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Quaternion Parametric Optimal Partition Invariancy Sensitivity Analysis in Linear Optimization

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This paper is a tribute to G.B. Dantzig the father of linear programming and the inventor of the simplex method.

Abstract

In this paper, we consider linear optimization problem in standard form with perturbation in both the right hand side and objective function data for which each of them includes a combination of two independent directions with different parameters. In this way, we have four independent parameters and refer to the problem as quaternion parametric programming. We are interested in identifying the region where optimal partition is invariant. This region is referred to as invariancy region. An algorithmic procedure is presented that is capable to identify the invariancy region includes the origin in polynomial time. It is proved that this region is a polyhedron as a convex set. A closed form of the optimal value function is obtained too.

Keyword: Linear programming, Optimal partition, Invariancy region, Optimal value function.

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1 Introduction

In real life it is possible not only the prices may change but the amount of supplying may change also. In this respect even the changes in price and also in supplying may happen in two different

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directions .e.g. Δc_1, Δc_2 and/or Δb_1 and Δb_2, since the management may provide the required quantities from two different sources with the same price but in an odd situation the prices may change independently. Thus this is the case that we concern the region of the changes so that the optimal partition remains optimal.

Consider the parametric linear optimization problem as

$$(P) \qquad \min\{(c+\epsilon_1\Delta c_1+\epsilon_2\Delta c_2)^T x|\ Ax=b+\lambda_1\Delta b_1+\lambda_2\Delta b_2,\ x\geq 0\},$$

where $A\in\mathbb{R}^{m\times n}, b\in\mathbb{R}^m$ and $c\in\mathbb{R}^n$ are known fixed data, $\epsilon_1,\epsilon_2,\lambda_1$ and λ_2 are real parameters, $\Delta b_1\in\mathbb{R}^m,\Delta b_2\in\mathbb{R}^m,\Delta c_1\in\mathbb{R}^n$ and $\Delta c_2\in\mathbb{R}^n$ are perturbation vectors, and $x\in\mathbb{R}^n$ is an unknown vector.

The dual of (P) is defined as

$$(D) \qquad \max\{(b+\lambda_1\Delta b_1+\lambda_2\Delta b_2)^T y|A^Ty+s=c+\epsilon_1\Delta c_1+\epsilon_2\Delta c_2,\ s\geq 0\},$$

where $y\in\mathbb{R}^m, s\in\mathbb{R}^n$ are the unknown vectors. Any vector $x\geq 0$ which satisfies $Ax=b+\lambda_1\Delta b_1+\lambda_2\Delta b_2$ is called a primal feasible solution of (P). Moreover, a vector (y,s) with $s\geq 0$ is called a dual feasible solution of (D) if it satisfies $A^Ty+s=c+\epsilon_1\Delta c_1+\epsilon_2\Delta c_2$. Obviously, each primal feasible solution x only depends on parameters λ_1 and λ_2 but not ϵ_1 and ϵ_2. Similarly, each dual feasible solution (y,s) varies depending on ϵ_1 andϵ_2 but not with λ_1 and λ_2. Therefore, we denote primal and dual feasible solutions of (P) and (D) by $x(\lambda_1,\lambda_2)$ and $(y(\epsilon_1,\epsilon_2),\ s(\epsilon_1,\epsilon_2))$, respectively. For any primal-dual feasible solution $(x(\lambda_1,\lambda_2),\ y(\epsilon_1,\epsilon_2),\ s(\epsilon_1,\epsilon_2))$, the weak duality property holds i.e.,

$$(c+\epsilon_1\Delta c_1+\epsilon_2\Delta c_2)^T x(\lambda_1,\lambda_2)\geq(b+\lambda_1\Delta b_1+\lambda_2\Delta b_2)^T y(\epsilon_1,\epsilon_2),$$

and equality holds if and only if they are optimal solutions (strong duality property [2]). In this way $x(\lambda_1,\lambda_2)^T s(\epsilon_1,\epsilon_2)=0$ holds for optimal solutions that is referred to as complementarity. If in addition to complementarity, $x(\lambda_1,\lambda_2)+s(\epsilon_1,\epsilon_2)>0$, then these solutions are called strictly complementary optimal solutions. It is worth mentioning that this kind of optimal solutions exist by Goldman-Tucker Theorem[9]. Let $(\mathcal{P})$ and $(\mathcal{D})$ denote the feasible solution sets of (P) and (D), respectively. Their optimal solution sets are denoted by $(\mathcal{P}^*)$ and $(\mathcal{D}^*)$, correspondingly.

The *optimal value function* is defined as

$$\Phi=(c+\epsilon_1\Delta c_1+\epsilon_2\Delta c_2)^T x(\lambda_1,\lambda_2)=(b+\lambda_1\Delta b_1+\lambda_2\Delta b_2)^T y(\epsilon_1,\epsilon_2),$$

where $\Delta b_1,\Delta b_2,\Delta c_1$ and Δc_2 are fixed perturbations and $(x(\lambda_1,\lambda_2),y(\epsilon_1,\epsilon_2),s(\epsilon_1,\epsilon_2))$ is a primal-dual optimal solution of problems (P) and (D). The support set of a nonnegative vector $v\in\mathbb{R}^n$ is defined as $\sigma(v)=\{i: v_i>0, 1\leq i\leq n\}$. The index set $\{1,2,\ldots,n\}$ can be partitioned into two subsets

$$\begin{aligned}
\mathcal{B}(\lambda_1,\lambda_2) &= \{i: x_i(\lambda_1,\lambda_2)>0 \text{ for a primal optimal solution } x(\lambda_1,\lambda_2)\},\\
\mathcal{N}(\epsilon_1,\epsilon_2) &= \{i: s_i(\epsilon_1,\epsilon_2)>0 \text{ for a dual optimal solution } (y(\epsilon_1,\epsilon_2),s(\epsilon_1,\epsilon_2))\}.
\end{aligned}$$

This partition is known as the *optimal partition* of the index set $\{1,2,\ldots,n\}$ for problems (P) and (D), and is denoted by

$$\pi(\lambda_1,\lambda_2,\epsilon_1,\epsilon_2)=(\mathcal{B}(\lambda_1,\lambda_2),\mathcal{N}(\epsilon_1,\epsilon_2)).$$

Since the optimal solution sets $(\mathcal{P}^*)$ and $(\mathcal{D}^*)$ are convex, optimal partition is unique.

