A simple algorithm for computing a zero of a nonlinear function of a variable in a given interval [a,b]

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In this technical report we present a simple algorithm for computing a zero of a nonlinear function in a given interval, that is find $x^* \in [a,b]$, such that

$$f(x^*) = 0,$$

where $f : \mathbb{R} \to \mathbb{R}$ is defined in interval $[a,b]$ and $a < b$. The idea of the algorithm is as follows:

Algorithm

1) Compute the values: $fa = f(a)$ and $fb = f(b)$.
2) If $f(a)f(b) > 0$, then apparently in the interval $[a,b]$ there is no zero of function f. Modify the bounds a and b and restart the procedure.
3) If $|f(a)| \le \varepsilon$ or $|f(b)| \le \varepsilon$, then one of the points a or b is a zero of f.
4) Otherwise compute the point $c = a + (b-a)/2$ and the value $f(c)$.
5) If $f(c) \ge 0$, then set $b = c$, $fb = fc$ and go to step 6; else $a = c$, $fa = fc$ and go to step 6.
6) Go to step 2. ♦

At every iteration the interval is halved. The middle point replaces one of the bounds of the interval according to the value of function f. Therefore, the length of the intervals is going to zero, thus showing the convergence of the algorithm. Initially, the length of the interval is $b-a$. At iteration k the length of the current interval is $(b-a)/2^k$.
Some examples illustrate the running of the algorithm.

Example 1
Consider the function

$$f(x) = x^2 - 4.$$

The plot of this function is presented in Figure 1. The results of the algorithm for different bounds a and b are as follows.

Experiment #1:
a= 0.0000000000000E+00 fa=-0.4000000000000E+01
b= 0.3000000000000E+01 fb= 0.5000000000000E+01
Solution: 0.2000000000015E+01 Function value: 0.5820766091347E-10
Number of iterations; 36

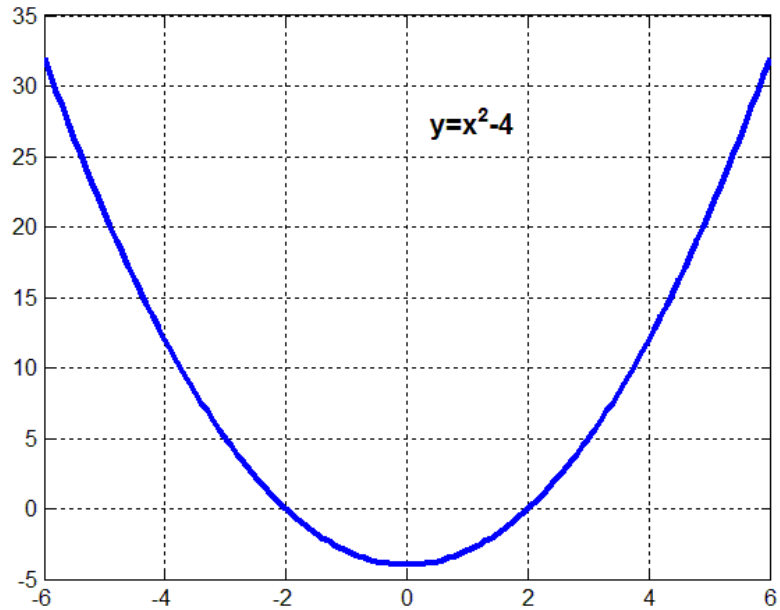

Fig. 1. Function $y = x^2 - 4$.

Experiment #2

a= 0.0000000000000E+00 fa=-0.4000000000000E+01
b= 0.6000000000000E+01 fb= 0.3200000000000E+02
Solution: 0.2000000000015E+01 Function value: 0.5820766091347E-10
Number of iterations; 37

Experiment #3

a= 0.1000000000000E+01 fa=-0.3000000000000E+01
b= 0.4000000000000E+01 fb= 0.1200000000000E+02
Solution: 0.1999999999985E+01 Function value: -0.5820766091347E-10
Number of iterations; 36

Experiment #4

a=-0.6000000000000E+01 fa= 0.3200000000000E+02
b= 0.0000000000000E+00 fb=-0.4000000000000E+01
Solution: -0.2000000000015E+01 Function value: 0.5820766091347E-10
Number of iterations; 37

