LEMMA 3.7. *Let HLCP be sufficient and assume $\mathcal{F}^*$ in* (2.7) *is nonempty, $\beta \in (1/2\,,1)$ and $c \ne 0$. Let $\lceil z, \tau \rfloor \in \mathcal{F}_{\overline{b}}$ (with $0 < \tau < \tau_0$) belonging to $\mathcal{D}_s(\beta)$.*

(i) If a starting point $\lceil z^0, \tau_0 \rfloor \in \mathcal{F}_{\overline{b}}$ satisfies (3.46), *then*

$$\zeta(z, c\,\tau\,\overline{b}) \le |c|\xi_b(1+4\kappa)\sqrt{\tau}n, \tag{3.48}$$

where $\xi_b := \frac{2}{\sqrt{\beta}}\left(2 + \frac{1}{\beta}\right)$.

(ii) If $\lceil z^0, \tau_0 \rfloor \in \mathcal{F}_{\overline{b}}$ satisfies

$$\zeta(z^0, r^0) \le \frac{\beta\sqrt{\tau_0}}{\sqrt{n}(1+4\kappa)(\max\{1,\zeta^*\} + \frac{1}{\beta})}, \tag{3.49}$$

then

$$\zeta(z, c\,\tau\,\overline{b}) \le |c|\sqrt{\tau}\sqrt{n}. \tag{3.50}$$

Proof. To prove (i), note that $\lceil z^0, \tau_0 \rfloor \in \mathcal{F}_{\overline{b}}$, $\overline{b} = -r^0/\tau_0$, and $Qx^* + Rs^* = b$ yield

$$Q\frac{c\,\tau}{\tau_0}(x^* - x^0) + R\frac{c\,\tau}{\tau_0}(s^* - s^0) = c\,\tau\,\overline{b}.$$

It follows from the definition of ζ and the assumption $z^0 \ge z^*$ that

$$\begin{aligned}
\zeta(z,\ c\tau\overline{b})^2 &\le \|\sqrt{\frac{s}{x}}\frac{c\,\tau}{\tau_0}(x^0 - x^*)\|^2 + \|\sqrt{\frac{x}{s}}\frac{c\,\tau}{\tau_0}(s^0 - s^*)\|^2 \\
&\le 4\frac{c^2\,\tau^2}{\tau_0^2}\left(\sum_i \frac{s_i}{x_i}(x_i^0)^2 + \sum_i \frac{x_i}{s_i}(s_i^0)^2\right) \\
&= 4\frac{c^2\,\tau^2}{\tau_0^2}\left(\sum_i \frac{1}{x_is_i}(s_ix_i^0)^2 + \sum_i \frac{1}{x_is_i}(x_is_i^0)^2\right) \\
&\le 4\frac{c^2\,\tau^2}{\tau_0^2}\|(xs)^{-1}\|_\infty(\|sx^0\|^2 + \|xs^0\|^2).
\end{aligned}$$

Consequently,

$$\zeta(z,\ c\tau\bar{b}) \le 2\frac{|c|\tau}{\tau_0}\|(xs)^{-1/2}\|_\infty\|\lceil sx^0,\ xs^0 \rfloor\| \le 2\frac{|c|\,\tau}{\tau_0}\|(xs)^{-1/2}\|_\infty\|(sx^0+xs^0)\|_1.$$

Combining this result with the estimate in (3.47) and using the fact that $\lceil x,\ s,\ \tau \rfloor \in \mathcal{D}_s(\beta)$ we obtain

$$\begin{aligned}\zeta(z, c\,\tau\,\bar{b}) &\le 2\left(2+\frac{\mu(z)}{\tau}\right)|c|(1+4\kappa)\|(xs)^{-1/2}\|_\infty n\tau \\ &\le \frac{2\left(2+\frac{1}{\beta}\right)|c|(1+4\kappa)}{\sqrt{\beta}}\,\frac{n\tau}{\sqrt{\tau}}.\end{aligned}$$

Inequality (3.48) now follows.