Karmarkar[7] initiated a method that solves linear optimization problems in polynomial time which are developed as interior point methods later on. An interior point method terminates at primal-dual strictly complementary optimal solution which is enable to identify associated optimal partition[6].

In this paper we want to identify the region where optimal partition is invariant. This study has been carried out for special cases. In quaternion parametric programming, if all parameters (correspondingly all perturbing direction) are zero but one, the problem is referred to as uni-parametric linear programming [1, 3, 8]. Moreover, the case when the right hand side and objective function data have one identical parameters, is investigated in [4, 5]. In these cases, the range of the parameter variation is an interval on the real line that is referred to as *invariancy interval*. The end points of these intervals are called *transition point*s. Here, we suppose that all four parameters varies independently and derive strong results for this general case that covers all previous ones.

The paper is organized as follows. In section 2, some fundamental concepts are presented. Section 3, devoted to present an algorithmic approach that is capable to identify the invariancy region. Section 4, talks about the representation of optimal value function. Examples presented in section 5 to illustrate the results.

2 Invariancy regions

Let optimal partition for unperturbed version of problems (P) and (D) is known as $\pi = (\mathcal{B}, \mathcal{N})$. Thus, the invariancy region is the set of parameter vectors $(\lambda_1, \lambda_2, \epsilon_1, \epsilon_2)$ where for the members of this set, $\pi(\lambda_1, \lambda_2, \epsilon_1, \epsilon_2) = (\mathcal{B}, \mathcal{N})$ holds. This region is nonempty, because the origin belongs to it. This invariancy region is denoted by $\mathcal{IR}$. For $\epsilon_1 = \epsilon_2 = 0$, problems (P) and (D) reduce to

$$\min\{c^T x | Ax = b + \lambda_1 \Delta b_1 + \lambda_2 \Delta b_2,\ x \geq 0\}, \tag{1}$$

and

$$\max\{(b + \lambda_1 \Delta b_1 + \lambda_2 \Delta b_2)^T y |\ A^T y + s = c,\ s \geq 0\}. \tag{2}$$

The invariancy region associated to these problems is denoted by $\mathcal{IR}_P$.

The following lemma shows that the set of dual optimal solutions $(\mathcal{D}^*)$ for problem (2) on the $\mathcal{IR}_P$ is invariant.

Lemma 1. *The set of dual optimal solution set $(\mathcal{D}^*)$ for problem (2) on $\mathcal{IR}_P$ is invariant.*

Proof. For two arbitrary pairs of parameters $(\bar{\lambda}_1, \bar{\lambda}_2)$, $(\tilde{\lambda}_1, \tilde{\lambda}_2) \in \mathcal{IR}_P$, let $(\bar{x}, \bar{y}, \bar{s})$ and $(\tilde{x}, \tilde{y}, \tilde{s})$ be given primal-dual optimal solutions of problems (1) and (2) at $(\bar{\lambda}_1, \bar{\lambda}_2)$ and $(\tilde{\lambda}_1, \tilde{\lambda}_2)$, respectively. From the assumption, we have

$$\pi(\bar{\lambda}_1, \bar{\lambda}_2) = (\mathcal{B}, \mathcal{N}) = \pi(\tilde{\lambda}_1, \tilde{\lambda}_2). \tag{3}$$

It is easy to verify that $(\bar{x}, \tilde{y}, \tilde{s})$ and $(\tilde{x}, \bar{y}, \bar{s})$ are primal-dual solutions (1) and (2) at $(\bar{\lambda}_1, \bar{\lambda}_2)$ and $(\tilde{\lambda}_1, \tilde{\lambda}_2)$, respectively. Moreover, the optimality properties $\bar{x}^T \tilde{s} = 0$ and $\tilde{x}^T \bar{s} = 0$ immediately follows from (3). The complete proof. □

On the other hand, for $\lambda_1 = \lambda_2 = 0$, problems (P) and (D) reduce to

$$\min\{(c + \epsilon_1 \Delta c_1 + \epsilon_2 \Delta c_2)^T x | Ax = b,\ x \geq 0\}, \tag{4}$$

$$\max\{b^T y | A^T y + s = c + \epsilon_1 \Delta c_1 + \epsilon_2 \Delta c_2,\ s \geq 0\}. \tag{5}$$

The invariancy region associated to these problems is denoted by $\mathcal{IR}_D$. The following lemma shows that the set of primal optimal solutions $(\mathcal{P}^*)$ on the $\mathcal{IR}_D$ is invariant. The proof is similar to the proof of Lemma 1 and is omitted.

Lemma 2. *The set of primal optimal solutions on invariancy region $\mathcal{IR}_D$ is invariant.*

The following lemma shows that the invariancy region in the context is convex. Thus, for identifying the region, one only need to determine its border.

Lemma 3. *The invariancy region $\mathcal{IR}$ is a convex set.*

Proof. Without loss of generality, we can suppose that the invariancy region, is not the singleton $\{(0,0,0,0)\}$. Let $(\bar{\lambda}_1, \bar{\lambda}_2, \bar{\epsilon}_1, \bar{\epsilon}_2)$ and $(\tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\epsilon}_1, \tilde{\epsilon}_2)$ be two arbitrary members of the region $\mathcal{IR}$. In addition, let $(\bar{x}, \bar{y}, \bar{s})$ and $(\tilde{x}, \tilde{y}, \tilde{s})$ be strictly complementary optimal solutions of problems (P) and (D) at these points, respectively. For

$$\begin{aligned} \lambda_1 &= \theta_1\bar{\lambda}_1 + (1-\theta_1)\tilde{\lambda}_1, \\ \epsilon_1 &= \theta_2\bar{\epsilon}_1 + (1-\theta_2)\tilde{\epsilon}_1, \\ \lambda_2 &= \theta_1\bar{\lambda}_2 + (1-\theta_1)\tilde{\lambda}_2, \\ \epsilon_2 &= \theta_2\bar{\epsilon}_2 + (1-\theta_2)\tilde{\epsilon}_2, \end{aligned}$$

where $\theta_1, \theta_2 \in (0,1)$. We define

$$\begin{aligned} x &= \theta_1\bar{x} + (1-\theta_1)\tilde{x}, \\ y &= \theta_2\bar{y} + (1-\theta_2)\tilde{y}, \\ s &= \theta_2\bar{s} + (1-\theta_2)\tilde{s}. \end{aligned}$$