Experiment #5

a=-0.6000000000000E+01 fa= 0.3200000000000E+02
b= 0.1000000000000E+01 fb=-0.3000000000000E+01
Solution: -0.2000000000007E+01 Function value: 0.2910383045673E-10
Number of iterations; 37

Experiment #6

a=-0.6000000000000E+01 fa= 0.3200000000000E+02
b=-0.1000000000000E+01 fb=-0.3000000000000E+01
Solution: -0.1999999999985E+01 Function value: -0.5820766091347E-10
Number of iterations; 36

Experiment #7

a=-0.6000000000000E+02 fa= 0.3596000000000E+04
b= 0.1000000000000E+01 fb=-0.3000000000000E+01
Solution: -0.2000000000020E+01 Function value: 0.8003553375602E-10
Number of iterations; 39

Experiment #8

a=-0.6000000000000E+01 fa= 0.3200000000000E+02
b= 0.6000000000000E+01 fb= 0.3200000000000E+02
Aparently in interval [-0.6000000000000E+01, 0.6000000000000E+01] NO Zero
Modify the interval by changing the bounds a and/or b

Example 2

Consider the function

$$f(x) = x^2 - 8x - 9.$$

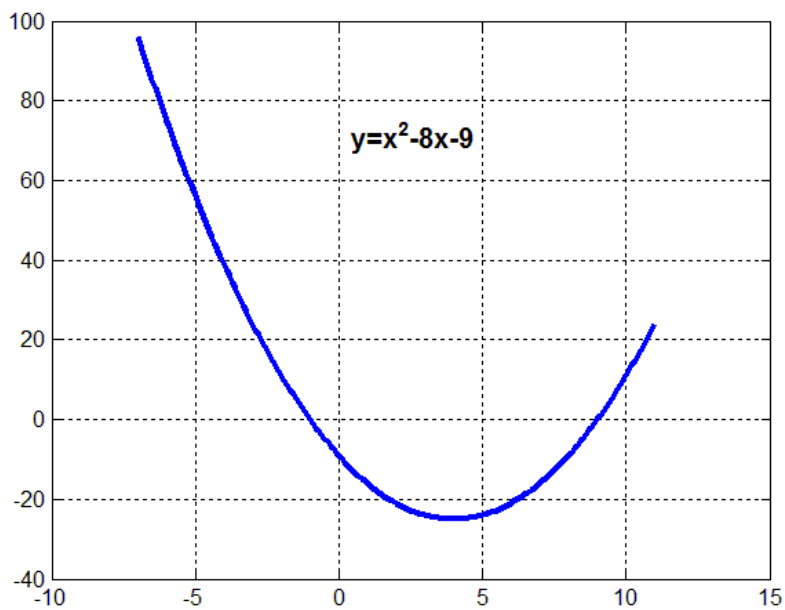

Fig. 2. Function $y = x^2 - 8x - 9.$

Experiment #1

a= 0.0000000000000E+00 fa=-0.9000000000000E+01
b= 0.2000000000000E+01 fb=-0.2100000000000E+02
Aparently in interval [0.0000000000000E+00, 0.2000000000000E+01] NO Zero
Modify the interval by changing the bounds a and/or b

Experiment #2

a= 0.0000000000000E+00 fa=-0.9000000000000E+01
b= 0.1200000000000E+02 fb= 0.3900000000000E+02
Solution: 0.9000000000000E+01 Function value: 0.0000000000000E+00
Number of iterations; 2

Experiment #3

a=-0.3000000000000E+01 fa= 0.2400000000000E+02
b= 0.1000000000000E+01 fb=-0.1600000000000E+02
Solution: -0.1000000000000E+01 Function value: 0.0000000000000E+00
Number of iterations; 1

Experiment #4

a=-0.3000000000000E+02 fa= 0.1131000000000E+04
b= 0.1000000000000E+02 fb= 0.1100000000000E+02
Aparently in interval [-0.3000000000000E+02, 0.1000000000000E+02] NO Zero
Modify the interval by changing the bounds a and/or b

Experiment #5

a=-0.3000000000000E+02 fa= 0.1131000000000E+04
b= 0.0000000000000E+00 fb=-0.9000000000000E+01
Solution: -0.9999999999991E+00 Function value: -0.9094947017729E-11
Number of iterations; 41

Example 3

Consider the function:

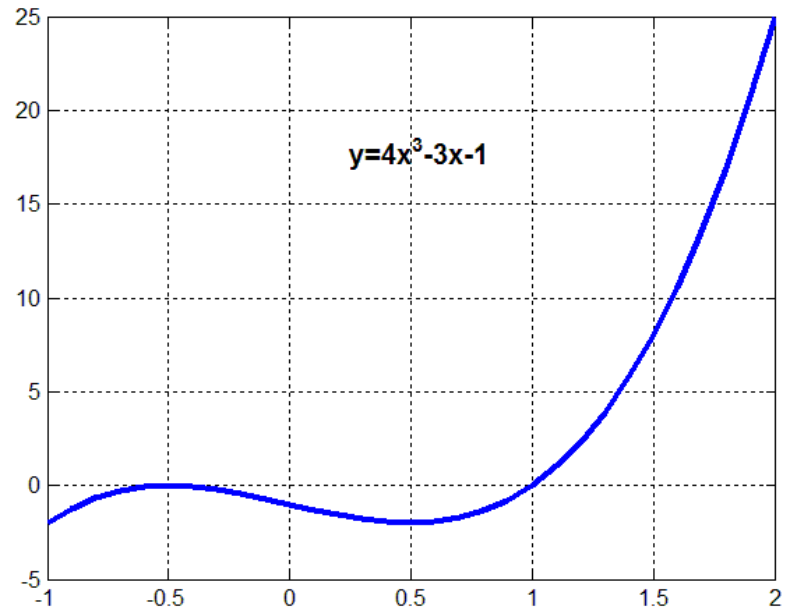

Fig. 3. Function $y = 4x^3 - 3x - 1$

Experiment #1

a= 0.0000000000000E+00 fa=-0.1000000000000E+01
b= 0.1500000000000E+01 fb= 0.8000000000000E+01
Solution: 0.1000000000007E+01 Function value: 0.6548361852765E-10
Number of iterations; 36

Experiment #2

a=-0.5000000000000E+00 fa= 0.0000000000000E+00
b= 0.1500000000000E+01 fb= 0.8000000000000E+01
Solution: -0.5000000000000E+00 Function value: 0.0000000000000E+00
Number of iterations; 0

Experiment #3

a=-0.2000000000000E+01 fa=-0.2700000000000E+02
b= 0.0000000000000E+00 fb=-0.1000000000000E+01
Aparently in interval [-0.2000000000000E+01, 0.0000000000000E+00] NO Zero
Modify the interval by changing the bounds a and/or b

Remark: Notice, at $x = -0.5$, the graph bounces off the x-axis. The algorithm is not able to find such points.

Example 4

Consider the function

$$f(x) = x^3 + 4x^2 - 4x - 16.$$

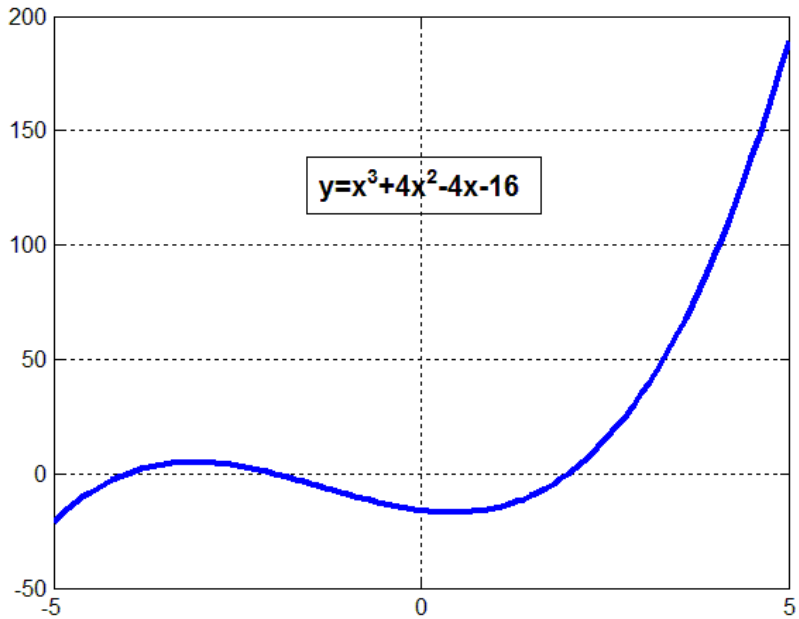

Fig. 4. Function $y = x^3 + 4x^2 - 4x - 16.$

Experiment #1

a=-0.5000000000000E+01 fa=-0.2100000000000E+02
b=-0.2500000000000E+01 fb= 0.3375000000000E+01
Solution: -0.3999999999996E+01 Function value: 0.4365574568510E-10
Number of iterations; 37