To prove (ii), apply Lemma 3.5 with $\tilde{b} = c\,\tau\,\bar{b}$, the fact that $\zeta(z^0, c\tilde{b}) = |c|\zeta(z^0, \tilde{b})$, and Lemma 3.6 consecutively to get

$$\begin{aligned}\zeta(z, c\,\tau\,\bar{b}) &\le |c|\frac{\tau}{\mu_0}\|(x\,s\,x^0 s^0)^{-1/2}\|_\infty(x^T s^0 + s^T x^0)\zeta(z^0, r^0) \\ &\le |c|\tau\,\|(x\,s\,x^0 s^0)^{-1/2}\|_\infty(1+4\kappa)\left(\max\{1,\zeta^*\}+\frac{\mu(z)}{\tau}\right)n\,\zeta(z^0,\tau^0).\end{aligned}$$

Since $\lceil x,\ s,\ \tau \rfloor$ and $\lceil x^0,\ s^0,\ \tau_0 \rfloor$ are in $\mathcal{D}_s(\beta)$ we have that

$$(xs)^{-1/2} \le \frac{1}{\sqrt{\beta\tau}}\ ,\quad (x_0 s_0)^{-1/2} \le \frac{1}{\sqrt{\beta\tau_0}}\quad\text{and}\quad \frac{\mu(z)}{\tau} \le \frac{1}{\beta}.$$

Using these facts we get that

$$\zeta(z, c\,\tau\,\bar{b}) \le \frac{n\tau}{\sqrt{\tau}}\,\frac{|c|(1+4\kappa)\left(\max\{1,\zeta^*\}+\frac{1}{\beta}\right)}{\beta\sqrt{\tau_0}}\zeta(z^0, r^0).$$

Therefore, if $\lceil z^0,\ \tau_0 \rfloor$ satisfies (3.49), equation (3.50) follows. ∎

3.2. Global Convergence.

LEMMA 3.8. *If the HLCP is $P_*(\kappa)$, then the direction $w = \lceil u,\ v \rfloor$ and the steplength $\overline{\theta}$ generated by the predictor step of our algorithm satisfy*

$$\frac{\|uv\|}{\tau} \le \left(\frac{1}{\sqrt{8}}+\kappa\right)\left(\frac{\sqrt{n}}{\sqrt{\beta}}+2\sqrt{2}\xi_b(1+4\kappa)n\right)^2 := C_l\,, \tag{3.51}$$

$$\overline{\theta} \ge \frac{2}{1+\sqrt{1+\frac{4C_l}{\gamma\beta}}}\,, \tag{3.52}$$

if the starting point satisfies (3.46),

and

$$\frac{\|uv\|}{\tau} \le \left(\frac{1}{\sqrt{8}}+\kappa\right)\left(1+\frac{1}{\sqrt{\beta}}\right)^2 n := C_s\,, \tag{3.53}$$

$$\overline{\theta} \geq \frac{2}{1+\sqrt{1+\frac{4C_s}{\gamma\beta}}}\,, \tag{3.54}$$

if the starting point satisfies (3.49).

Proof. From the bound for $\|uv\|$ in Lemma 3.4, and since in the predictor we have that $\|\tilde{a}\| = \|(xs)^{1/2}\|$, and $\tilde{b} = -r = \tau\overline{b}$ we obtain

$$\begin{aligned}\|uv\| &\leq \left(\frac{1}{\sqrt{8}}+\kappa\right)\left(\left\|(xs)^{1/2}\right\| + 2\sqrt{2}\zeta(z,\tilde{b})\right)^2\\ &\leq \left(\frac{1}{\sqrt{8}}+\kappa\right)\left(\sqrt{n\mu} + 2\sqrt{2}\zeta(z,\tau\overline{b})\right)^2.\end{aligned}$$