Obviously, (x, y, s) is a primal-dual feasible solution of problems (P) and (D) at $(\lambda_1, \lambda_2, \epsilon_1, \epsilon_2)$. Moreover, $\sigma(x) = \sigma(\bar{x}) \cup \sigma(\tilde{x}) = \mathcal{B}$ and $\sigma(s) = \sigma(\bar{s}) \cup \sigma(\tilde{s}) = \mathcal{N}$, that proves the optimality of this solution for problems (P) and (D), as well as having the optimal partition $\pi(\lambda_1, \lambda_2, \epsilon_1, \epsilon_2) = (\mathcal{B}, \mathcal{N})$. The proof is complete. $\square$

Now, we state a fundamental theorem that demonstrate the relationship between the invariancy region $\mathcal{IR}$ and two invariancy regions $\mathcal{IR}_P$ and $\mathcal{IR}_D$. This theorem plays a main role in identifying the invariancy region.

Theorem 4. *Consider the problems (P) and (D). Let $\mathcal{I}R_P$ and $\mathcal{I}R_D$ be the invariancy regions for corresponding primal and dual problems. Then,*

$$\mathcal{I}R = \mathcal{I}R_D \times \mathcal{I}R_P.$$

Proof. Let $\pi = (\mathcal{B}, \mathcal{N})$ be the optimal partition of the index set $\{1, 2, \ldots, n\}$ for problems (P) and (D). Moreover, let (x^*, y^*, s^*) be a strictly complementary optimal solution of these problems. Thus, $\sigma(x^*) = \mathcal{B}$, and $\sigma(s^*) = \mathcal{N}$.
First, we prove that $\mathcal{I}R_D \times \mathcal{I}R_P \subseteq \mathcal{I}R$. Let $((\bar{\lambda}_1, \bar{\lambda}_2), (\tilde{\epsilon}_1, \tilde{\epsilon}_2)) \in \mathcal{I}R_D \times \mathcal{I}R_P$. Thus, there is a strictly complementary optimal solution for problems (P) and (D) at $(\bar{\lambda}_1, \bar{\lambda}_2)$, say $(\bar{x}, \bar{y}, \bar{s})$ with optimal partition π. Analogously, there is a strictly complementary optimal solution $(\tilde{x}, \tilde{y}, \tilde{s})$ for these problems at $(\tilde{\epsilon}_1, \tilde{\epsilon}_2)$ with the same optimal partition. By Lemmas 1 and 2, one can consider $(\bar{x}, \tilde{y}, \tilde{s})$ as a strictly complementary optimal solution of these problems at $(\bar{\lambda}_1, \bar{\lambda}_2, \tilde{\epsilon}_1, \tilde{\epsilon}_2)$. Since, $\sigma(\bar{x}) = B$, and $\sigma(\tilde{s}) = N$, the inclusion $\mathcal{I}R_D \times \mathcal{I}R_P \subseteq \mathcal{I}R$ is concluded.
One the other hand, to prove $\mathcal{I}R \subseteq \mathcal{I}R_D \times \mathcal{I}R_P$, let $(\bar{\lambda_1}, \bar{\lambda_2}, \tilde{\epsilon_1}, \tilde{\epsilon_2}) \in \mathcal{I}R$. Thus, there is a strictly complementary optimal solution for (P) and (D) at this point, say $(\bar{x}, \tilde{y}, \tilde{s})$ where $\sigma(\bar{x}) = \mathcal{B}$ and $\sigma(\tilde{s}) = \mathcal{N}$. It is easy to verify that $(\bar{x}, y^*, s^*)$ is a strictly complementary optimal solution of problems (P) and (D) at $(\bar{\lambda_1}, \bar{\lambda_2})$, with optimal partition $\pi = (\mathcal{B}, \mathcal{N})$. Thus, $(\bar{\lambda_1}, \bar{\lambda_2}) \in \mathcal{I}R_D$. Similarly, $(x^*, \tilde{y}, \tilde{s})$ is an optimal solution of these problems at $(\tilde{\epsilon_1}, \tilde{\epsilon_2})$, with the same optimal partition. Thus, $(\bar{\epsilon_1}, \bar{\epsilon_2}) \in \mathcal{I}R_P$. The proof is complete. $\square$

According to the Theorem 4, to identify the invariancy region $\mathcal{IR}$, it is enough to determine the invariancy regions $\mathcal{IR}_D$ and $\mathcal{IR}_P$. It will be shown that all auxiliary linear optimization problems for obtaining recent invariancy region can be solved in polynomial time by an interior point method, thus the invariancy region $\mathcal{IR}$ can be identified in polynomial time as well.

Remark 5. Theorem 4 says that the invariancy region $\mathcal{I}R$ is a convex set in a space of dimension four. It is easy to verify that if $(\epsilon_1, \epsilon_2) = (\lambda_1, \lambda_2)$, then $\mathcal{I}R = \mathcal{I}R_P \cap \mathcal{I}R_D$. On the other hand, if either $\mathcal{I}R_D$ or $\mathcal{I}R_P$ is the singleton $\{(0,0)\}$, then the invariancy region $\mathcal{I}R$ is in two dimensional subspace (including either (λ_1, λ_2) or (ϵ_1, ϵ_2)). On the other hand, if both invariancy regions $\mathcal{I}R_P$ and $\mathcal{I}R_D$ are singleton $\{(0,0)\}$, then $\mathcal{I}R = \{(0,0,0,0)\}$. Observe that the boundaries of the region (including lines and points) are in special case, optimal partition at there defers from the optimal partition at interior points of the region.