Experiment #2

a=-0.2500000000000E+01 fa= 0.3375000000000E+01
b= 0.0000000000000E+00 fb=-0.1600000000000E+02
Solution: -0.2000000000007E+01 Function value: 0.5820766091347E-10
Number of iterations; 36

Experiment #3

a= 0.0000000000000E+00 fa=-0.1600000000000E+02
b= 0.3000000000000E+01 fb= 0.3500000000000E+02
Solution: 0.2000000000004E+01 Function value: 0.8731149137020E-10
Number of iterations; 38

Experiment #4

a=-0.5000000000000E+01 fa=-0.2100000000000E+02
b= 0.3000000000000E+01 fb= 0.3500000000000E+02
Solution: 0.2000000000000E+01 Function value: 0.0000000000000E+00
Number of iterations; 3

Experiment #5

a=-0.5000000000000E+01 fa=-0.2100000000000E+02
b= 0.0000000000000E+00 fb=-0.1600000000000E+02
Aparently in interval [-0.5000000000000E+01, 0.0000000000000E+00] NO Zero
Modify the interval by changing the bounds a and/or b

Example 5

Consider the function

$$f(x) = \frac{1}{(x-0.3)^2 + 0.01} + \frac{1}{(x-0.9)^2 + 0.04} - 6.$$

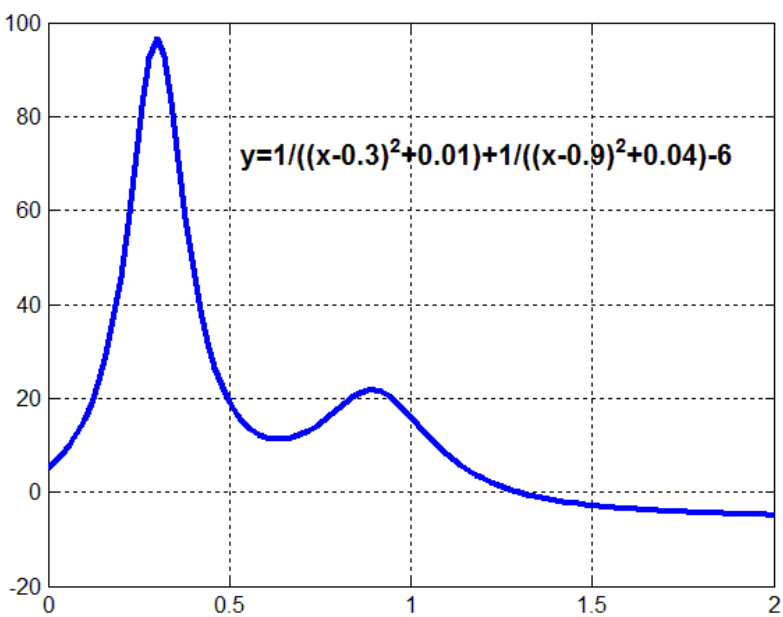

Fig. 5. Function $y = \frac{1}{(x-0.3)^2 + 0.01} + \frac{1}{(x-0.9)^2 + 0.04} - 6.$

Experiment #1

a= 0.0000000000000E+00 fa= 0.5176470611825E+01
b= 0.2000000000000E+01 fb=-0.4855172413194E+01
Solution: 0.1299549683616E+01 Function value: -0.3890576749654E-10
Number of iterations; 38

Experiment #2

a= 0.0000000000000E+00 fa= 0.5176470611825E+01
b= 0.1500000000000E+01 fb=-0.2810344821892E+01
Solution: 0.1299549683614E+01 Function value: 0.1135092020377E-11
Number of iterations; 38

Experiment #3

a= 0.1000000000000E+01 fa= 0.1600000035852E+02
b= 0.1500000000000E+01 fb=-0.2810344821892E+01
Solution: 0.1299549683616E+01 Function value: -0.3890576749654E-10
Number of iterations; 36