Now, from Lemma (3.7), $\zeta(z,\tau\overline{b})$ is bounded from above and we obtain the following bounds for $\|uv\|$.
If the starting point satisfies (3.46) then we have

$$\begin{aligned}\|uv\| &\leq \left(\frac{1}{\sqrt{8}}+\kappa\right)\left(\sqrt{n\mu} + 2\sqrt{2}\xi_b(1+4\kappa)\sqrt{\tau}n\right)^2\\ &\leq \left(\frac{1}{\sqrt{8}}+\kappa\right)\left(\frac{\sqrt{n}}{\sqrt{\beta}} + 2\sqrt{2}\xi_b(1+4\kappa)n\right)^2\tau\,,\end{aligned} \tag{3.55}$$

and we obtain

$$\frac{\|uv\|}{\tau} \leq \left(\frac{1}{\sqrt{8}}+\kappa\right)\left(\frac{\sqrt{n}}{\sqrt{\beta}} + 2\sqrt{2}\xi_b(1+4\kappa)n\right)^2 := C_l.$$

Using the previous inequality in the definitions of $\overline{\theta}$ and $\overline{\theta}^1$ we get

$$\overline{\theta} \geq \frac{2}{1+\sqrt{1+4C_l\frac{1}{\gamma\beta}}}\,.$$

Exactly in the same way, if the starting point satisfies (3.49) we'll have that

$$\|uv\| \leq \left(\frac{1}{\sqrt{8}}+\kappa\right)\left(\sqrt{n\mu} + \sqrt{\tau}\sqrt{n}\right)^2\,, \tag{3.56}$$

and now

$$\frac{\|uv\|}{\tau} \leq \left(\frac{1}{\sqrt{8}}+\kappa\right)\left(1+\sqrt{\frac{\mu}{\tau}}\right)^2 n \leq \left(\frac{1}{\sqrt{8}}+\kappa\right)\left(1+\frac{1}{\sqrt{\beta}}\right)^2 n := C_s.$$

Using the previous inequality in the definitions of $\overline{\theta}$ and $\overline{\theta}^1$ we get

$$\overline{\theta} \geq \frac{2}{1+\sqrt{1+4C_s\frac{1}{\gamma\beta}}}\,.$$

∎

THEOREM 3.9. *Let HLCP be sufficient and solvable. The duality gaps and residuals of the iteration sequence generated by Algorithm 1 converge to zero, i.e.,*

$$\lim_{k\to\infty} \mu_k = 0, \ \lim_{k\to\infty} r^k = 0.$$

Proof. After the corrector step, from the definitions of θ_+ and τ it follows that

$$\tau_+ = \overline{\tau} = (1-\overline{\theta})\tau. \tag{3.57}$$

Hence

$$\mu(z_k) \le \frac{\tau_k}{\beta} = \frac{1}{\beta}\prod_{i=1}^{k}(1-\overline{\theta}_i)\tau_0. \tag{3.58}$$

Concerning r_k we observe that

$$r_+ = \overline{r} = Q\overline{x} + R\overline{s} - b = Qx + Rs - b + \overline{\theta}(Q\overline{u} + R\overline{v}) = r - \overline{\theta}r = r(1-\overline{\theta}).$$

Hence we have

$$r_k = \prod_{i=1}^{k}(1-\overline{\theta}_i)r_0. \tag{3.59}$$

If $\lceil x_0, s_0 \rfloor$ satisfies (3.46), from (3.52) we deduce

$$\mu(z_k) \le \frac{1}{\beta}\left(1 - \frac{2}{1+\sqrt{1+\frac{4C_l}{\gamma\beta}}}\right)^k \tau_0\,,$$

and

$$r_k \le \left(1 - \frac{2}{1+\sqrt{1+\frac{4C_l}{\gamma\beta}}}\right)^k r_0.$$

If $\lceil x_0, s_0 \rfloor$ satisfies (3.49), from (3.54) we obtain

$$\mu(z_k) \le \frac{1}{\beta}\left(1 - \frac{2}{1+\sqrt{1+\frac{4C_s}{\gamma\beta}}}\right)^k \tau_0\,,$$

and

$$r_k \le \left(1 - \frac{2}{1+\sqrt{1+\frac{4C_s}{\gamma\beta}}}\right)^k r_0.$$

