Measuring Asymmetries by Fourier Series

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In memoriam Prof. Dan Butnariu.

Abstract

Symmetry is not only a fundamental concept, but also a tool, in many fields. For instance, it is a cornerstone of Modern Physics. If we assume that physical systems have a high degree of symmetry (or at least, in an approximate way), then it is possible to simplify the equations that describe them. Also, the tenacious search for a unified description may be guided by the notion that a valid (and therefore, preferable) theory would be the one most symmetrical.

One of the proposed ways to express quantitatively the symmetry degree of shapes is through the coefficients of Fourier series.

Also, we analyze a geometrical construct which gives an efficient measure of the Level Asymmetry for shapes and in general, for any fuzzy set.

Keywords: Fuzzy Measures, Bayesian Nets, Fourier Analysis, Computer Vision, Artificial Intelligence.
Mathematics Subject Classification: 68727, 03E72, 26E50.

1. Introduction

Quoting Hermann Weyl, "an object is said to be *symmetrical,* if one can subject it to a certain operation, and it appears exactly the same after the operation as before. Any such operation is called a *symmetry* of the object" [13].

Symmetry is a fundamental concept and also a useful tool in almost any scientific or artistic field. For instance, it is a cornerstone not only of Modern Physics, but also of apparently unconnected areas as Music. In fact, these two particular fields do intersect in the *Physics of Sound.*

If we assume that physical systems have a high degree of symmetry (at least approximative), then it is possible to simplify the equations describing them.

Also, in the obstinate search for a unified description for elementary particle, the clue is the equivalence between valid (and therefore, desirable) theory and the more symmetrical possible theory.

[1] AMO - Advanced Modeling and Optimization. ISSN: 1841-4311

Usually, Symmetry, and in parallel, Asymmetry, are considered as two sides of the same coin: an object will either be totally symmetric, or totally asymmetric, relative to a pattern object. There would be no intermediate situations of partial symmetry or partial asymmetry. But this dychotomical classification is too simple, and lacks of a necessary and realistic gradation. For this reason, it is convenient to introduce “shade regions”, modulating the degree of symmetry (a fuzzy concept).

So, defining symmetry as a continuous feature, we get to a more complex definition, but more useful in many essential fields, as Computer Vision. Its interest is therefore not only theoretical, but also applied in A I.

When we consider an isolated physical system, its symmetry properties are closely related to the conservation laws which characterize such a system.

The great mathematician *Emmy Noether* gives a clear description of this relation, in two theorems, establishing that (first theorem) *"each symmetry of a physical system implies that some physical property of that system is conserved".* And conversely (second theorem), *"each conserved quantity (into a system) has a corresponding symmetry".*

2. Groups in Action

The transformations describing physical symmetries form a mathematical algebraic structure, known as Group.

Group Theory may be considered as the *Mathematics of Symmetry.*

In many physical situations, symmetries are also isometries, that is, transformations that preserve distances. Although many books have appeared on this subject, [15, 16] stand out as classical reference.

Let O be a general object (image, signal...). We may suppose it is 1-D, 2-D or 3-D.

The *Symmetry Group of O,* denoted as $G(O)$, is composed of *all the isometries under which invariance is preserved,* considering the composition as group operation.

Therefore, *the Symmetry Group is a subset of the Isometry Group,* that is,

$$G\,(O) \subset Iso\,(O)\,.$$

So, it is possible to construct new plausible computational tools which allow the automatic transition from theoretical concepts on Symmetry/Asymmetry to applications in the real world. And with this, it is the feasible to construct and manipulate a collection of nearest shapes: given an object O, we will define SD, the Symmetry Distance of the shape to its reference pattern.

In this way we are quantifying the distance from Symmetry of a shape as continuous feature, instead of a discrete one. Before we had either exact coincidence or just difference. Now, it is possible to distinguish gradual differences to its Symmetrical shape.

This distance from Symmetry in shape will be defined [6] as the minimum mean squared distance required to *move* points from the original shape, in order to obtain a symmetrical shape.

So, *SD* is the *minimum effort required to turn a given shape into a symmetric shape.*

3. Remarks on Group Theory

Among others, certain groups are very useful for Modern Physics. For instance, the set of all proper rotations around any axis of a sphere (or ball) is the *Special Orthogonal Group,* denoted $SO(3)$. The number 3 refers to the dimension of the subjacent space. $SO(3)$ is a special type of group, a *Lie Group.* Its rotations can be about any angle. Such rotations preserve distances on the surface of the ball.