The next two lemmas denote relationship between primal (dual) optimal solution set at boundary points and interior points of $\mathcal{IR}_P$ ($\mathcal{IR}_D$). We present a simple proof for the first one and the proof of the other goes similarly.

Lemma 6. *Let the invariancy region $\mathcal{IR}_P$ with the optimal partition $\pi = (\mathcal{B}, \mathcal{N})$ be known . Moreover, let $(\bar{\lambda}_1, \bar{\lambda}_2)$ be a parameter value on the boundary of this region and $\bar{\pi} = (\bar{\mathcal{B}}, \bar{\mathcal{N}})$ denotes associated optimal partition. Then $\bar{\mathcal{B}} \subset \mathcal{B}$.*

Proof. It is straightforward to verify that $(\bar{\lambda}_1, \bar{\lambda}_2)$ is the optimal solution of the following problem

$$\min(\max)\{\lambda_1 : A_{\mathcal{B}} x_{\mathcal{B}} - (\Delta b_1 + \frac{\bar{\lambda}_2}{\bar{\lambda}_1}\Delta b_2)\lambda_1 = b,\ x_{\mathcal{B}} \geq 0\},$$

and the statement follows immediately. □

Lemma 7. *Let the invariancy region $\mathcal{IR}_D$ with the optimal partition $\pi = (\mathcal{B}, \mathcal{N})$ be known . Moreover, let $(\bar{\epsilon}_1, \bar{\epsilon}_2)$ be a parameter value on the boundary of this region and $\bar{\pi} = (\bar{\mathcal{B}}, \bar{\mathcal{N}})$ denotes associated optimal partition. Then $\bar{\mathcal{N}} \supset \mathcal{N}$.*

3 Algorithmic approach to identify invariancy regions

According to Theorem 4, it is enough to find the regions for (λ_1, λ_2) and (ϵ_1, ϵ_2), independently. Let us consider identifying of the region for (λ_1, λ_2). We investigate the case $\lambda_2 = \alpha\lambda_1$. In this case, the problem reduces to uni-parametric problem and one can immediately find the maximum(or minimum) value of λ as follows [8]:

$$\lambda_u = \max(\min)\{\lambda_1 :\ A_{\mathcal{B}} x_{\mathcal{B}} - (\Delta b_1 + \alpha\Delta b_2)\lambda_1 = b,\ x_{\mathcal{B}} \geq 0\}. \tag{6}$$

The following result can be concluded directly from the convexity of the optimal solution set.

Lemma 8. *Let $(\bar{\lambda}_1, \bar{\lambda}_2)$ and $(\tilde{\lambda}_1, \tilde{\lambda}_2)$ be two arbitrary points with identical optimal partition $\pi = (\mathcal{B}, \mathcal{N})$. Then, for any point at the line segment between these two points, optimal partition is invariant. Moreover, there are two points $\Lambda^+ = (\lambda_1^+, \lambda_2^+)$ and $\Lambda_- = (\lambda_1^-, \lambda_2^-)$ (these points might be at infinity) on this line that optimal partition is invariant on the interior point of the line segment jointing points Λ^+ and Λ^-.*

Proof. First part of lemma is trivial by the convexity of the optimal solution set. Observe that the representation of the line including two points $(\bar{\lambda}_1, \bar{\lambda}_2)$ and $(\tilde{\lambda}_1, \tilde{\lambda}_2)$ is

$$\lambda_2 = \frac{\tilde{m}\tilde{\lambda}_1 - \bar{m}\bar{\lambda}_1}{\tilde{\lambda}_1 - \bar{\lambda}_1}\lambda_1 + \frac{\bar{m} - \tilde{m}}{\tilde{\lambda}_1 - \bar{\lambda}_1}\tilde{\lambda}_1\bar{\lambda}_1,$$

where $\tilde{m}$ and $\bar{m}$ are two real numbers satisfying in

$$\bar{\lambda}_2 = \bar{m}\bar{\lambda}_1 \text{ and } \tilde{\lambda}_2 = \tilde{m}\tilde{\lambda}_1, \tag{7}$$

respectively. To identify two points Λ^+ and Λ^-, it is enough to find the maximum and minimum of λ_1 over the feasible set

$$\{\lambda_1 : A_{\mathcal{B}} x_{\mathcal{B}} - (\Delta b_1 + \frac{\tilde{m}\tilde{\lambda}_1 - \bar{m}\bar{\lambda}_1}{\tilde{\lambda}_1 - \bar{\lambda}_1}\Delta b_2)\lambda_1 = b + \frac{\bar{m} - \tilde{m}}{\tilde{\lambda}_1 - \bar{\lambda}_1}\tilde{\lambda}_1\bar{\lambda}_1\Delta b_2,\ x_{\mathcal{B}} \geq 0\}, \tag{8}$$

respectively and using (7). The proof is complete. □

Remark 9. If the value of λ_1 in 8 is infinite then the region is unbounded as it is shown in figures 2 and 4.

Recall that the (half-)lines obtained in Lemma 8 are transition (half-)lines when $\bar{\lambda}_1$ and $\tilde{\lambda}_1$ are two points that calculated by (6) with appropriate values of α. The intersection of two consequent transition (half-)lines is referred to as transition points.

Let us consider a case when optimal partitions at two points $(\bar{\lambda}_1, \bar{\lambda}_2)$ and $(\tilde{\lambda}_1, \tilde{\lambda}_2)$ are not identical. The next lemma explains this situation. Suppose that is given a transition point. One can find its immediate neighboring transition point as well as the transition line joining them as follows. Let two points $(\bar{\lambda}_1, \bar{\lambda}_2)$ and $(\tilde{\lambda}_1, \tilde{\lambda}_2)$ be obtained by solving (6) leading to two different optimal partitions. Then either

Case 1, They belong to two different transition lines; *or*

Case 2, At least one of these points is a transition point.