In both cases we observe that

$$\lim_{k\to\infty} \mu_k = 0 \ \text{ and } \ \lim_{k\to\infty} r_k = 0.$$

∎

3.3. Polynomial complexity. In this section we analyze the computational complexity of our algorithm.

THEOREM 3.10. *Algorithm 1 is well defined and*

$$\begin{aligned} \mu_k &\leq \frac{1}{\beta}\left(1-\frac{1}{\frac{192\sqrt{2}}{\beta(1-\beta)}(1+4\kappa)^2n^{3/2}}\right)^k \mu_0, \\ r_k &\leq \left(1-\frac{1}{\frac{192\sqrt{2}}{\beta(1-\beta)}(1+4\kappa)^2n^{3/2}}\right)^k r_0, \quad k=1,2,\ldots \end{aligned} \tag{3.60}$$

if the starting point satisfies (3.46),
and

$$\begin{aligned} \mu_k &\leq \frac{1}{\beta}\left(1-\frac{1}{\frac{20}{\beta(1-\beta)}(1+4\kappa)n}\right)^k \mu_0, \\ r_k &\leq \left(1-\frac{1}{\frac{20}{\beta(1-\beta)}(1+4\kappa)n}\right)^k r_0, \quad k=1,2,\ldots \end{aligned} \tag{3.61}$$

if the starting point satisfies (3.49).

Proof. If the starting point satisfies (3.46), from Lemma 3.8 and the definition of γ it follows that

$$\overline{\theta}_k \geq \frac{2}{1+\sqrt{1+\frac{4C_l}{\gamma\beta}}} \geq \frac{2}{1+\frac{192\sqrt{2}}{\beta(1-\beta)}(1+4\kappa)^2n^{3/2}} \geq \frac{1}{\frac{192\sqrt{2}}{\beta(1-\beta)}(1+4\kappa)^2n^{3/2}}.$$

Therefore, (3.60) follows from (3.58) and (3.59).
If the starting point satisfies (3.49), using the definition of γ and Lemma 3.8 it follows that

$$\overline{\theta}_k \geq \frac{2}{1+\sqrt{1+\frac{4C_s}{\gamma\beta}}} \geq \frac{2}{1+\frac{20}{\beta(1-\beta)}(1+4\kappa)n} \geq \frac{1}{\frac{20}{\beta(1-\beta)}(1+4\kappa)n}.$$

Hence, (3.61) follows from (3.58) and (3.59).
The proof is complete. ∎

The next corollary is an immediate consequence of the above theorem.

COROLLARY 3.11. *Algorithm 1 produces a point* $\lceil z^k, \tau_k \rfloor \in \mathcal{D}_s(\beta)$ *with* $x^{k\,T} s^k \leq \varepsilon$ *in at most*

$$O\left((1+\kappa)^2 n^{3/2} \log\left(x^{0\,T} s^0/\varepsilon\right)\right)$$

iterations, if the starting point satisfies (3.46),
and

$$O\left((1+\kappa) n \log\left(x^{0\,T} s^0/\varepsilon\right)\right)$$

iterations, if the starting point satisfies (3.49).

Proof. We will prove only one of the results of this corollary, since the other one can be done in the same way. Let $L_\varepsilon := \log \frac{x^{0T}s^0}{\varepsilon}$.

If the starting point satisfies (3.49), from Theorem 3.10 it follows that

$$x^{k\,T} s^k = n\mu_k \le \frac{1}{\beta}\left(1 - \frac{1}{\frac{20}{\beta(1-\beta)}(1+4\kappa)n}\right)^k x^{0\,T} s^0.$$

Hence $x^{k\,T} s^k \le \varepsilon$ whenever

$$k \log\left(1 - \frac{1}{\frac{20}{\beta(1-\beta)}(1+4\kappa)n}\right) \le \log\left(\frac{\varepsilon\beta}{x^{0\,T} s^0}\right).$$

Since $\log(1-t) \le -t$ on $(0,1)$ the above inequality holds if

$$k \ge \frac{20(1+4\kappa)n}{\beta(1-\beta)} \log \frac{x^{0\,T} s^0}{\varepsilon\beta}.$$

The desired result follows noticing that

$$\frac{x^{0\,T} s^0}{\varepsilon\beta} = \frac{\exp L_\varepsilon}{\beta} \le \exp\left(L_\varepsilon + \frac{1}{\beta}\right).$$