Experiment #4

a= 0.1000000000000E+01 fa= 0.1600000035852E+02
b= 0.1050000000000E+02 fb=-0.5979543248959E+01
Solution: 0.1299549683615E+01 Function value: -0.2889599670652E-10
Number of iterations; 40

Experiment#5

a= 0.1000000000000E+01 fa= 0.1600000035852E+02
b= 0.1000500000000E+04 fb=-0.5999997999599E+01
Solution: 0.1299549683612E+01 Function value: 0.3781241986189E-10
Number of iterations; 47

Fortran program:

```
C==============================================================
C A simple algorithm for computing zero of a nonlinear function
C of a varioable in a given interval [a,b]
C
C Neculai Andrei
C March 28, 1975
C==============================================================


        subroutine func(x,f)
        real*8 x,f

c       f = x**2 - 4.d0
c       f = x**2 - 8.d0*x - 9.d0
c       f = 4.d0*x**3 - 3.d0*x - 1.d0
c       f = x**3 + 4.d0*x**2 - 4.d0*x - 16.d0

        f = 1.d0/((x-0.3d0)**2+0.01) + 1.d0/((x-0.9d0)**2+0.04) -6.d0

        return
        end


c============================================ Main program
c                                             ==============
        real*8 a,b,c, x,f
        real*8 fa,fb,fc,fx
        real*8 epsm
        integer iprint
```

```
        epsm=0.0000000001d0
        iprint = 0

C Searching interval
c ==================
        a = 1.d0
        b = 1000.5d0

      open(unit=4,file='zero.out',status='unknown')

        iter=0

        call func(a,fa)
        call func(b,fb)
        write(4,50) a,fa
50      format(4x,'a=',e20.13,5x,'fa=',e20.13)
        write(4,51) b,fb
51      format(4x,'b=',e20.13,5x,'fb=',e20.13,/)

          if(fa*fb .gt. 0.d0) then
        write(4,100) a,b
100     format(4x,'Aparently in interval [',e20.13,',',e20.13,'] NO Zero')
        write(4,101)
101    format(4x,'Modify the interval by changing the bounds a and/or b')
        go to 998
          end if

c-------------------------------------------------------------
10      continue

        if(dabs(fa) .le. epsm .or. dabs(fb) .le. epsm) go to 999

        iter=iter+1

        if(fa .le. 0.d0) then

          c = a + (b-a)/2.d0
          call func(c,fc)

          if(fc .ge. 0.d0) then
            b = c
            fb = fc
            if(iprint .eq. 1) then
              write(4,99) iter
99                format(4x,'iter='i5)
                  write(4,110) a,b,c
110               format(4x,' a='e20.13,4x,' b=',e20.13,4x,' c=',e20.13)
                  write(4,111) fa,fb,fc
111               format(4x,'fa='e20.13,4x,'fb=',e20.13,4x,'fc=',e20.13)
                  write(4,112) b-a
112           format(4x,'fa<0. b-a=',e20.13)
           end if
            go to 10
          else
            a = c
            fa = fc
            if(iprint .eq. 1) then
              write(4,99) iter
                  write(4,110) a,b,c
                  write(4,111) fa,fb,fc
              write(4,112) b-a
           end if
            go to 10
          end if
        end if


        if(fa. ge. 0.d0) then

          c = a + (b-a)/2.d0
```

```
          call func(c,fc)

          if(fc .ge. 0.d0) then
            a = c
            fa = fc
            if(iprint .eq. 1) then
              write(4,99) iter
                  write(4,110) a,b,c
                  write(4,111) fa,fb,fc
              write(4,113) b-a
113               format(4x,'fa>0. b-a=',e20.13)
           end if
            go to 10
          else
            b  = c
            fb = fc
            if(iprint .eq. 1) then
              write(4,99) iter
                  write(4,110) a,b,c
                  write(4,111) fa,fb,fc
              write(4,112) b-a
           end if
            go to 10
          end if
        end if

999     continue

        if(dabs(fa) .le. epsm) x = a
        if(dabs(fb) .le. epsm) x = b
        call func(x,fx)

        write(4,120) x,fx
120     format(2x,'Solution: ',e20.13,4x,'Function value: ',e20.13)
        write(4,121) iter
121     format(2x,'Number of iterations; ',i7)

998     continue

        stop
        end
c=========================================================================
c Last line
```

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