To clarify the situation, we choose a convex combination called `midpoint` of these two points and identify the optimal partition at it.
For Case 1, if this optimal partition is identical with $\pi = (\mathcal{B}, \mathcal{N})$, the optimal partition at origin, then trivially, these two points belong to two different transition lines. The following lemma says that one can identify in polynomial time a transition line that contains one of these points.

Lemma 10. *Let two points $(\bar{\lambda}_1, \bar{\lambda}_2)$ and $(\tilde{\lambda}_1, \tilde{\lambda}_2)$ be given by solving appropriate problems as (6), with different optimal partitions. Then transition lines containing these points can be identified in polynomial time.*

Proof. Without loss of generality, we prove the statement for $(\bar{\lambda}_1, \bar{\lambda}_2)$, the proof for the other point goes analogously. Let $(\hat{\lambda}_1, \hat{\lambda}_2)$ be a midpoint between $(\bar{\lambda}_1, \bar{\lambda}_2)$ and $(\tilde{\lambda}_1, \tilde{\lambda}_2)$ defined as

$$\hat{\lambda}_1 = \frac{\bar{\lambda}_1 + \tilde{\lambda}_1}{2}, \quad \hat{\lambda}_2 = \frac{\bar{\lambda}_2 + \tilde{\lambda}_2}{2}.$$

Let $\hat{\alpha} = \frac{\hat{\lambda}_2}{\hat{\lambda}_1}$. We solve problem (6) for $\hat{\alpha}$ and identify the corresponding optimal partition as $\hat{\pi} = (\hat{\mathcal{B}}, \hat{\mathcal{N}})$. If $\hat{\pi}$ is identical with the optimal partition at $(\bar{\lambda}_1, \bar{\lambda}_2)$, then two points $(\hat{\lambda}_1, \hat{\lambda}_2)$ and $(\bar{\lambda}_1, \bar{\lambda}_2)$ belong to a single transition line and one can identify it by the procedure presented in the proof of Lemma 8. Otherwise, we update as

$$(\hat{\lambda}_1, \hat{\lambda}_2) \to (\tilde{\lambda}_1, \tilde{\lambda}_2),$$

and continue the procedure of this proof till for corresponding $\bar{m}$ and $\tilde{m}$, $|\bar{m} - \tilde{m}| < \varepsilon$ holds, where ε is a reasonable computational tolerance. It is straightforward to verify that the procedure terminates in almost $k = \lceil \log_2 \frac{|\bar{m}_0 - \tilde{m}_0|}{\varepsilon} \rceil$ iterations, where $\bar{m}_0$ and $\tilde{m}_0$ are slopes corresponding to initial points $(\bar{\lambda}_1, \bar{\lambda}_2)$ and $(\tilde{\lambda}_1, \tilde{\lambda}_2)$, respectively. It is obvious that when the procedure terminates at exactly k iteration, then $(\bar{\lambda}_1, \bar{\lambda}_2)$ is a transition point. In this case, one can consider $\tilde{m} = \bar{m}_0 + 2\varepsilon$ and $\bar{m} = \bar{m}_0 + 4\varepsilon$ and run the procedure presented in the proof of Lemma 8 to identify the transition line containing $(\bar{\lambda}_1, \bar{\lambda}_2)$. The proof is complete. □

For Case 2, if the optimal partition on this midpoint is identical with the optimal partition at either $(\bar{\lambda}_1, \bar{\lambda}_2)$ or $(\tilde{\lambda}_1, \tilde{\lambda}_2)$, then the other one is a transition point. In this situation, one can apply the procedure presented in the proof of Lemma 10, to identify a transition line containing this transition point. If the optimal partition defers from optimal partitions at both points $(\bar{\lambda}_1, \bar{\lambda}_2)$ and $(\tilde{\lambda}_1, \tilde{\lambda}_2)$, then both of them are transition points and the line segment joining them is a transition line.

Now we are ready to combine these results in a single algorithm.

Algorithm

Step 1: Consider two real values $\bar{\alpha}$ and $\tilde{\alpha}$ and solve (6) for these values. Denote corresponding points with $(\bar{\lambda}_{u1}, \bar{\lambda}_{u2})$ and $(\tilde{\lambda}_{u1}, \tilde{\lambda}_{u2})$.

Step 2: Denote optimal partitions at $(\bar{\lambda}_{u1}, \bar{\lambda}_{u2})$ and $(\tilde{\lambda}_{u1}, \tilde{\lambda}_{u2})$, with $\bar{\pi}$ and $\tilde{\pi}$, respectively.

Step 3: If $\bar{\pi} = \tilde{\pi}$, then do as the proof of Lemma 8.

Step 4: If $\bar{\pi} \neq \tilde{\pi}$, then do as the proof of Lemma 10.

Step 5: Having a transition point, run the procedure at the end of Lemma 10, till reaching to the first obtained transition point.

4 The optimal value function on an invariancy region

In this section, we investigate the behavior of the optimal value function. The following theorem presents the representation of the optimal value function on the invariancy region $\mathcal{IR}$.

Theorem 11. *The optimal value function Φ is linear in terms of each parameter and it is of degree two totally form on the invariancy region $\mathcal{I}R$.*