Now, we easily see that $x^{k\,T} s^k \le \varepsilon,\ \forall\, k \ge K_\varepsilon$, where

$$K_\varepsilon := \frac{20\left(1 + \frac{1}{\beta L_\varepsilon}\right)}{\beta(1-\beta)}(1+4\kappa)nL_\varepsilon.$$

∎

3.4. Superlinear convergence. In this section we will prove that our algorithm is quadratically convergent for nondegenerate problems. The proof of the superlinear convergence is based on the following lemma which is a consequence of the result about the analyticity of the central path from [26].

LEMMA 3.12. *([26])If HLCP is sufficient and nondegenerate then there is a positive constant α such that the vectors u, v computed in the predictor step at each iteration of the algorithm, satisfy*

$$\|\, u \,\|_2 \le \sqrt{\alpha}\,\mu\,,\ \|\, v \,\|_2 \le \sqrt{\alpha}\,\mu\,.$$

With the help of the lemma above we obtain the following result.

THEOREM 3.13. *If HLCP is sufficient and nondegenerate then the sequences of the complementarity gaps (μ_k), feasibility measures (τ_k), and residuals $\left(r^k\right)$, $i = 1, \ldots, n$, produced by our algorithm, converge Q-quadratically to zero.*

Proof. Since after the predictor step we are in $\mathcal{D}_s(\beta)$, from the previous Lemma we have that

$$\|\, u \,\|_2 \le \sqrt{\alpha}\,\mu \le \frac{\sqrt{\alpha}}{\beta}\,\tau\,,\ \|\, v \,\|_2 \le \sqrt{\alpha}\,\mu \le \frac{\sqrt{\alpha}}{\beta}\,\tau\,. \tag{3.62}$$

From the discussion on the predictor step we have that

$$\begin{aligned} \overline{\theta} \;&\ge\; \tfrac{2}{1+\sqrt{1+4a_1}} = \tfrac{2}{1+\sqrt{1+4\frac{\|uv\|}{\gamma\beta\tau}}} \\ &\ge\; \tfrac{2}{1+\sqrt{1+4\frac{\alpha\tau}{\gamma\beta^3}}}\,. \end{aligned} \tag{3.63}$$

Hence,

$$1-\overline{\theta} \leq \frac{\sqrt{1+\frac{4\alpha\tau}{\gamma\beta^3}}-1}{1+\sqrt{1+4\frac{\alpha\tau}{\gamma\beta^3}}} = \frac{\frac{4\alpha\tau}{\gamma\beta^3}}{\left(1+\sqrt{1+4\frac{\alpha\tau}{\gamma\beta^3}}\right)^2} \leq \frac{\alpha}{\gamma\beta^3}\tau\,. \tag{3.64}$$

Since $\tau_{k+1} = (1-\overline{\theta}_k)\tau_k$, $\beta\tau_k \leq \mu_k \leq \frac{\tau_k}{\beta}$ and $r_{k+1} = (1-\overline{\theta}_k)r_k$ we have

$$\begin{aligned} \tau_{k+1} &= (1-\overline{\theta}_k)\tau_k \leq \frac{\alpha}{\gamma\beta^3}\tau_k{}^2\,, \\ r_{k+1} &= (1-\overline{\theta}_k)r_k \leq \frac{\alpha}{\gamma\beta^3}\tau_k r_k = \frac{\alpha\tau_0}{\gamma\beta^3 r_0}r_k{}^2\,, \\ \mu_{k+1} &\leq \frac{1}{\beta}\tau_{k+1} \leq \frac{\alpha}{\gamma\beta^4}\tau_k{}^2 \leq \frac{\alpha}{\gamma\beta^6}\mu_k{}^2\,. \end{aligned} \tag{3.65}$$