Another useful group is the *Lorentz Group, s*pecially, when dealing with relativistic questions. It is the set of all Lorentz transformations and it is possible to generalize it to the *Poincaré Group.*

In the Standard Model, *Gauge Symmetry* is based on the group

$$SU\,(3) \times SU\,(2) \times U\,(1)$$

because three fundamental forces are described by it

- the symmetry of the $SU\,(3)$ describes the *strong force,*
- the $SU\,(2)$ group describes *weak interactions*
- the $U\,(1)$ group describes the *electromagnetic force.*

In the early 70's, a new class of symmetry was discovered: *Supersymmetry.* Sometimes abbreviated as *SUSY,* is useful when working with fundamental fields and space-times. Initial research on it was related to *String Theory.* In fact, it was created to include *fermions* in such Theory. So, for each *boson* (a particle of integral spin, that transmits a force), there is a corresponding *fermion* (a particle of half-integral spin).

From a mathematical point of view, *SUSY* describes complex fields with *holomorphy* (complex differentiability in each point of an open subset of C). So, it provided useful models of more "realistic" theories.

Each finite group can be deconstructed into *"atoms of symmetry"*, also called *"simple groups"*.

According to the *Jordan-Hölder theorem,*

Any two deconstructions of a finite group always give the same collection of simple groups.

Most finite simple groups fit into a table.

But we can find 26 exceptions to such a classification. They are the so called *sporadic groups.*

One of them is the so-called *"Monster"* [2]. In fact, it is the largest of them, containing all but 6 of the other groups. Its size would be expressed by a number of 54 digits.

It was created by Griess (1982), at Princeton, but their name proceeds from Conway.

It has more of 1,050 symmetries, and exists into a space of

$$196,883 = 47 \times 59 \times 71$$

dimensions. Such factors are the largest divisors of such cardinal of dimensionality. Also containing 8×10^{53} elements.

Nevertheless, it is a "simple group", in the sense that it does not have normal subgroups, other than itself and the identity element.

One of its "applications" so far would be giving the best way for packing spheres in 24 dimensions. That is related with the *packing problem,* and the *Kepler conjecture,* later studied by C. F. Gauss, according to which

"No packing of spheres of the same radius, in three dimensions, has a density greater than the face-centred (hexagonal) cubic packing".

A very curious associated phenomenom is the *Moonshine,* between the Monster and a certain sequence of numbers.

Conway and Norton (1979), and then, Frenkel, Lepowsky and Meurman searched the connections among the Monster and String Theory. Also Borcherds analyzed such relations, creating a *Monster Lie Algebra,* and receiving for this the Field Medal in 1998.

4. Measuring Symmetry by Fourier Analysis

Let us recall some basic theoretical aspects of *Fourier Series.* It is a well-known fact that *Jean Baptiste Joseph Fourier* (1768-1830) studied the mathematical theory of heat conduction, establishing the partial differential equations that govern heat diffusion, and solving them by infinite series of trigonometric functions. They are now called *Fourier Series.*

Fourier saw that any periodic signal is a linear composition of sinusoids, or sine waves. So, developing a periodic function (f) by an infinite sum of terms, each one of them being an expression only in sines, or only cosines functions, or perhaps in both, each one provided with a particular "weight", a_k or b_k,its so-called *coefficients of Fourier*

$$f = a_0 + \sum_{k=1}^{\infty} \{a_k \ \cos \ (k\varphi) + b_k \ \sin \ (k\varphi)\}$$

or equivalently,

$$f = \sum_{k=0}^{\infty} \left\{ a_k \ \cos \ \left(2\pi k \frac{t}{T}\right) + b_k \ \sin \ \left(2\pi k \frac{t}{T}\right) \right\}$$

if we consider the *phase angle*

$$\varphi = 2\pi \frac{t}{T}$$

But sometimes, we can see

$$f = a_0 + \sum_{k=1}^{\infty} \left\{ a_k \ \cos \ \left(2\pi k \frac{t}{T}\right) + b_k \ \sin \ \left(2\pi k \frac{t}{T}\right) \right\}$$

Or also

$$f = a_0 + \sum_{k=1}^{\infty} \left\{ a_k \ \cos \ (k\omega_0 t) + b_k \ \sin \ (k\omega_0 t) \right\}$$

Observe in all these equations the range of the corresponding summatory, $k \in \mathbf{N}$ or $k \in \mathbf{N}^*$.

Therefore, a periodic function, $f(t)$, is representable by an infinite sum of sine and/or cosine functions that are harmonically related. So, the frequency of any trigonometric term is an *harmonic* (a multiple integral) of the fundamental frequency of the periodic function.