Proof. Let $(\bar{\lambda}_1, \bar{\lambda}_2, \bar{\epsilon}_1, \bar{\epsilon}_2)$, $(\tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\epsilon}_1, \tilde{\epsilon}_2)$, and $(\hat{\lambda}_1, \hat{\lambda}_2, \hat{\epsilon}_1, \hat{\epsilon}_2)$, be three arbitrary elements in the invariancy region that are not on a single line. Let $(\bar{x}, \bar{y}, \bar{s}), (\tilde{x}, \tilde{y}, \tilde{s})$ and $(\hat{x}, \hat{y}, \hat{s})$ be arbitrary primal-dual optimal solutions at those points, respectively. For

$$\lambda_1 = \hat{\lambda}_1 - \theta_1 \Delta\bar{\lambda}_1 - \theta_2 \Delta\tilde{\lambda}_1, \tag{9}$$
$$\lambda_2 = \hat{\lambda}_2 - \theta_1 \Delta\bar{\lambda}_2 - \theta_2 \Delta\tilde{\lambda}_2, \tag{10}$$
$$\epsilon_1 = \hat{\epsilon}_1 - \theta_3 \Delta\bar{\epsilon}_1 - \theta_4 \Delta\tilde{\epsilon}_1, \tag{11}$$
$$\epsilon_2 = \hat{\epsilon}_2 - \theta_3 \Delta\bar{\epsilon}_2 - \theta_4 \Delta\tilde{\epsilon}_2, \tag{12}$$

where $0 < \theta_1, \theta_2, \theta_3, \theta_4 < 1, \Delta\bar{\lambda}_1 = \hat{\lambda}_1 - \bar{\lambda}_1, \Delta\tilde{\lambda}_1 = \hat{\lambda}_1 - \tilde{\lambda}_1, \Delta\bar{\epsilon}_1 = \hat{\epsilon}_1 - \bar{\epsilon}_1, \Delta\tilde{\epsilon}_1 = \hat{\epsilon}_1 - \tilde{\epsilon}_1$. We define:

$$\begin{aligned} x^* &= \hat{x} - \theta_1 \Delta\bar{x} - \theta_2 \Delta\tilde{x}, \\ y^* &= \hat{y} - \theta_3 \Delta\bar{y} - \theta_4 \Delta\tilde{y}, \\ s^* &= \hat{s} - \theta_3 \Delta\bar{s} - \theta_4 \Delta\tilde{s}. \end{aligned}$$

It can be easily investigated that (x^*, y^*, s^*) is a primal-dual optimal solution for (P) and (D). The optimal value function at this solution is,

$$\Phi = (b+\lambda_1\Delta b_1+\lambda_2\Delta b_2)^T y^* = a_0+a_1\theta_1+a_2\theta_2+a_3\theta_3+a_4\theta_4+a_5\theta_1\theta_3+a_6\theta_1\theta_4+a_7\theta_2\theta_3+a_8\theta_2\theta_4, \tag{13}$$

where

$$\begin{aligned}
a_0 &= (b + \lambda_1 \Delta b_1 + \lambda_2 \Delta b_2)^T \hat{y}, \\
a_1 &= -(\Delta\bar{\lambda}_1 \Delta b_1^T \hat{y} + \Delta\bar{\lambda}_2 \Delta b_2^T \hat{y}), \\
a_2 &= -(\Delta\tilde{\lambda}_1 \Delta b_1^T \hat{y} - \Delta\tilde{\lambda}_2 \Delta b_2^T \hat{y}), \\
a_3 &= -(b + \hat{\lambda}_1 \Delta b_1 + \hat{\lambda}_2 \Delta b_2)^T \Delta\bar{y}, \\
a_4 &= -(b + \hat{\lambda}_1 \Delta b_1 + \hat{\lambda}_2 \Delta b_2)^T \Delta\tilde{y}, \\
a_5 &= (\Delta\bar{\lambda}_1 \Delta b_1 + \Delta\bar{\lambda}_2 \Delta b_2)^T \Delta\bar{y}, \\
a_6 &= (\Delta\bar{\lambda}_1 \Delta b_1 + \Delta\bar{\lambda}_2 \Delta b_2)^T \Delta\tilde{y}, \\
a_7 &= (\Delta\tilde{\lambda}_1 \Delta b_1 + \Delta\tilde{\lambda}_2 \Delta b_2)^T \Delta\bar{y}, \\
a_8 &= (\Delta\tilde{\lambda}_1 \Delta b_1 + \Delta\tilde{\lambda}_2 \Delta b_2)^T \Delta\tilde{y}.
\end{aligned}$$

On the other hand, solving equations (9), (10) for θ_1, θ_2 and equations (11), (12) for θ_3 and θ_4 lead to

$$\begin{aligned}
\theta_1 &= \alpha_1 + \beta_1 \lambda_1 + \gamma_1 \lambda_2, \\
\theta_2 &= \alpha_2 + \beta_2 \lambda_1 + \gamma_2 \lambda_2, \\
\theta_3 &= \alpha_3 + \beta_3 \epsilon_1 + \gamma_3 \epsilon_2, \\
\theta_4 &= \alpha_4 + \beta_4 \epsilon_1 + \gamma_4 \epsilon_2,
\end{aligned}$$

where

$$\begin{aligned}
\alpha_1 &= \frac{\hat{\lambda}\Delta\tilde{\lambda}_2 - \hat{\lambda}_2 \Delta\tilde{\lambda}_1}{\Delta\bar{\lambda}_1 \Delta\tilde{\lambda}_2 - \Delta\bar{\lambda}_2 \Delta\tilde{\lambda}_1}, \\
\beta_1 &= -\frac{\Delta\tilde{\lambda}_2}{\Delta\bar{\lambda}_1 \Delta\tilde{\lambda}_2 - \Delta\bar{\lambda}_2 \Delta\tilde{\lambda}_1}, \\
\alpha_2 &= \frac{\hat{\lambda}_2 \Delta\bar{\lambda}_1 - \hat{\lambda}_1 \Delta\bar{\lambda}_2}{\Delta\bar{\lambda}_1 \Delta\tilde{\lambda}_2 - \Delta\bar{\lambda}_2 \Delta\tilde{\lambda}_1}, \\
\beta_2 &= \frac{\Delta\bar{\lambda}_2}{\Delta\bar{\lambda}_1 \Delta\tilde{\lambda}_2 - \Delta\bar{\lambda}_2 \Delta\tilde{\lambda}_1}, \\
\gamma_1 &= \frac{\Delta\tilde{\lambda}_1}{\Delta\bar{\lambda}_1 \Delta\tilde{\lambda}_2 - \Delta\bar{\lambda}_2 \Delta\tilde{\lambda}_1}, \\
\gamma_2 &= -\frac{\Delta\bar{\lambda}_1}{\Delta\bar{\lambda}_1 \Delta\tilde{\lambda}_2 - \Delta\bar{\lambda}_2 \Delta\tilde{\lambda}_1}.
\end{aligned}$$