The proof is complete. ∎

4. Infeasible First Order Predictor-Corrector Algorithm, Independent of κ. Algorithm 1 depends on a given parameter $\kappa \geq \chi(Q,R)$ because of the choice of γ from (3.8). As we pointed out in the introduction, in many applications it is very expensive to compute the handicap $\chi(Q,R)$ or to find a good upper bound for it [28]. Therefore we try to modify our algorithm in order to make it independent of κ. The idea which was first used in [18] is very simple. We start the algorithm with $\kappa = 1$ i.e we follow the steps of Algorithm 1 for this value of κ. If at a certain iteration the corrector fails to produce a point in $\mathcal{D}_s(\beta)$, then we conclude that the current value of κ is too small. We double the value of κ and restart Algorithm 1 from the last point produced in $\mathcal{D}_s(\beta)$. Clearly we have to double the value of κ at most $\lceil \log_2 \chi(Q,R) \rceil$ times. This leads to the following algorithm.

Algorithm 2

Given $\beta \in (1/2,\, 1)$ and $\lceil\, z^0,\, \mu(z^0)\,\rfloor \in \mathcal{D}_s(\beta)$:

  Set $\mu_0 \leftarrow \mu(z^0)$, $\tau_0 \leftarrow \mu_0$, $k \leftarrow 0$ and $\kappa \leftarrow 1$;

  repeat

    Compute γ from (3.8);

    (predictor step)

    Set $z \leftarrow z^k$;

    r_1. Compute predictor direction (3.6);

    r_2. Compute predictor steplength (3.7);

    r_3. Compute $\lceil\, \overline{z}\,,\, \overline{\tau}\,\rfloor$ from (3.9);

    If $\mu(\overline{z}\,) = 0$ then STOP: $\overline{z}$ is an optimal solution;

    If $\lceil\, \overline{z}\,,\, \overline{\tau}\,\rfloor \in \mathcal{D}_s(\beta)$, then set $z^{k+1} \leftarrow \overline{z}$, $\mu_{k+1} \leftarrow \mu(\overline{z}\,)$, $\tau_{k+1} \leftarrow \overline{\tau}$,

      $k \leftarrow k+1$ and RETURN;

    (corrector step)

    r_4. Compute corrector direction (3.10);

    r_5. Compute corrector steplength (3.12);

    r_6. Compute $\lceil\, z^+,\, \tau^+\,\rfloor$ from (3.13);

    if $\lceil\, z^+,\, \tau^+\,\rfloor \in \mathcal{D}_s(\beta)$, set $z^{k+1} \leftarrow z^+$, $\mu_{k+1} \leftarrow \mu(z^+)$, $\tau_{k+1} \leftarrow \tau^+$,

      $k \leftarrow k+1$ and RETURN;

    else, set $\kappa \leftarrow 2\kappa$ and $z^{k+1} \leftarrow z^k$, $\mu_{k+1} \leftarrow \mu(z^k)$, $\tau_{k+1} \leftarrow \tau_k$,

      $k \leftarrow k+1$ and RETURN;

  until some stopping criterion is satisfied.

Using Theorem 3.10, Corollary 3.11, and, Theorem 3.13 we obtain the following result.

THEOREM 4.1. *Algorithm 2 produces a point* $\lceil z^k, \tau_k \rfloor \in \mathcal{D}_s(\beta)$ *with* $x^{k\,T} s^k \leq \varepsilon$ *in at most*

$$O\left((1+\kappa)^2 n^{3/2} \log\left(x^{0\,T} s^0/\varepsilon\right)\right)$$

iterations, if the starting point satisfies (3.46),
and

$$O\left((1+\kappa) n \log\left(x^{0\,T} s^0/\varepsilon\right)\right)$$

iterations, if the starting point satisfies (3.49).
Also, if the HLCP is nondegenerate then the sequences (μ_k), (τ_k), *and* $\left(r^k\right)$, $i = 1, \ldots, n$, *produced by our algorithm, converge Q-quadratically to zero.*