In the aforementioned equations, we have

$a_0, \ a_k, \ b_k :$ Fourier coefficients, reachable from $f(t)$ by

$$a_0 = \frac{1}{T} \int_{t_0}^{t_0+T} f(t)\, dt$$
$$a_k = \frac{2}{T} \int_{t_0}^{t_0+T} f(t) \ \cos(k\omega_0 t)\, dt$$
$$b_k = \frac{2}{T} \int_{t_0}^{t_0+T} f(t) \ \sin(k\omega_0 t)\, dt$$

where

$$\omega_0 = \frac{2\pi}{T}$$

is the fundamental frequency of f.

T is its *period,* and $k\omega_0$ the k-th harmonic of $f(t)$.

For the subsequent analysis, it is convenient to see the effect of symmetry on the Fourier coefficients. It depends, obviously, on the type of symmetry.

In the case of f is an *even function,* that is, when

$$f(t) = f(-t), \ \forall t$$

we find these reductions, where some of the coefficients disappear

$$a_0 = \frac{2}{T}\int_0^{T/2} f(t)\, dt$$

$$a_k = \frac{4}{T}\int_0^{T/2} f(t)\ \cos(k\omega_0 t)\, dt$$

$$b_k = 0$$

In the case of f is an *odd function,* that is, when

$$f(t) = -f(-t)\,,\ \forall t$$

we find these reductions, where some of the coefficients disappear

$$a_0 = 0$$

$$a_k = 0$$

$$b_k = \frac{4}{T}\int_0^{T/2} f(t)\ \sin(k\omega_0 t)\, dt$$

In the case of f possessing a *half-wave symmetry,* that is, when

$$f(t) = f\left(t - \frac{T}{2}\right),\ \forall t$$

we find these reductions

$$a_0 = 0$$

$$a_k = \begin{cases} 0,\ if\ k\ even \\ \frac{4}{T}\int_0^{T/2} f(t)\ \cos(k\omega_0 t)\, dt,\ if\ k\ odd \end{cases}$$

$$b_k = \begin{cases} 0,\ if\ k\ even \\ \frac{4}{T}\int_0^{T/2} f(t)\ \sin(k\omega_0 t)\, dt,\ if\ k\ odd \end{cases}$$

And finally, in the case of f possessing a *quarter-wave symmetry,* that is, when f has half-wave symmetry, and moreover it has symmetry about the mid-point of the positive and negative half-cycles, we may distinguish between two subclasses,

f is even. Then,

$$a_0 = 0$$

$$a_k = \begin{cases} 0,\ for\ k\ even \\ \frac{8}{T}\int_0^{T/2} f(t)\ \cos(k\omega_0 t)\, dt,\ if\ k\ odd \end{cases}$$

(in both cases, because of the half-wave symmetry)

$$b_k = 0$$

(in this case, because the even character of f)

f *is odd.* Then,

$$a_0 = 0$$
$$a_k = 0$$
(in both cases, because f is odd))
$$b_k = \begin{cases} 0, \; if \; k \; even \\ \frac{8}{T} \int_0^{T/2} f(t) \; \sin(k\omega_0 t) \, dt, \; if \; k \; odd \end{cases}$$
(in this case, because the half-wave subjacent symmetry of f).

We start from an object, shape or form F*,* where generally we refer to its boundary, when it is a 3-dimensional construct.

About the symmetry, we know that symmetry is never perfect in the real world. Therefore, perfect symmetry is an imaginary, ideal reference, product of mathematically creative minds.

So, we are considering the actual symmetry, G_a, corresponding to an imperfect form, F_a, as opposed to ideal symmetry, G_i, associated to its "perfect" form, F_i.

In fact,

$$G_a \textit{ is a subgroup of } G_i$$

When we say: *"the form F has symmetry G"*, we are expressing that the form F belongs to the set $S(G)$. Such set, $S(G)$, contains all the shapes which are invariant under transformations of the symmetry group, G.

This can be denoted

$$F \in S(G)$$

We may define a space of all the possible objects, or shapes, denoted by

$$\mathbf{X} = \{X_i\}_{i \in \mathbf{N}}$$

In this way, we can assign to each element of $\mathbf{X}$ a crisp set containing all objects which fulfil the conditions of G.

If we denote this set as $\mathbf{S}(G)$, we have the mapping

$$G \to \mathbf{S}(G)$$

For this reason, we may introduce a *membership function,*

$$\mu_G : \mathbf{X} \to [0,1]$$
$$X \to \mu_G(X) \equiv \mu(G, X)$$
$$\textit{with } X \in \mathbf{X}$$

This characterises the membership degree of the shape X to the set $\mathbf{S}(G)$. That is, its degree of fulfilment of symmetry requirements which contain G.