Substitution of the values $\theta_1, \theta_2, \theta_3, \theta_4$ in (13) leads to

$$\Phi(\lambda_1, \lambda_2, \epsilon_1, \epsilon_2) = b_0 + b_1 \lambda_1 + b_2 \lambda_2 + b_3 \epsilon_1 + b_4 \epsilon_2 + b_5 \epsilon_1 \lambda_1 + b_6 \epsilon_2 \lambda_1 + b_7 \epsilon_1 \lambda_2 + b_8 \epsilon_2 \lambda_2,$$

where

$$\begin{aligned}
b_0 &= a_0 + \alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3 + \alpha_4 a_4 + \alpha_1 \alpha_3 a_5 + \alpha_1 \alpha_4 a_6 + \alpha_2 \alpha_3 a_7 + \alpha_2 \alpha_4 a_8, \\
b_1 &= \beta_1 a_1 + \beta_2 a_2 + \beta_1 \alpha_3 a_5 + \beta_1 \alpha_4 a_6 + \beta_2 \alpha_3 a_7 + \beta_2 \alpha_4 a_8, \\
b_2 &= \gamma_1 a_1 + \gamma_2 a_2 + \gamma_1 \alpha_3 a_5 + \gamma_1 \alpha_4 a_6 + \gamma_2 \alpha_3 a_7 + \gamma_2 \alpha_4 a_8, \\
b_3 &= \beta_3 a_3 + \beta_4 a_4 + \beta_3 \alpha_1 a_5 + \beta_4 \alpha_1 a_6 + \beta_3 \alpha_2 a_7 + \beta_4 \alpha_2 a_8, \\
b_4 &= \gamma_3 a_3 + \gamma_4 a_4 + \gamma_3 \alpha_1 a_5 + \gamma_4 \alpha_1 a_6 + \gamma_3 \alpha_2 a_7 + \gamma_4 \alpha_2 a_8, \\
b_5 &= \beta_1 \beta_3 a_5 + \beta_1 \beta_4 a_6 + \beta_2 \beta_3 a_7 + \beta_2 \beta_4 a_8, \\
b_6 &= \beta_1 \gamma_3 a_5 + \beta_1 \gamma_4 a_6 + \beta_2 \gamma_3 a_7 + \beta_2 \gamma_4 a_8, \\
b_7 &= \beta_3 \gamma_1 a_5 + \beta_4 \gamma_1 a_6 + \beta_3 \gamma_2 a_7 + \beta_4 \gamma_2 a_8, \\
b_8 &= \gamma_1 \gamma_2 a_5 + \gamma_1 \gamma_4 a_6 + \gamma_2 \gamma_3 a_7 + \gamma_2 \gamma_4 a_8,
\end{aligned}$$

that completes the proof. $\square$

5 Illustrative example

In this section, examples are presented to illustrate the obtained results.

Example 12. *Consider the problem as follows*

$$\begin{array}{lcccccccccc}
\min & -11x_1 & -2x_2 & +x_3 & -3x_4 & -4x_5 & -x_6 & & & & \\
s.t & 5x_1 & +x_2 & -x_3 & +2x_4 & +x_5 & & & & = & 12 \\
 & -14x_1 & -3x_2 & +3x_3 & -5x_4 & & +x_6 & & & = & 2 \\
 & 2x_1 & +\frac{1}{2}x_2 & -\frac{1}{2}x_3 & +\frac{1}{2}x_4 & & & +x_7 & & = & \frac{5}{2} \\
 & 3x_1 & +\frac{1}{2}x_2 & +\frac{1}{2} & x_3 & +\frac{3}{2}x_4 & & & +x_8 & = & 3 \\
 & x_1, & x_2, & x_3, & x_4, & x_5, & x_6, & x_7, & x_8 & \geq & 0.
\end{array}$$

It is easy to verify that $\pi = (\mathcal{B}, \mathcal{N}) = (\{1,2,3,5,6\}, \{4,7,8\})$ is the optimal partition of the index set {1, 2, 3,..., 8}. Let $\Delta b_1 = (\frac{5}{3}, \frac{101}{6}, \frac{-5}{6}, \frac{-3}{4})^T$ and $\Delta b_2 = (\frac{-17}{6}, \frac{32}{3}, \frac{-1}{6}, \frac{-1}{2})^T$ be perturbing directions. Running Algorithm leads to the region depicted in Figure 1.

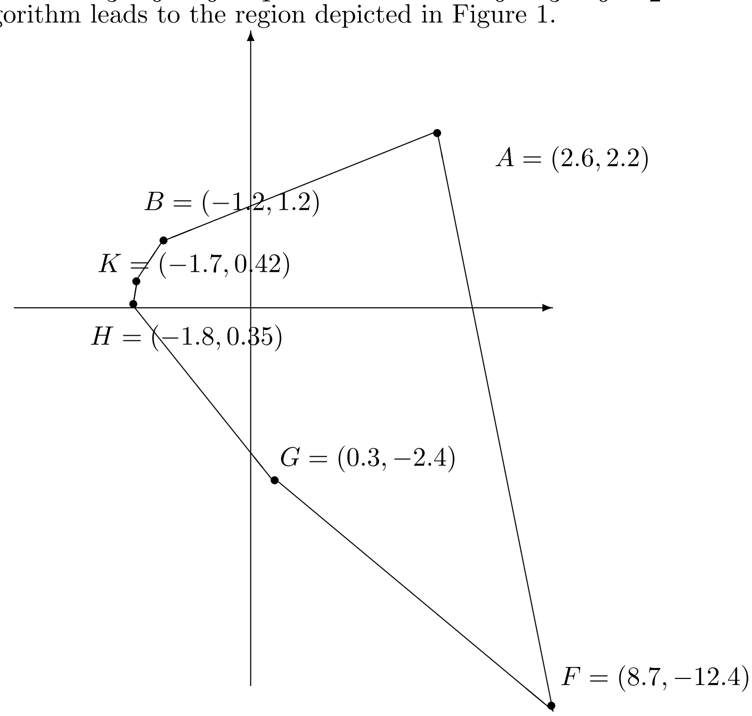

Figure 1:The invariancy region obtained for Example 1.