Proof. Suppose the starting point satisfies (3.49). We can treat the other case exactly in the same way.
Let κ^{max} be the largest value of κ used in Algorithm 2. It is obvious that $\kappa^{max} < 2\chi(Q,R)$. Suppose that at iteration k of Algorithm 2 we have $\kappa < \chi(Q,R)$. If the corrector step is accepted, i.e. if $z^+ \in \mathcal{D}_s(\beta)$, then $z^{k+1} = z^+$ and if not then we increase κ, but its value will never exceed $\kappa^{max} < 2\chi(Q,R)$. On the other hand, if $\kappa \geq \chi(Q,R)$ then the corrector step is never rejected during the following iterations. When the corrector works for values of $\kappa < \chi(Q,R)$, in the predictor step, τ will reduced as much as if $\kappa = \chi(Q,R)$ would have been used. In both cases, by inspecting the proof of the polynomial complexity of Algorithm 1, we can easily see that

$$\mu_{k+1} \leq \frac{1}{\beta}\left(1 - \frac{1}{\frac{64}{\beta(1-\beta)}(1+4\kappa^{max})n}\right)^k \mu_0 \leq \frac{1}{\beta}\left(1 - \frac{1}{\frac{64}{\beta(1-\beta)}(1+8\chi(Q,R))n}\right)^k \mu_0 \,.$$

Since there can be at most $\log_2 \kappa^{max}$ rejections we obtain our complexity result. In case the problem is nondegenerate, since there are only a finite number of corrector rejections, by inspecting the proof of the superlinear convergence of Algorithm 1, it follows that the quadratic convergence holds for the second algorithm also. ∎

Even if $\chi(Q,R)$ is known, it is not certain that Algorithm 1 with $\kappa = \chi(Q,R)$ is more efficient than Algorithm 2 on a particular problem. It may happen that the corrector step in Algorithm 2 is accepted for smaller values of κ at some iterations, and those iterations will produce a better reduction of the complementarity gap.

The author would like to thank his advisor, Dr. Florian A. Potra, for his many constructive comments on this paper.

REFERENCES

[1] M. Anitescu, G. Lesaja, and F. A. Potra. Equivalence between different formulations of the linear complementarity problem. *Optimization Methods & Software*, 7(3):265–290, 1997.

[2] J. F. Bonnans and C. C. Gonzaga. Convergence of interior point algorithms for the monotone linear complementarity problem. *Mathematics of Operations Research*, 21:1–25, 1996.

[3] J. F. Bonnans and F. A. Potra. On the convergence of the iteration sequence of infeasible path following algorithms for linear complementarity problems. *Mathematics of Operations Research*, 22(2):378–407, 1997.

[4] Marco Colombo and Jack Gondzio. Further development of multiple centrality correctors for interior point methods. Technical report, School of Mathematics, The University of Edinburgh, Mayfield Road, Edinburgh EH9 3JZ, U.K., October 2005.