Hence, we have different situations,

- *full membership:* when $\mu_G(X) = 1$
- *null membership* (or not membership at all): $\mu_G(X) = 0$
- *partial membership:* $0 < \mu_G(X) < 1$

In the 2-D case, generalizable to 3-D and higher [12, 13], we may consider the forms and their boundaries closed surfaces in $\mathbf{R}^3$. It is feasible to describe them by selecting a convenient coordinate system. So, we obtain a form function, denoted here by

$$R(\alpha,\ \beta)$$

Here, we put

$$\begin{gathered} R \equiv radius \\ \alpha \equiv azimuthal\ angle \\ and \\ \beta \equiv polar\ angle \\ relative\ to\ the\ axis\ where\ is\ higher\ the\ symmetry \end{gathered}$$

being its corresponding ranges

$$\begin{gathered} R > 0 \\ 0 \leq \alpha < 2\pi \\ 0 \leq \beta \leq \pi \end{gathered}$$

The centre must be chosen so as highest symmetry is reached in this point. Then, it may be the centroid of the shape.

So, we obtain the subsequent equation

$$R(\alpha) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \{a_k \cos k\alpha + b_k \sin k\alpha\}$$

Which can be translated to a sinusoidal expression,

$$R(\alpha) = \frac{\varsigma_0}{2} + \sum_{k=1}^{\infty} \varsigma_k \sin(k\alpha + \alpha_k)$$

Or also, if it is expressed in cosinusoidal way,

$$R(\alpha) = \frac{\xi_0}{2} + \sum_{k=1}^{\infty} \xi_k \cos(k\alpha - \alpha_k)$$

Such expansion always exists, when the function f is periodic and fulfils some conditions by Dirichlet.

Now, it is already possible to introduce the measure of *Degree of Symmetry,* in our modelling process.

Suppose a given form, F, which presents certain symmetries, described by the symmetry group G. Then, the form must contain only terms which are compatible with G, whereas incompatible term must disappear. If we denote $\delta(F, G)$ the symmetry degree of F relative to G, we define

$$\delta(F, G) = \frac{\sum^{K}_{\substack{i>0 \\ compatible}} c_i}{\sum_{j=1}^{K} c_j}$$

That is, the proportion between the sum of coefficients of every term compatible with G over the sum of all coefficients of form F. So, describing their relative contribution to F of terms compatible with G.

Clearly,

$$0 \leq \delta(F, G) \leq 1$$

Then, it induces a natural

Classification of Fuzzy Asymmetries

according to their value,

$$\delta(F, G) = 0: \ MISSING\ SYMMETRY$$

When G cannot be found in F

$$\delta(F, G) \gtrapprox 0: \ HIDDEN\ SYMMETRY$$

When F bears not perceptible relation to G, but $\delta(F, G) > 0$

$$\delta(F, G) > 0: DISTANT\ SYMMETRY$$

When F is notably deviated from G, but with a clear relation between F and G

$$\delta(F, G) < 1: \ APPROXIMATE\ SYMMETRY$$

When there are little deviations of F from G

$$\delta(F,G) \lessapprox 1: \ APPARENT\ SYMMETRY$$

When there are only imperceptible deviations of F from G

$$\delta(F,G) = 1: \ ACTUAL\ SYMMETRY$$

When F strictly fulfills the symmetry conditions of G.

5. A new geometrical modelling

Now, we need some previous and very essential concepts [2, 13].

In Fuzzy Measure Theory (connected with such idea in Classical/Crisp Measure Theory), a *fuzzy atom* is a fuzzy measurable set which has positive fuzzy measure, and contains no "smaller" set of positive fuzzy measure.

Formally expressed:

Given a fuzzy measurable space, (X, Ω), and a finite fuzzy measure, μ, on that space, a fuzzy set $A \subset \Omega$ is called a *(fuzzy) atom,* if

$$\mu(A) > 0$$

and for any measurable fuzzy subset,

$$B \subset A$$

with

$$\mu(A) > \mu(B)$$

it holds

$$\mu(B) = 0$$

A fuzzy measure which has no atoms is called a *non-atomic fuzzy measure.* That is, a fuzzy measure is *non-atomic,* when

$$\forall A\ fuzzy\ measurable\ with\ \mu(A) > 0,$$
$$\exists\ a\ measurable\ fuzzy\ subset,\ B \subset A,$$
$$such\ that: 0 < \mu(B) < \mu(A)$$

Then, a non-atomic fuzzy measure with at least one positive value has an infinite number of different values into $[0,1]$.