Example 13. *Consider the problem as follows*

$$\begin{array}{rl}
\min & -x_1 - x_2 - 2x_3 \\
s.t: & x_1 + x_2 + 2x_3 + x_4 = 1 \\
 & x_1 - x_2 + x_5 = 1 \\
 & x_1 + x_2 + x_3 + x_6 = 1 \\
 & x_1, x_2, x_3, x_4, x_5, x_6 \geq 0.
\end{array}$$

It is easy to verify that $\pi = (\mathcal{B}, \mathcal{N}) = (\{1,2,3,5,6\}, \{4\})$ is the optimal partition of the index set {1, 2, 3,..., 6}. Let $\Delta b_1 = (1,2,1)^T$ and $\Delta b_2 = (2,0,-3)^T$ be perturbing directions. Running Algorithm leads to the region depicted in Figure 2.

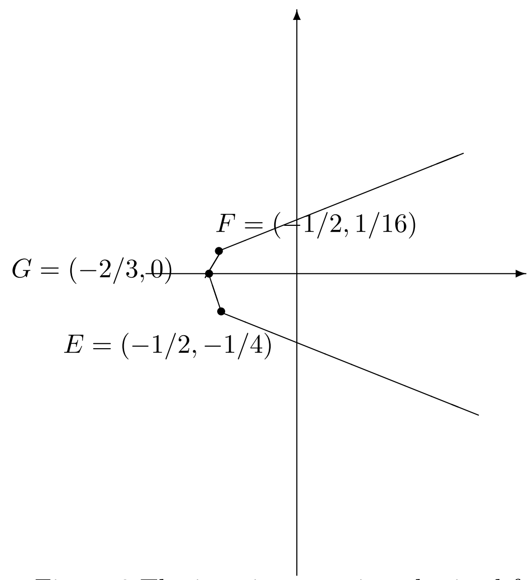

Figure 2:The invariancy region obtained for Example 2.

Example 14. *Consider the example 13 with perturbation vectors* $\Delta b_1 = (1, \frac{1}{2}, \frac{1}{4})^T$ *and* $\Delta b_2 = (-1, 2, \frac{-3}{4})^T$*. Running Algorithm leads to the region depicted in Figure 3.*

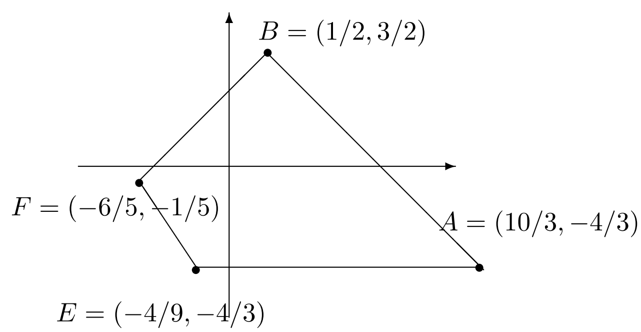

Figure 3:The invariancy region obtained for Example 3.

Example 15. *Consider the problem as follows*

$$\begin{aligned} \min \quad & -2x_1 - x_2 \\ s.t: \quad & x_1 + x_2 + x_3 = 4 \\ & x_1 + 2x_2 + x_4 = 6 \\ & 2x_1 + x_2 + x_5 = 6 \\ & x_1 + x_6 = 3 \\ & x_1, x_2, x_3, x_4, x_5, x_6 \geq 0. \end{aligned}$$

It is easy to verify that $\pi = (\mathcal{B}, \mathcal{N}) = (\{1, 2, 3, 4, 6\}, \{5\})$ is the optimal partition of the index set $\{1, 2, 3, ..., 6\}$. Let $\Delta b_1 = (1, 2, -1, 1)^T$ and $\Delta b_2 = (2, -1, 1, 2)^T$ be perturbing directions. Running Algorithm leads to the region depicted in Figure 4.

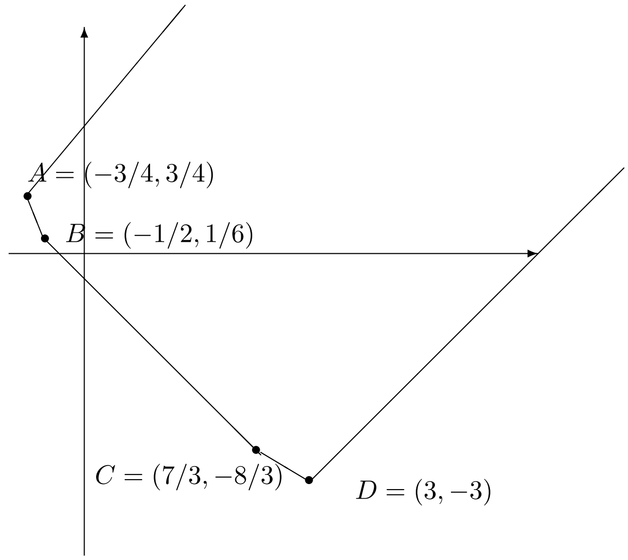

Figure 4:The invariancy region obtained for Example 4.

In the Figures 1, 2, 3 and 4, lines determining the invariancy regions are called transition lines and intersection of these lines are transition points. Consider the point $(\bar{\lambda}_1, \bar{\lambda}_2) = (2, 2)$ on the transition line AB in Figure 1, it is easy to verify that the optimal partition in this point is

$$\pi = (\mathcal{B}, \mathcal{N}) = (\{2, 5, 6\}, \{1, 3, 4, 7, 8\}).$$

Thus $\bar{\mathcal{B}} \subset \mathcal{B}, \mathcal{N} \subset \bar{\mathcal{N}}$. This result is in agreement with Lemmas 6 and 7. By the step 2 of the Algorithm, Lemmas 6 and 7 are true in other transition lines of this region and also in the invariancy regions of other examples.

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