[5] R. W. Cottle, J.-S. Pang, and R. E. Stone. *The Linear Complementarity Problem.* Academic Press, Boston, MA, 1992.
[6] J. Ji, F. A. Potra, and S. Huang. Predictor-corrector method for linear complementarity problems with polynomial complexity and superlinear convergence. *J. Optim. Theory Appl.*, 85(1):187–199, 1995.
[7] J. Ji, F. A. Potra, and R. Sheng. A predictor–corrector method for solving the P_*-matrix LCP from infeasible starting points. *Optimization Methods and Software*, 6:109–126, 1995.
[8] M. Kojima, N. Megiddo, T. Noma, and A. Yoshise. *A Unified Approach to Interior Point Algorithms for Linear Complementarity Problems*, volume 538 of *Lecture Notes in Comput. Sci.* Springer-Verlag, New York, 1991.
[9] X. Liu and F. A. Potra. Corrector-predictor methods for sufficient linear complementarity problems in a wide neighborhood of the central path. *SIAM J. Optim.*, 17(3):871–890, 2006.
[10] S. Mehrotra. Quadratic convergence in a primal-dual method. *Math. Oper. Res.*, 18(3):741–751, 1993.
[11] J. Miao. A quadratically convergent $O((1+k)\sqrt{n}L)$-iteration algorithm for the $P_*(k)$-matrix linear complementarity problem. *Mathematical Programming*, 69:355–368, 1995.
[12] S. Mizuno, M. J. Todd, and Y. Ye. On adaptive-step primal-dual interior-point algorithms for linear programming. *Mathematics of Operations Research*, 18(4):964–981, 1993.
[13] R. D. C. Monteiro and S. J. Wright. Local convergence of interior-point algorithms for degenerate monotone LCP. *Computational Optimization and Applications*, 3:131–155, 1994.
[14] F. A. Potra. Efficient hybrid algorithms for finding zeros of convex functions. *Journal of Complexity*, 10:199–215, 1994.
[15] F. A. Potra. An infeasible interior–point predictor-corrector algorithm for linear programming. *SIAM Journal on Optimization*, 6(1), 1996.
[16] F. A. Potra. An $O(nL)$ infeasible interior–point algorithm for LCP with quadratic convergence. *Annals of Operations Research*, 62:81–102, 1996.
[17] F. A. Potra. A superlinearly convergent predictor-corrector method for degenerate LCP in a wide neighborhood of the central path with $O(\sqrt{n}L)$-iteration complexity. *Math. Program.*, 100(2, Ser. A):317–337, 2004.
[18] F. A. Potra and X. Liu. A predictor-corrector method for $P_*(\kappa)$ HLCP in a wide neighborhood of the central path. Technical Report Ser. A, electronic, 2004.
[19] F. A. Potra and X. Liu. Predictor-corrector methods for sufficient linear complementarity problems in a wide neighborhood of the central path. *Optim. Methods Softw.*, 20(1):145–168, 2005.
[20] F. A. Potra and R. Sheng. A large-step infeasible–interior–point method for the P_*-matrix LCP. *SIAM Journal on Optimization*, 7(2):318–335, 1997.
[21] F. A. Potra and J. Stoer. On a class of superlinearly convergent polynomial time interior point methods for sufficient LCP. Technical Report TR2008-02, Mathematics and Statistics, UMBC, February 2008. http://www.math.umbc.edu/~kogan/technical_papers/index2008.html.
[22] M. Salahi, M.R. Peyghami, and T. Terlaky. New complexity analysis of IIPMs for linear optimization based on a specific self-regular function. *European Journal of Operational Research*, 186:466–485, 2008.
[23] R. Sheng and F. A. Potra. A quadratically convergent infeasible-interior-point algorithm for LCP with polynomial complexity. *SIAM Journal on Optimization*, 7(2):304–317, 1997.
[24] J. Stoer. High order long-step methods for solving linear complementarity problems. *Ann. Oper. Res.*, 103:149–159, 2001. Optimization and numerical algebra (Nanjing, 1999).
[25] J. Stoer and M. Wechs. Infeasible-interior-point paths for sufficient linear complementarity problems and their analyticity. *Math. Programming*, 83(3, Ser. A):407–423, 1998.
[26] J. Stoer, M. Wechs, and S. Mizuno. High order infeasible-interior-point methods for solving sufficient linear complementarity problems. *Math. Oper. Res.*, 23(4):832–862, 1998.
[27] H. Väliaho. P_*-matrices are just sufficient. *Linear Algebra and its Applications*, 239:103–108, 1996.
[28] H. Väliaho. Determining the handicap of a sufficient matrix. *Linear Algebra Appl.*, 253:279–298, 1997.
[29] Y. Ye and K. Anstreicher. On quadratic and $O(\sqrt{n}L)$ convergence of predictor-corrector algorithm for LCP. *Mathematical Programming*, 62(3):537–551, 1993.
[30] Y. Ye, O. Güler, R. A. Tapia, and Y. Zhang. A quadratically convergent $O(\sqrt{n}L)$-iteration algorithm for linear programming. *Mathematical Programming*, 59(2):151–162, 1993.
[31] Y. Zhang, R. A. Tapia, and J. E. Dennis. On the superlinear and quadratic convergence of primal–dual interior point linear programming algorithms. *SIAM Journal on Optimization*,

2(2):304–324, 1992.

[32] Y. Zhang, R. A. Tapia, and F. A. Potra. On the superlinear convergence of interior point algorithms for a general class of problems. *SIAM J. on Optimization*, 3(2):413–422, 1993.