Being as starting point a set A, such that

$$\mu(A) > 0$$

From this, we may construct a decreasing sequence of fuzzy measurable subsets,

$$A_1 \supset A_2 \supset A_3 \supset \ \ldots$$

such that

$$\mu(A_1) > \mu(A_2) > \mu(A_3) > \ldots$$

Non-atomic fuzzy measures have a continuum of values.

Because if μ is a non-atomic fuzzy measure, and A is a fuzzy measurable set, with

$$\mu(A) > 0$$

then

$$\forall b \in \mathbf{R},\ which\ holds\ 0 < b < \mu(A),$$
$$\exists\ a\ measurable\ fuzzy\ subset,\ B \subset A,$$
$$such\ that\ \mu(B) = b$$

The precedent result is due to W. Sierpinski [12].

And it is a clear reminiscent of the *Intermediate Value Theorem* for continuous functions.

So, for instance, if we consider our [5],

Corollary: Let (E, μ) a *finite fuzzy metric space,* and $\{A_i\}_{i=1}^{n}$ a *DCC* or contractive chain of enchained fuzzy subsets (or subworlds into the universe $U \supset A$), according to Noether condition, all them containing the monatomic fuzzy set (or world) A, that is,

$$A \subset \ldots \subset A_{i+1} \subset A_i \subset U,\ \forall i \in \{1, 2, \ldots, n\}$$

being

$$lim_{i \to \infty}\ A_i = A$$

Then, we have

$$[L_a\ (A_i)] = 0,\ in\ the\ atom,$$
$$or\ monatomic\ world,\ A_i = A$$
$$[L_a\ (A_i)] = 1,\ in\ other\ worlds$$

Where L_a denotes our Asymmetry Level Fuzzy Measure, which will be finite, according hypothesis.

In fact,

$$\Re_{L_a} \subseteq [0,1]$$

Obviously, it verifies the *Decreasing Chain Condition (DCC),* which for instance, characterizes the Artinian modules.

Geometrically, the situation (relative to such symmetric character) should be modelled by a contractive set, or decreasing collection, verifying the *DCC* condition, therefore a chain of subworlds, each one inserted in the precedent, and where each one, but the last, shows asymmetries, whereas at the end, in the limit, the total symmetry appears.

Because our construct holds the *Decreasing Chain Condition (DCC),* it may be stationary from a certain step,

$$\exists m \in \mathbf{N} : \; A_m = A_{m+1} = A_{m+2} = \; ...$$

To solve this problem, either we can admit the symmetry as discontinuous function, and so we see without problems that

$$ASYMMETRY \rightarrow ASYMMETRY \rightarrow ASYMMETRY \rightarrow \; ...$$
$$... \; \rightarrow ASYMMETRY \rightarrow SYMMETRY$$

Or we may assign a certain value as level of symmetry or asymmetry (by duality), with a definition suggested by the belonging degree of elements to fuzzy sets; or equivalently, as a level of satisfaction of some condition or property, defined so in the limit it is possible to obtain the state of complete symmetry,

$$A_1 \supseteq A_2 \supseteq A_3 \supseteq \; ... \; \supseteq A_n \supseteq \; ... \; \supseteq A = \{a\}$$

So, for instance, with the contractivity condition taken from the concept of cardinality,

$$c(A_1) \geq c(A_2) \geq c(A_3) \geq \; ... \; \geq c(A_n) \geq \; ... \; \geq c(A) = 1$$

Also we can suppose, simplifying, that each world has a cardinal number one less than the precedent world.

Once classified in decreasing order, reaching some degree of homogeneity among elements, it is possible to introduce the function "symmetry level" (or asymmetry level, by duality). Respectively denoted L_s and L_a.

With an increasing sequence of values in their sequence, depending on the cardinality of the selected world at each step, until converging to one from the left (as symmetry value, corresponding with the totally symmetrical scenario), in the limit, when we "arrive" to the monoatomic world, aforementioned and denoted as A,

$$\{A_n\}_{n \in N} \to A$$

Also we can introduce fuzzy concepts to represent the uncertainty in measures: triangular fuzzy numbers, triangular shaped fuzzy numbers, trapezoidal fuzzy numbers, trapezoidal shaped fuzzy numbers, and so on, with their corresponding operations, mutual relationships and therefore using fuzzy mathematics, after a process of fuzzification of shapes.

6. Conclusion

From this construction a new Normal Fuzzy Measure is feasible, called *Asymmetry Level Measure,* and denoted L_a, as we described in detail in some of our papers [*see* 7] .

Such framework allows the possibility of solving the temporal asymmetry problem.

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